Probability Foundations for Electrical Engineers
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Lecture – 31
Examples: 2 Random Variables

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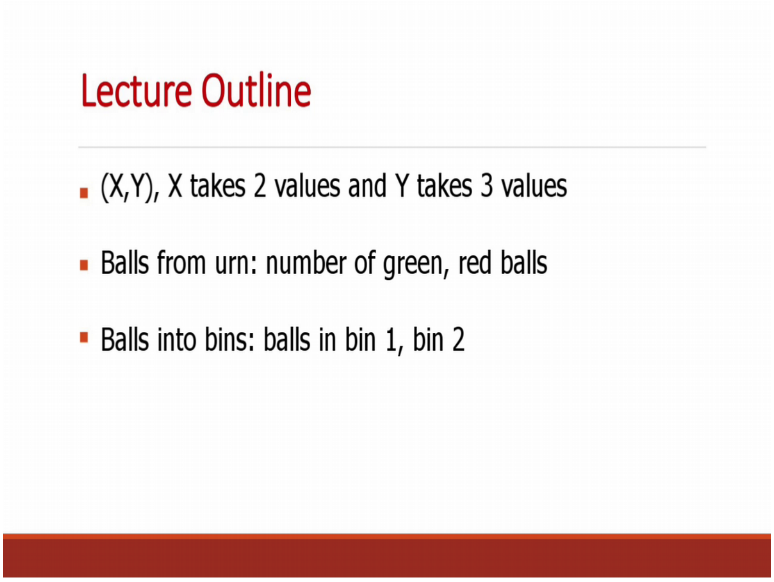

So, in this lecture we are going to be seeing examples of two random variables. So, I am sure you have looked at Professor Aravinds lectures on two random variables, and the meaning of joint PMF, and I will give you some very basic simple examples of joint PMFs, how to write them down, how to think of them, how in some simple situations, how to evaluate the joint PMF. It is not significantly different in principle from what you have learnt already in terms of events, but except that the nomenclature, what you call a random variable in distribution etcetera may be a little bit new. So, let us see a few examples.

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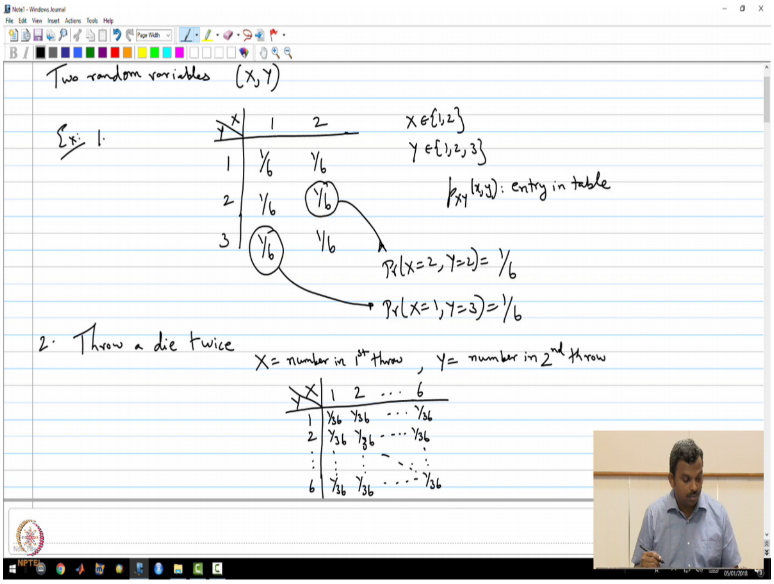

So, I will start with a simple case of two random variables X Y, and let us say they take values like this. So, I like drawing the two random variable case in some sort of a table. So, I would put a table like this and say X takes value here, Y takes values here maybe X takes value is let us say 1 and 2, maybe Y takes 3 values 1 2 3. So, this is what does this table tell you; this table tells you that X takes values 1 and 2 and Y takes values 1 2 3, and each entry in this table will be p x y of x comma y.

