**Probability Foundations for Electrical Engineers**
**Prof. Aravind R**
**Department of Electrical Engineering**
**Indian Institute of Technology, Madras**

**Lecture - 60**
**pdf Method**

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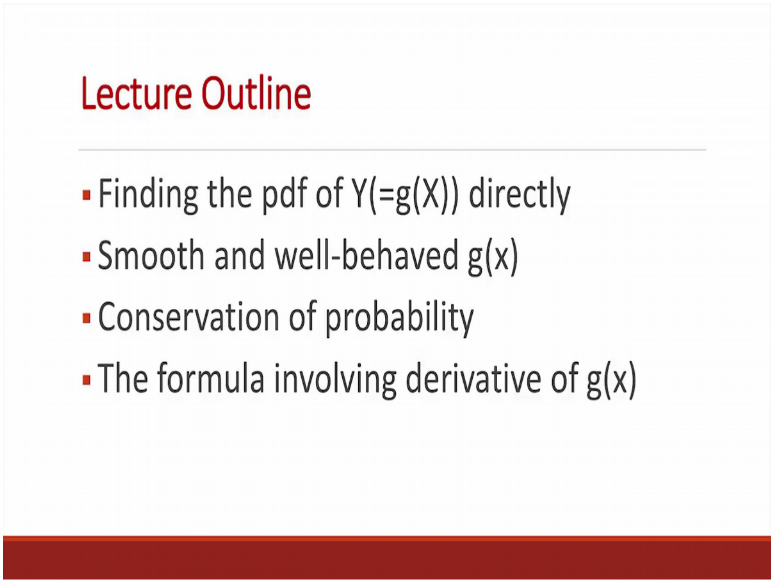


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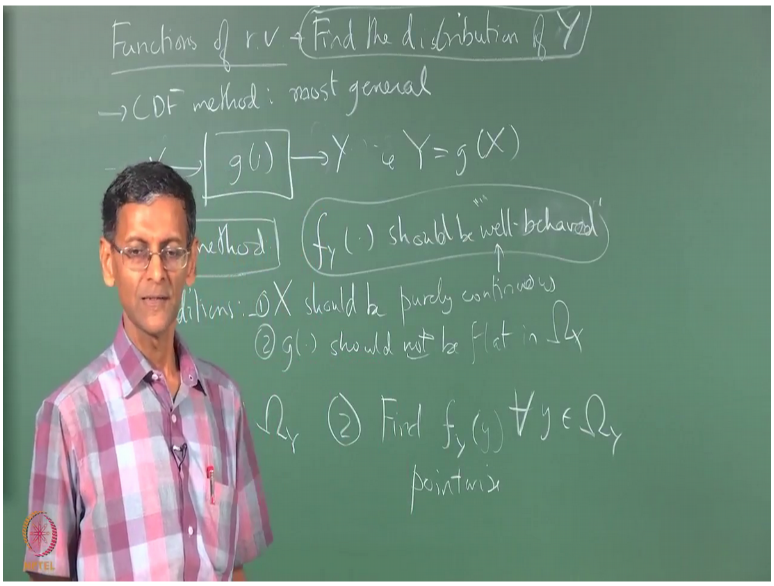

Today we saw how the CDF method can be used in a very simple example to determine the distribution of the output random variable Y. So, the goal and all of this is to find the distribution of Y where there is yeah as a CDF or a pdf it does not matter, right.

So, today we are going to cover this or talk about this pdf method for which of course, they are assumption is that the y does have a well behaved pdf, is not it. So, what are the conditions for this well X should. So, the conditions are and write them down. So, that there is no doubt about this X should be purely continuous then g should be, g should not have flat segments; g should be reasonably well behaved also you do not want to take some crazy g, but we let us say we are not interested in looking at totally a crazy g which is the jumps up and down arbitrarily we are going to rule those kinds of things out.

Say for, the main con constraint on g said it should not have flat portions in omega x should not be because this is obviously, something where which is allowed in practice and so we have to write it down separately right. So, if these two conditions are satisfied then this f y will be properly definable and computable also and we can look at how we might want to do fine get it directly instead of going through the CDF route, is not it. As usual we are going to make use of the conservation of probability principle to find this, anyway.

