Probability Foundations for Electrical Engineers
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Lecture - 43
Real Line as Sample Space

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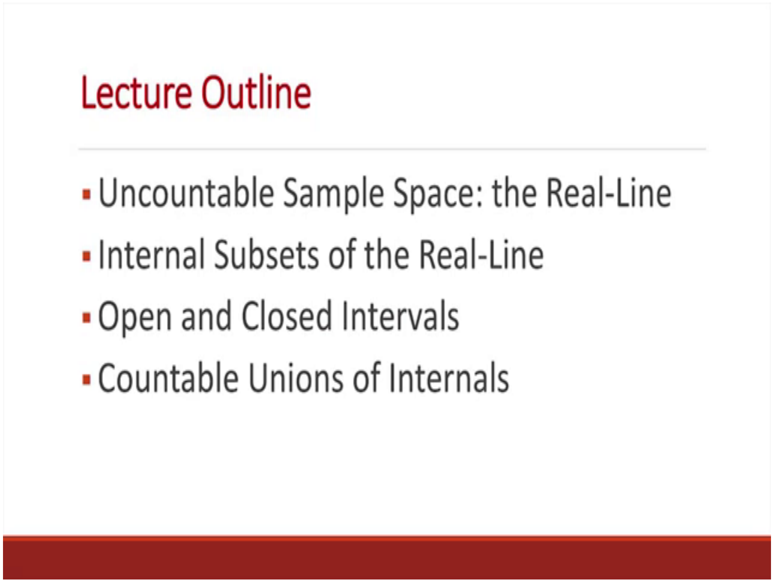

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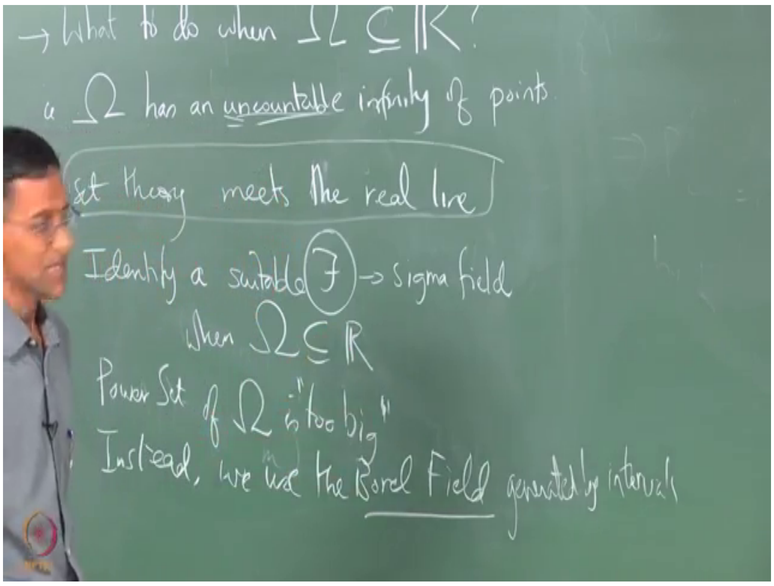

What if, what to do shall we say when your sample space is some portion of the real line? So, all our discrete ideas will have to be changed right. We cannot do the usual discrete stuff we have to do.

In other words I mean well and truly you have an uncountable infinity right. So, interval of the real line has an uncountable infinity of points right, any interval a b closed or opened does not matter. So, we have to come up with a theory of start with the theory of sets on the real line. So, you can think of it as set theory has to. So, far applied only to discrete spaces you know it means the real line now. So, what does it do? So, you can think write this. So, as I said, you have to be aware prepared for thinking about unions and intersections of; of what? Of points taken from the realm that is not enough right, you need to also look at intervals of the real line.

So, what is first of all the first question we need to answer is what is the script f right. The field identify as a suitable remember we talked about this doral if a sigma field sorry we call it a sigma field go back to your first or second lecture. And also he gave examples of omega being less than being a portion of the real line edge spinning pointer is the most commonly encountered one of the simplest things to look at and also you can think of as I said in the first lecture you can think of darts or either 2D or 1D either whatever and we assume that we can measure the point to infinite precision essentially right. Whatever outcome we get if we want it to be well and truly want a real number make sure to infinite position. So, there is no we do we do not want to force this to be a countable set of points only we want it to be as free as it can possibly be. So, that is the omega.