So, that would be the entry here. So, for instance a very simple way to write this down is to say 1 by 6, 1 by 6, 1 by 6, 1 by 6, 1 by 6, 1 by. So, this is a very simple example some sort of a uniform distribution, there are 2 random variables here X and Y, and they take values 1 comma 1 with probability 1 by 6. So, what that is what it means what is that what are these things means. So, for instance this entry here means that probability that X equals 2 and Y equals 2 is 1 by 6.

The meaning of this comma is and like Professor Aravind also mentioned a probability that X equals 2 and Y equals 2 is 1 by 6; there for instance this entry means probability that X equals 1, Y equals 3 this 1 by 6. So, this is the way in which one thinks of 2 random variables and taking values together. So, any other case in which you have 2 random variables you can write it like this. So, you can write X taking values along the columns, and Y taking values along the rows and write it down and then write down the probability of these things. So, this is a very simple nice example. So, what I am going to do now is give you some situations based on our continuing series of examples, tossing a coin, throwing a die etcetera and then try and come up with distributions for 2 random variables. So, that is the next few examples that we are going to see. So, let us begin with. So, this is a first example let me do an example with throwing a die twice, this is also a very very simple example and I will say X as the number in first toss first throw, and Y is the number in second throw.

So, you have a dice you threw it once, threw it twice, the first time you got some number that I am going to call as X. So, this X could be 6 possibilities 1, 2, 3, 4, 5, 6 and Y is the number that you got in the second toss second throw, and that is also going to be from 1 to 6. And we have seen before that the probability that X takes a particular number comma Y takes the particular number is actually 1 by 36, there are 36 different possibilities each possibility is equally likely, you get the 1 by 36 there. So, if I have to write it down like in this table.

So, I am going to put X here, we are going to put Y here and then I will have numbers from 1 2 dot dot dot 6, likewise here I will have 1 2 dot dot dot 6 what will be the number in each of these places it will be 1 by 36 right. This is an example of a joint PMF that occurred from this simple experiment likewise you can fill up this table it is as a 6 by 6 table; each number is 1 by 36. So, this is how we come up with very simple examples of joint distributions.

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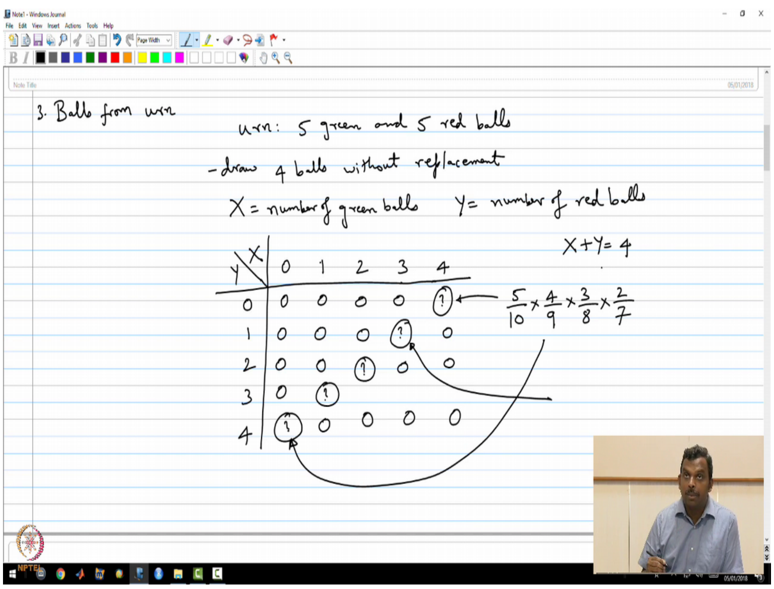

One can complicate things a little bit more let me give you another example. Let us look at the next example is balls from an urn; yes let us say we have an urn which has 5 green and 5 red balls. So, this could be anything else also I am just giving you a number and let us say we draw 4 balls without replacement. So, all these things are quite crucial, I am going to draw 4 balls 1 after another without replacing them and then I am going to call X as the number of green balls, and I am going to call Y as the number of red balls alright. So, now, I want to make this table once again, I am going to put X here I am going to put Y here then start making this table. So, what are the possible values that X can take?