The first step anyway you have to find omega omega y, this you cannot do without emphasized yesterday. And then this second step is to find f the density function for all y in omega y, point wise. In other words you are finding this output pdf point by point, but there could be an in uncountable infinity of points here, but if you paramit; you know if you just take it for an arbitrary y and omega y and if you can find as a function of y then you have solve the problem, is not it. But that does not take away the fact that you are actually defining it point wise. And what is interesting here is that there could be count a small number of points for which you cannot find f y because of reasons that will become clear as we take up some examples.

So, for some point you know in omega y you may not be able to find or one unique value of the pdf, but so, but such points do not matter when you have a function it does not matter if one or two points you cannot specified exactly like for example, if you take a unit step it. Does not matter, if you define u 0 as 1 or half 0 is not it in most cases is no, case that I can think of you need to worry about what is the exact value I said is that 0 you can define it to be whatever you want that is per convenience.

Like that here also they will be one or two points where you might write I count to a small might even say a finite number of points where it is not possible to exactly determine it using this procedure that we are going to look list out now, but that is ok.

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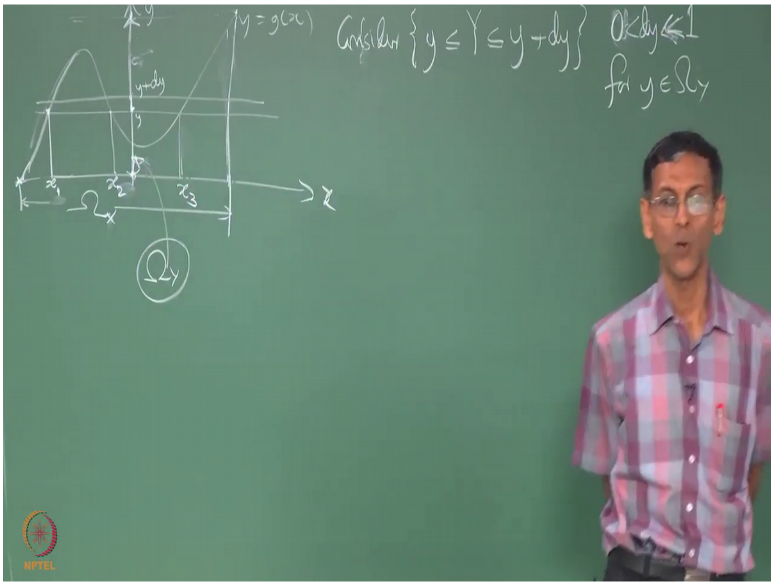


So, what we are going to do is take the general case I mean general in the sense I am not going to I am just going to take some arbitrary g which is right. So, let me say that this is the origin this is y equal to g of y, let me just say y and this is y equal to g of x and since a function g which is not necessarily monotonically increasing or decreasing, but can go up and down.

And let us say that for the sake of argument for, that we have omega x maybe between here and here just. So, I want me you know maybe I should take this, let me take this point above this so that I have some variation (Refer Time: 05:35). Of course, you have can you are free to assume that omega x is goes on forever, but then I have not drawn g in that fashion. So, I am limiting the omega x to the portion where I have drawn the g is not it. I do not want to get myself into trouble by saying omega x goes on like this and I am not defined g yet beyond this.

Anyway. So, let us focus just on this simple case where this is a finite interval and this is the g in that interval right. So, what is omega y? Omega y is obviously, the widest interval of values that you get on y and it is for the non monotonic case, it is not. So, simple you have to go and look at the variation of the function in conjunction with omega x and see, here we you might say that this is the smallest value and this might be the largest value just this particular case. So, this is the omega y let me write it like that, the certain values taken by y is from here to here.

So, I know if I pick some value small y in that interval omega y which corresponds to drawing a straight line like that. So, I am interested in the event that consider let me start here consider this event y less small y less than capital Y less than y plus dy for some small dy greater than 0. So, always in these things this dy is larger than 0 and its much smaller than 1 and this is a typical calculus differential, positive differential.