So, what is the suitable sigma field for such cases? The answer is you cannot use first thing what can be, a power set of such a omega is too big again this is deep result in Maths and I am not going to give any justification for this. The power set of such an omega whether it is any sum, if it is for any submittal of R of course, for R it is whatever any interval of R. If you try to look at all possible subsets of that they say that the Math books say clearly that it is too big and it cannot be used because we cannot you know give a probability assign probabilities to all of these subsets in a consistent way that is what they say and that is again the no book that I have looked at ever gives a nice explanation of why it is too big, so we will have just leave it at that right. Hopefully some of you might take some analysis courses in math later analysis or measure theory or some such thing if you do that you will encounter this statement again and they may encounter some and some more details about this.

So, what everybody does is scale down from a power set a set of all possible subsets of such an omega to what is called the Borel field. We use the so called Borel field, Borel is the name of a mathematician by the way the Borel field generated by intervals now to flesh out what I mean by say by generated by intervals by this term generated by intervals supposing you start with assuming you know let omega equal to R itself after all any subset of R you can easily any saw any portion of R you can easily generate and you can do the same thing it is not a problem right.

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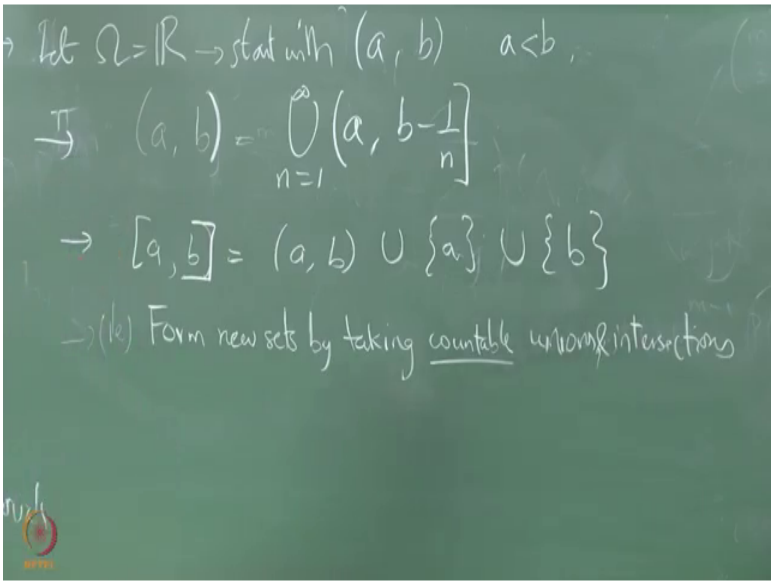

So, for simplicity let assume that you have the entire real line you have to write you have to generate a bar this up, but you know do this exact action of generating this Borel field. What we do is start with some open interval a b, where a b a is less than b and both are real numbers. Any a b and then you take unions, intersections and so on of such a b.

So, if you start with a b, how do I get the closed interval? Let me write a couple of things here then, this here this is an interval which does not inc load b, but in closed on either includes a nor b, but if I want to include sorry no I did not mean. Let me write it, write and not surrounded there does not follow from this. If I want to generate an open interval from the closed interval, I will do things I can do things like this it is not there is no then here I will erase the then, I have this countable union of that is if I take these close intervals for different values of n, if I take the union of all of these see this is what I mean by saying that theory means a real line. This is maybe new to some of you this kind of manipulation, but this is what we see in books right.

This kind of union gives you can get links this a sequence of close intervals each of this is a nested closed interval, you can see that for as you make n bigger and bigger you are actually making this the interval slightly larger and larger. And ultimately in the limit as n goes to and n is an integer positive integer you get the open interval a b. So, this is a basically what the real nice is books talk about is the connection between open and closed right.

How do you. So, in a way that is to get open from close how do you get close from open? Close from open is very simply got by the union of this a b union of what; u a and union b, these are singleton point just this points just one point a b. So, this is the closed interval on both ends. So, I am just giving you these examples to show how you can manipulate these intervals. Of course, this is you we are not just going to do this we are going to take a complement of a b, we are going to take the unions of a are disjoints intervals and so on and so forth right.

So, basically if you if you do this in other words let me see that I write it in English and then also write. You repeatedly form new sets by taking countable unions and intersections and complements, form thing is we are allowed to take down to build unions like this. In fact, there is something else you can do with it. Where does a countable intersection arise? Countable intersection I can also write one more thing here it is not under you know something similar to these two, but not exactly related.