If I take 4 balls without replacement, X could be 0 right I might have no green balls at all the balls I took might be red or it could be 1, or it could be 2, or it could be 3 or it could be 4 same way with Y. So, Y could be 0, Y could be 1, Y could be 2, Y could be 3, Y could be 4 I took a symmetric sort of case that you had 5 green balls in 5 red balls, now supposing I had taken 5 green balls and only 3 red balls, and I had picked up 4 balls without replacement then the situation will be different now what how many red balls you can have you can have at most 3 red balls right.

So, things become a little bit more murky there, but let us say we pick a case like this. So, now, I want to write down probabilities for each of these possibilities. So, now, let us I mean when you try to fill out these kind of tables and fill out numbers with probabilities, you to you have to first figure out the what the easy cases are, and figure out what the difficult cases are.

Now see one of the nice things about this problem is, I know that I picked 4 balls. So, if I tell you the value of X right; if X is going to be say 1, I had 1 green ball how many red balls should I necessarily have. I should definitely have 3 red balls right. So, in this case these 2 random variables X and Y are connected in a very nice manner right the ball could be either green or red. And if I draw 4 balls and if I have X number of green balls Y should be 4 minus x. So, we have this nice relationship that X plus Y equals 4. So, for any other possibility where X plus Y is not 4, the probability has to be 0 right. So, if you look at p x y of X comma Y it just going to be 0 here, 0 here, 0 here, 0 here it might be something non zero here.

And also there is a certain symmetry here you can argue. So, the symmetry is extremely powerful when you try to write down probabilities, there is a symmetry here now whatever happens with X the same thing should happen with Y right. So, if you exchange X and Y you should get the same answer. So, this matrix so, to speak has to be symmetric, what do you have on the top side should be the same as what you have on the bottom side right. So, you cannot have any other way. So, you will have 0 here, 0 here, I do not know what the 0 comma 4 will be, I will write down possible values later on. So, let us start putting out these guys something here, this is going to be 0. So, this is 0, this could be something here, this is 0, this is 0. So, you see clearly only along this line you can even have nonzero probabilities do you agree. So, any other possibility has 0 probabilities.

So, this joint distribution is actually working out in a very very simple fashion nevertheless it is good to see how to write it down in some some examples. So, what about values in the middle, what is the probability that you will have 0 red balls and 4 green balls, this guy this entry here is going to be the first ball has to be green ok, given that the first ball was green, the second ball should also be green, third ball should also be green, and fourth ball should also be green. So, that will be the number that comes there is it correct. So, now, you might want to think of how to put in the entry here, and the entry here and the entry here and the entry here, what will be the entry here? And here will be the same as this thing right what about this one? This one will require slightly more involved calculation right.

So, you have to figure out exactly 1 red ball and 3 green balls; have to come up when you draw 4 balls without replacement. So, you have to figure out where the red ball will come, it is a little bit more complicated. So, I am not going to do this computation for you, I will leave it as an exercise you can do it later. So, same thing with same thing with these values, but this is how the non zero values will be. So, the non zero values will be along this diagonal, and they will all add up to 1 and there will be symmetric in some fashion. So, you can quickly try and write down the probability. So, this was a case in which you actually had 2 random variables, but nevertheless it was almost like only 1 random variable because there 2 of them were tightly coupled. So, if you take a slightly more complicated situation, you might get more interesting things. So, let us take more interesting sort of scenarios, where you have two random variables again I will do this balls from example.