So, that is an event which says that I am looking at these two horizontal lines and I am looking at the probabilities or rather the event that the output is in that interval. So, what do I want to do with this interval? I want to map it back into x. So, I want to look at what into what values x should take so that y takes values in the interval y to y plus dy, it is as simple as that.

And if I map this, this you know is all I have to say is the probability of this is obtainable from the description on x somehow, it is not too difficult to do because it turns out that if I have let us say if I take y not y plus dy, if I take y and I have how many roots can I find here. So, this is a root this is a root x 1 x 2 unfortunately this omega is this come in the way, but I think I can write it without too much clutter x 1, x 2, x 3 I have 3 roots for this particular value of y.

So, for, so if I take y in omega y; obviously, it has have at least one root by the by definition otherwise you may have something wrong with omega y, is not it.

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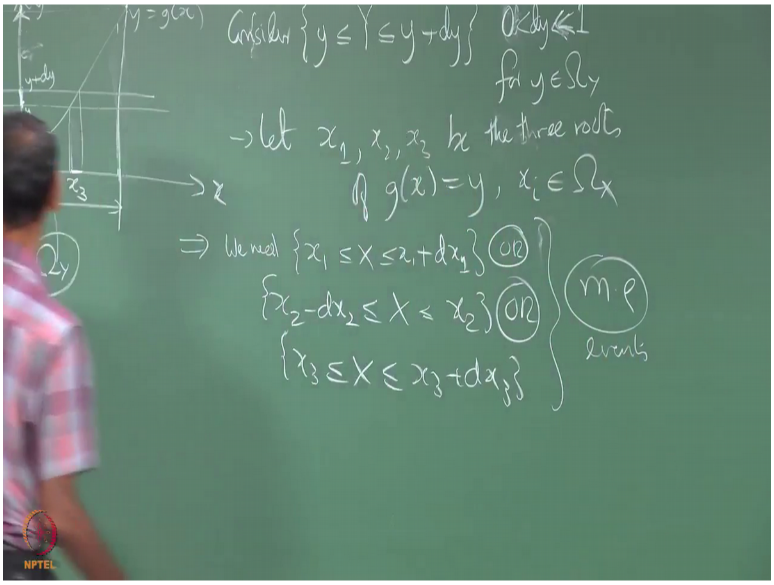


So, let x 1, x 2, x 3 of course, it can be more, but let me say for the time being, but we have 3 roots that is the whole all the 3 roots, there are 3 roots of what equation this g x equal to y. The 3 real roots where all of these x I has to belong to omega x you are not interested in roots of the equation outside omega x. There has to be at least 1, but 2 obviously, to make it more general I am assuming more than 1 and in this picture this is 3.

So, now what should what variation should x have so that y is between capital y is between y and small y and y plus dy? The variations in x can be written as a union of disjoint events mutually exclusive events which are what; if I want to be here then the input should be what, it should be either here or here or here and the 3 events in the input side are mutually exclusive. So, the key point is this as far as this picture is concerned is which this implies I suppose I can start writing it here yes. So, we need in this case x 1 smaller than and I am assuming that g is decreasing at x 2, but increasing at x 1 and x 3. So, that I have x 1 and I have here I need to write dx 1, dx 2, dx 3, but if I start marking that here it becomes too messy. So, let me not mark it. I will just highlight it with some chalk here and you all know what I mean when write dx 1, dx 2, dx 3.

Now, of course, you cannot assume that all 3 are let me write it; in this with respect to this particular diagram I am going to write I will write it like this or I will write x 2 minus dx 2 to keep dx 2 positive because the function is decreasing at x 2. This is;

probably you know the function you might there is no loss of generality in saying I look at the function and see what sign I have to give for the dx at that point.

So, one of these 3 events have to has to happen for this to happen and these are all me. In other words if x is going to sit in this interval it cannot are the same time sit here or and sit here disjoint intervals are always me basic would be fundamental property, I mean disjoint intervals, right.