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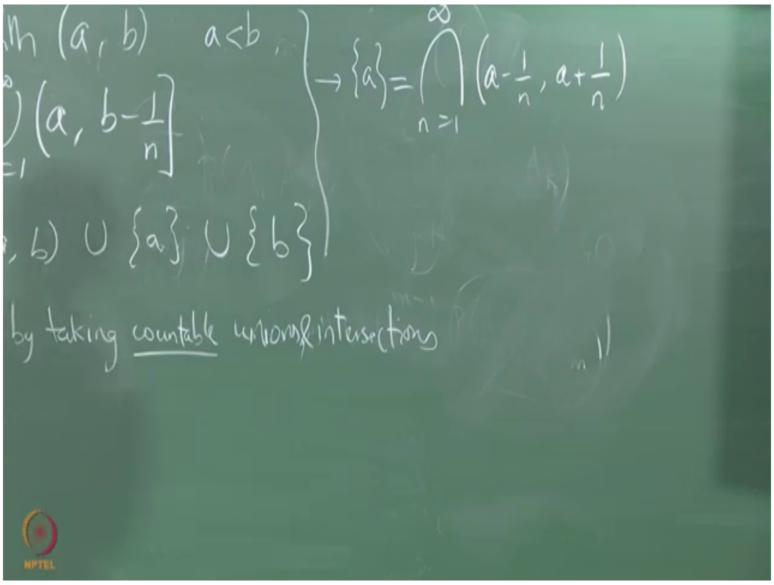

This singleton point a for example, is the countable intersection of what? Supposing I want to write the singleton point as a countable intersection or some that is I want to take an interval and shrink it to a single point, what should I do? I shall have to take the interval a minus 1 by n comma a plus 1 by n is open interval and if I do this n equal to 1 to infinity this is it shrinks to this one point. Note that this a is inside this interval for all n. So, this is an example of a countable intersection and of course, complements. So, if you of all intervals open close, does not matter. I have already shown you how open and close are obtainable by simple countable unions.

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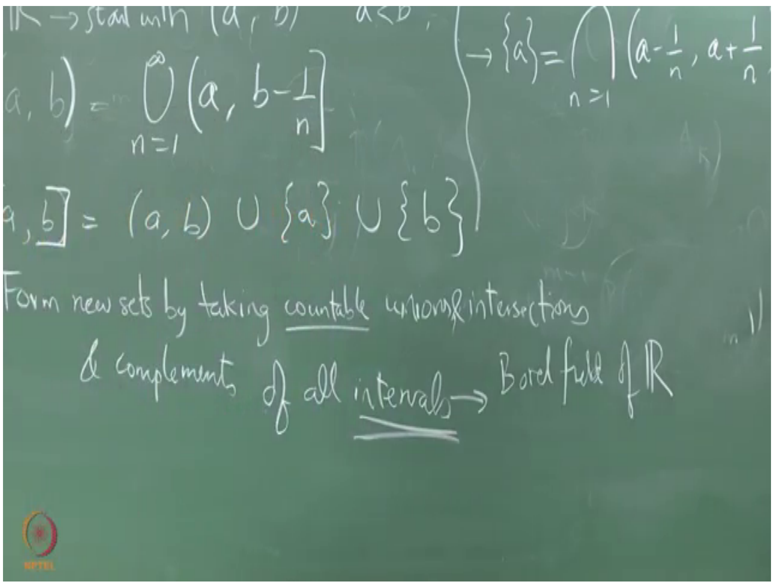

So, if you do all this you get what people have called as a Borel field. So, the key idea is intervals which was new, they jump now from the single points to the isolated individual points that we had earlier. So, this is called the Borel field of the real line. And notice it has nothing blue probability at all. This is a far more general concept which is used in a bunch of different construction arguments involving the real line is nothing to do not specific it properly. Probability happens to on top of this Borel field that is all right.

So, what does the Borel field contain then? If you look at it, it contains all open intervals let me I should list them out clearly one more time. Before listing them out let me also point out that no interval can be formed as a countable union or the point inside it right. You can say is a b of this kind is this a countable union of all the points inside a b no because the number of points has not been any interval a b is uncountable. So, you cannot express this interval a b as a countable union of this kind of the points inside the interval. So, that is in other words.

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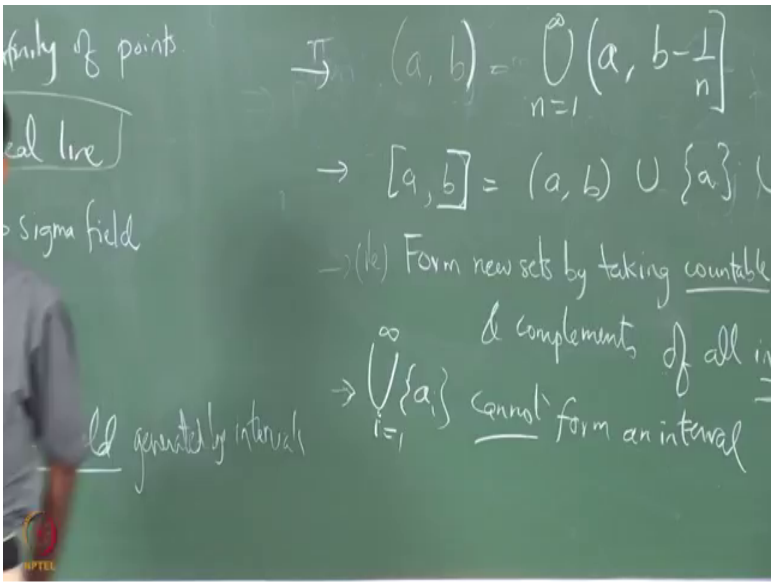