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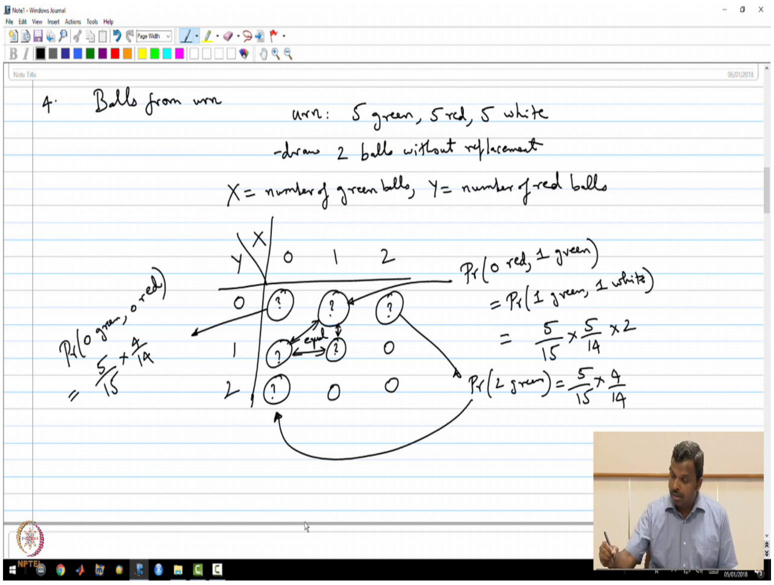

So, let me have an urn now I will have 3 balls. So, I will have let us say 5 green, 5 red, let us say 5 white balls. Now the things you get a little bit more interesting and let us say I draw I will make my job a little bit easier I say I will draw 2 balls, without replacement now. And then I will let X be the number of green balls, just like before, and I will let Y be the number of red balls. Now in this case I do not have this relationship that X plus Y is equal to 2, because I also had these white balls right.

So, if every single possibility will exist and then you can have more interesting probability distribution function; you can have let us say if you make this table here X can take values 0, 1, 2 Y can take values 0, 1, 2, but none of these things are going to be 0 right all of these things can be nonzero and one needs to calculate these probabilities. So, let us say.

So, so hopefully this is clear to you because there are 3 different colors of balls, and I am drawing 2 balls without replacement that is then I have only 2 random variables right. X is number of green balls Y is number of red balls. So, every single possibility can happen right, I can have 0 green balls 0 red balls right this entry. I can have 0 green balls 0 red balls right probability of 0 green 0 red right that is the probability here. So, what will the probability work out to that will be the probability that I have 2 white balls, I draw 2 balls both of them have to be white.

So, that would mean 5 by 15 times 4 by 14 right for first ball has to be white conditional on that the next ball should also be white. So, I have 4 by 14 that will be the entry here. So, likewise you can try and fill in other entries it is things will get a little bit more complicated as you compute these probabilities. So, for instance if you want to compute this guy. So, this is the probability of 0 or red and 1 green right, that is the probability now this could happen in multiple ways I might have 1 green and 1 white or I could have 1 green. So, there should be no red I can have 1 green and 1 white right that is the only possibility sorry.

No matter 2 possibilities there. So, we repeat this. So, probability of 0 red and 1 green what should happen here is, this is the same as probability of having 1 green and 1 white its. So, the only 2 balls I am drawing, if I do not want any red, one has to be green the other has to be white how can this happen this can happen in 2 ways either the green could come first and the white could come next or the white could come first and the green could come next.

So, in either way the probability will be 5 by 15 times 5 by 14, and like I said this can happen in 2 ways because the green could come first and a white could come next or the white could come first and the green could come next. So, they understood. So, that is the probability. So, likewise you have to figure out what this probability will be this one also is actually quite easy, I think this is one this probability of 2 green that is the same as probability of 2 white that is just 5 by 15 times 4 by 14.

So, likewise you can fill up all these entries, it is not too difficult because I am drawing only 2 balls you can fill it up. So, what is the kind of a moral of a story here? So, suppose we want to define 2 random variables, and you want a sample space to support those 2 random variables, and you want those 2 random variables to be interesting in some way the sample space should be very very rich as in it should be richer than 2 possibilities.