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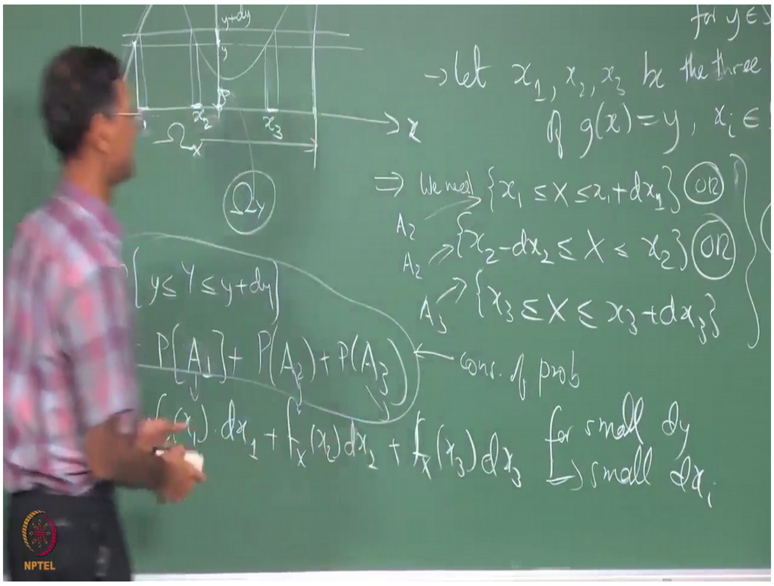


So, therefore, what is conservational probability say in this case? The probability of that is going to be equal to exactly equal to what.

Student: Sum of (Refer Time: 13:01).

The sum of these 3. Well, let me write this as a yeah A 1 plus as I will call this as A 1 rather than keep writing it all over again. So, I have 3 events here I have A 1 plus A 2 plus A 3 this is most important equation would be need. This is again let me this is conservation of probability. If you do this properly for all the points why you are guaranteed that the unit probability in x is going to be transferred to even probability in y; so that you should not leave out any of the output points or output intervals all output intervals that can be mapped vig should be accounted for.

Why is it now easy to or supposing you make dy very small and we are assuming that g is well behaved and it in that; so the assumption we make is that g is a smooth curve. So, that as dy goes to 0 all the dxi also go to 0 right. So, g we are going to impose some smoothness condition on g also, so that we can make that statement. So, if the, if d by small d x I are also going to be small and each of these the first order approximation say how do you write this P of A 1, P of A 2 and so on. So, it turns out that this P of A 1 for example, is what if for small d x 1.

Student: (Refer Time: 14:53) f x of x (Refer Time: 14:56).

Right. So, it is a density function at x 1 f x at x 1 multiplied by.

Student: dx 1.

Dx 1 now dx 1 is positive, but all these dx's are positive that is another reason why I we want to keep them that way. I cannot write if I am allowed this dx to have sign then I have to worry about is it going to be positive or negative. So, I have written it. So, that it is all positive.

Student: (Refer Time: 15:18).

So, what is about, what about P of A 2? Similarly likewise this is, this is f x by x 2 into d again dx 2 is positive and it does not matter whether in the interval it is to the left of x 2 or it is right of x 2. If dx 2 small enough is the same then P of A 3 will be f x of x 3, dx 3. Now, note that how are the magnitudes of dx 1, dx 2, dx 3 can all be different why because the slopes can be.

Student: Different.

Different, there is no reason that this slope is the same as say here it may look the same see, it is difficult you know to draw and make this make the slopes look very different on the I is a especially here or maybe I you know I drew this too much like a sine wave right. So, maybe you should draw it with some highly asymmetric kind of rising and falling. So, you can see for yourselves that even in a rough diagram you can show you can see that dx 1 cannot be the same as dx 2 in general.

What about the left hand side? The left hand side by may know by the definitional density function is just the density of y and f and y small of capital Y and small y or density of capital Y and small y multiplied by dy; dy we have assumed to be some positive quantity small positive quantity to begin with.

So, what happens? I am just writing equals here this equals is only when let me say for small l qualified, for very small dy which implies small dxi. So, what we have here? Here I have 3, I know I have 3 roots I am writing it as 3, right.