This union of over i, of a i or x i cannot form an interval. Let us say i equal to 1 infinite does not matter alright. No countable collection of points can be combined to form an interval. This is one more very important fact relating to the real lines. So, this much background of real analysis we need we cannot do without. But the good news is that we are not going to go back to this anymore just we are just setting the stage and then once we realize that is intervals we need to focus on we grab that and say we can we know how to deal with intervals. So, we are sort of done in a way that is true. So, the focus shifts from points discrete points to intervals is that ok, all right.

So, what is the Borel field then contain? It contains all points it can let me, we will use this F. Let me call this F the Borel field right.

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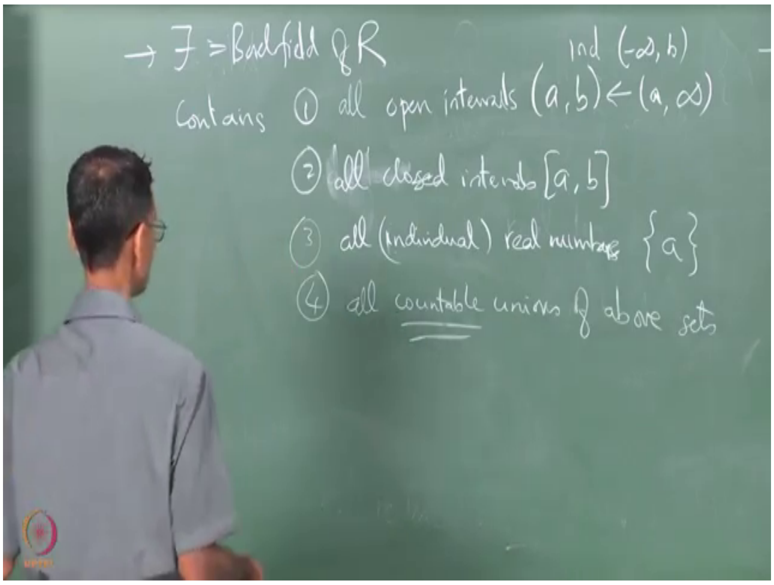

What does it contain? All open intervals a b, now why I should I separate out these open intervals and closed intervals because the infinity when you look at infinity it occurs only as an open interval. So, this includes minus infinity b including minus infinity b and a comma infinity. So, typically infinity occurs only as an open interval not as a closed interval because in it is a concept in some sense in such intervals. You are just saying the set of all real numbers smaller than b here just to write that you are writing infinity as a kind of a place holder minus infinity to place holder.

Similarly, the set of all real numbers bigger than a is this. So, typically you do not include you know I say the real line ends at infinity that it is not that, usually that is not the way in which it is understood. So, the closed intervals are all closed intervals of course, for finite a b and of course, the half open half closed I do not want to get. It also includes all points all individual real numbers right. Why because, what did I how did I get this by taking a countable intersection right. So, all real numbers individually are included in F, not just the intervals points are also included, this and all unions of the about sets right.

So, this is maybe I am not claiming this is completely exhaustive, I think it is most I mean, mostly anything that we want to any interval that we want or a union of intervals I think should be should fall into one of these categories right. If you have for example, a b union c d that will be perfectly valid a union and prove disjointed or even overlapping intervals, it does not matter right. So, this since they are the integers for example, are a special case of real numbers this countable union of the integers like the set of all integers is included, but that is not the only thing you can have a countable union of irrational numbers for example, things of that sort, they are also included.

So, it is in fact, very hard to locate a set which is not in this collection. A set a subset of the real line which is not is actually not easy to construct a subset of the real line is which is not in this collection. But apparently such subsets do exist which is what we alluded to earlier in that it the power set of the real line would obviously include those also, but those are not trivial at all to construct right. But as I said I do not I mean this is not a match mathematics class so we cannot stray too far from our path which is just you know apply look at applied probability and not these fundamental questions of our you know what is consistent with the countable union, what is not consistent and so on right.

As long as we can work with this we will and we are going to stick to this, and all the things that we do we will keep going around here. So, all discrete probability with integers for example, is included here. That is what I want to say. So, this includes all the sets of integers that we have seen so far.