So, for instance in the previous example we had only 2 colored balls, and when I say 1 is green the other is red really together they really only have one dimension. So, you saw in the previous example that you had only one dimension, it is not really both of them are not really giving you nonzero probabilities. On the other hand when my sample space is rich when I have 3 different colors and then I am drawing only 2 balls and I have only 2 random variables here all these entries become nonzero. So, you can go ahead and fill up these entries maybe there will be an assignment problem based on this how to fill up the entries here.

So, this is how you compute PMFs for joint distribution of 2 random variables, how do you find the PMF. You have to just write down each event carefully and write them hopefully that is clear to you I am not going this in full detail, everyone have written down three in fact, actually using this you can pretty much fill up most of the possibilities. In fact, 1 entry here so, you can say this entry will be 0 this entry will be 0 right this entry will also be 0 right. So, because you cannot have 2 green and 2 red I mean only two balls are being drawn. So, I think most of these problems are answered. In fact, this same these 2 entries will be equal, and then this same number will come here and this is the only number that you have to compute one can quite easily compute this number. So, this will be a nice triangular sort of PMF in this simple example.

So hopefully you can complete this problem, this is the only thing that is missing 1 comma 1 that is also very easy it will be similar to this expression and you will get it. Think about how to computed one very very important check that you have to do is after you compute the PMF for this joint distribution, you have to add everything up and check that you get 1. If you do not get 1 something is wrong with your calculation. So, that is one way to quickly verify whether your answer is correct or wrong. So in fact, I was just now thinking that all these 3 numbers have to be equal. So, you would get the same thing. So, let us see if our. So, let us complete this example and maybe I can check if this is possible or not. So, let me erase these guys and actually put in the values and then check for you if these guys are going to work out or not so, let us see.

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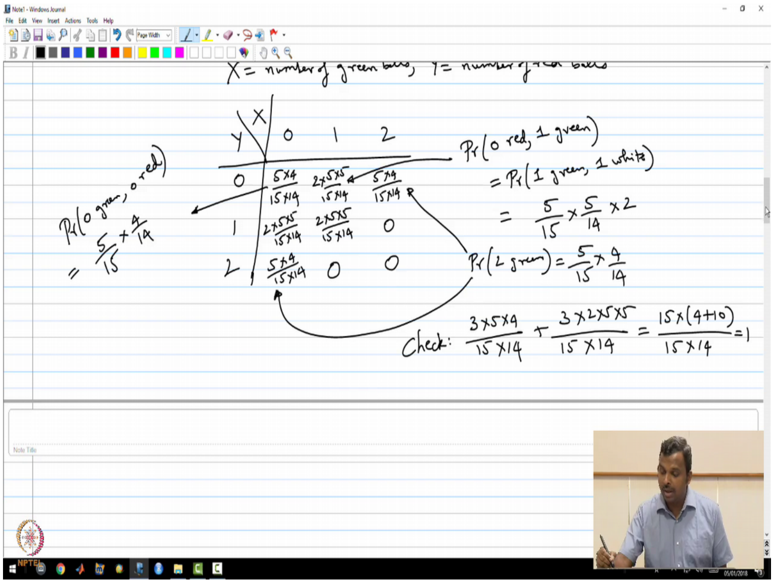

So, this is going to be 5 into 4 carefully 5 into 4 by 15 into 14, all these 3 entries are going to be 2 times 5 times 5 by 15 into 14, all these 3 entries 2 times 5 times 5 by 15 into 14. These 2 entries are again going to be the same thing right. So, 5 by 5 times 4 by 15 into 14. So, this is the answer. So, like I said we have to check if this is a valid PMF.