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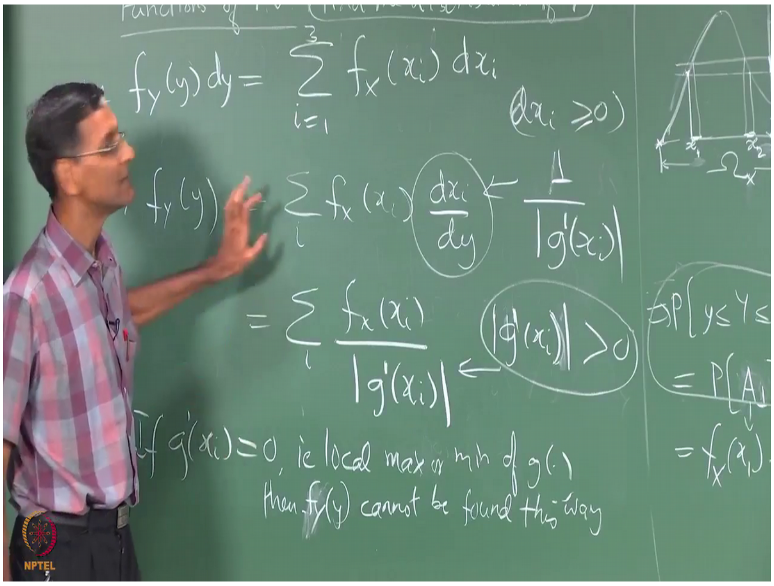


So, here dxi is all (Refer Time: 18:03) and dy is strictly positive not 0. So, I can divide by d sorry d y and get a formula for that output density that I want. So, this is going to be I just write sigma over i, could i has a at least 1, but some number in general more than 1. What is this quantity by from your favorite calculus class of whenever it was?

Student: (Refer Time: 18:52).

It is easy.

Student: Derivative.

Derivative in or 1 by the reciprocal a derivative at x I not only that is a magnitude of the derivative, I do not care about the sign I only care about the magnitude, I only care about magnitude derivative not the sign. So, it is written like this 1 by g prime x I why does this the derivative come into the picture comes into the picture because here fundamentally continuous quantities and if the derivative is large; that means, a small variation here can produces a much larger variation here conversely the derivative small you need a larger variation here to get a smaller variation here.

So, that is why unlike in the discrete case in the continuous case you are dealing with intervals and when you are dealing with intervals and functions there is no escaping some calculus. But we have we have had this kind of calculus I mean the computation is ever since Newton this root on differential calculus so that it should not be any problem for us and therefore, you will write this as sigma over i, f x of x i divided by mod of g dash the derivative at x i. This is basically the formula we want. And we are going to assume of course, that this, the derivative is not 0 that is in other words if you have the 0 derivative this is what I was alluding to earlier you cannot really directly apply this formula. And this if you go out here and say to where is a local maximum of g, at that point it is on its not you cannot easily map some dy to any dx. First of all the first order math said I mean the models if we have done here may say the d of r has to be 0 at a maximum of g it does not even actually make sense to give a variation dy when you have a hitting a maximum or minimum.

So, but if you have any finite number of such maxima and minima it does not matter. You do not have to calculate f y you can say f y I do not want to define it at those finite you know at the small number of points at which g has a local maximum minimum or local extremum. So, d is g 1 this to be this magnitude (Refer Time: 21:44) is be strictly larger than 0 for this computation to work g prime x i has to be positive more a magnitude. So, if equal 0 ie local extremum max or min of g; what happens if then this method then f y cannot be found using this method, this way.

Actually it turns out in many situations that is this equation does not make sense is what I am writing here. In many situations it turns out that such local maximum minimum of g actually result in the density function going out to infinity which we already have some example one example of. We did one example when we talked about that arc sign pdf we will come back to this here again that 1 by square root of pi, pi into 1 minus root 1 by x squared or something that is going to come as a of a result of a transformation or a function of right.

But again let me not I mean just for your like the camera and for your notes, this is not an issue I repeat when there is only a finite number of such points to worry about. And we have ruled out or we have said this method anyway will not work if there is a flat portion if there is a flat portion, then there is an infinite number of such points in some sense, there is an infinite number of roots uncountable infinite. Now, supposing g is flat somewhere then you are not supposed to use this method anyway, f y in fact will not have a good meaning. So, therefore, since we have when we started off assuming that there are no such there is no such case you can you can say safely there is only going to be a finite number of maxima and minima, and therefore it does not matter if you do not find a fire judge points.