So, how do I know it is a valid PMF, this is the check for that, let us add all of these things. If you add all of these things you would get 3 into 5 into 4 by 15 into 14 plus 3 into 2 into 5 into 5, 5 by 15 into 14. Now comes the moment of truth is this going to add up to 1. So, in the numerator if you see 5 into 3 appears as common here. So, I can take that as a common factor. So, if I do that 5 into 3 is 15 times what do I have? 4 plus 10 which is 14 right divided by 15 into 14 and that gives me 1 fair enough this is a valid PMF. So, once again I want to emphasize in this example that the compared example 3 and example 4. So, when you have a smaller sample space and you want to define too many random variables in them, you will keep getting redundant situations nothing interesting will happen.

But when you have very big sample spaces with a lot of possibilities, then you can define 2 random variables and you can get non trivial examples so in fact, you can do more interesting things with such example such as I mentioned maybe the assignment problems will be interesting then.

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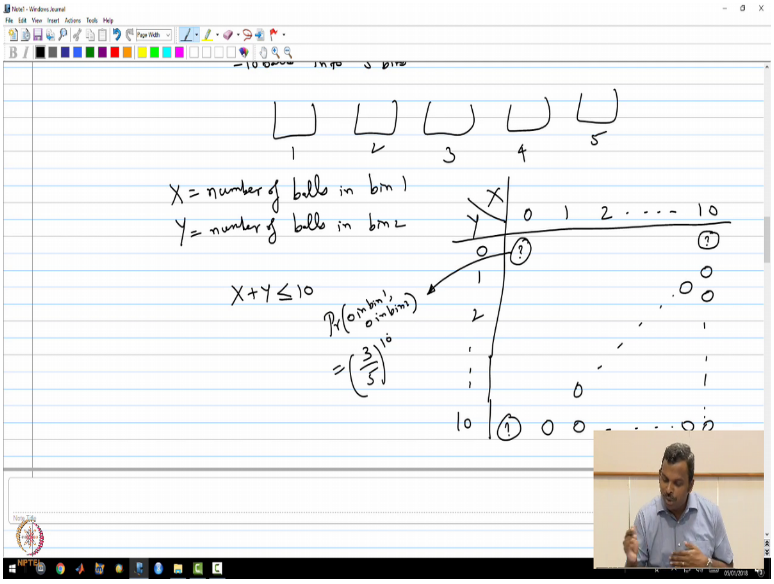

So, I think I want to close this particular lecture with one more example with the I will be very brief with this example, but balls and balls into bins is also a very very rich example. So, you can define quite a few random variables, let us say you have 10 balls into 5 bin, this situation is rich enough that you can define a lot of examples of interest for us one particular example might be X is the number of balls in bin 1, and Y is the number of balls 2.

So, here again if you were to draw this table here, you would have an interesting possibility here 0 1 2 all the way up to 10, 0 1 2 all the way up to 10 and you can see clearly X plus Y has to be I can say is less than or equal to 10 right. So, even though X can take values from 0 to 10, Y can take values from 0 to 10, X plus Y together cannot be greater than 10 because there are only 10 balls that are being thrown. So, what will happen when you have a condition like that is, when X is 10 Y can be something can be nonzero here, but everything below that will be 0. Same thing here something non 0 here and everything to the right of that it is going to be 0, same thing with any other possibility.

So, along this triangular part, this part will always go off to 0. So, this is kind of similar to the balls in the urn situation. So, all these guys will be 0, but what about this upper triangular part? You have to do some careful calculation to compute that. So, this is the interesting thing that we can do here, the calculations here are not easy at all you have to pay a lot of attention and carefully do it, I will just show you one calculation which is the easiest one. So, the easiest one is this guy what is that? Probability of 0 and bin 1 comma 0 in bin 2, this is easy bin 1 is 0 bin 2 is 0. So, every ball should fall either in the third bin fourth bin or fifth bin. So, that is just 3 by 5 raise to the power 10. An easy calculation I did that if you want to do anything else I will come and to do it, it is a little bit more complicated.

So, anyway hopefully I gave you enough illustrations of simple scenarios in which 2 random variables occur in a natural fashion, and I also showed you an interesting case in which other random variables are non trivial why the sample space should be rich enough to support 2 random variable we will stop this lecture.