Probability Foundations for Electrical Engineers
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Lecture - 61

Examples

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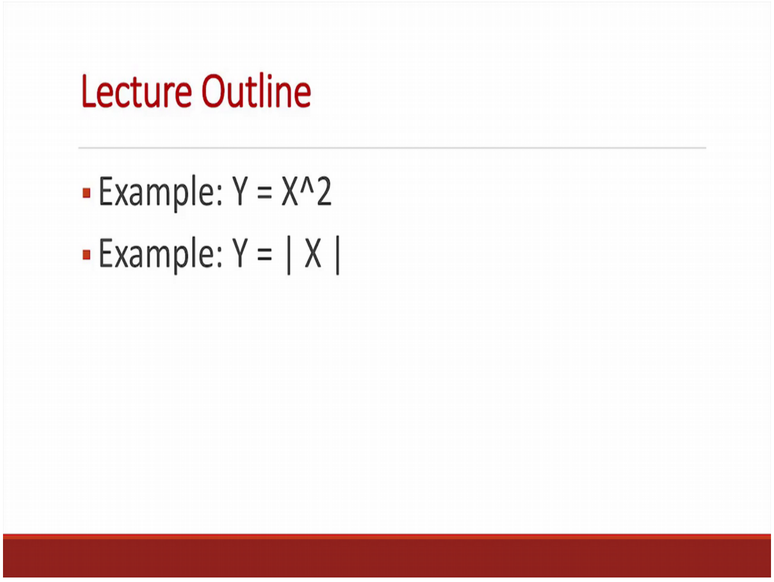

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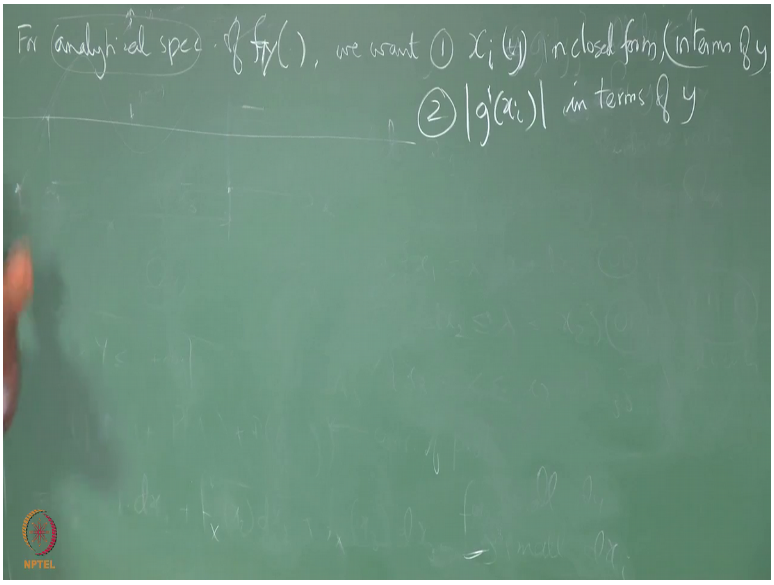

So, what do we need in for anyhow for the analytical method to complete, to be able to write to do this with just your pen and paper without doing any actual you know number crunching we need what. For an analytical specification let me say f y of the output density we want 1, two things we want one x i as some function g to be expressed in closed form x i y in closed we want this x i as a function of y you want it closed form for example, the roots of a quadratic equation you know how to do that. You also want the derivative this g prime of x i as a function of y, in terms of y let me say the arbitrary y that you pick again there is also in terms of y i should have written it like that.

So, everything has to be in other words this whole hand side has to be in terms of y that hand side then if that once that happens you pretty much define your output density for any small y that you care about. So, when I mean (Refer Time: 01:57), I do not know if this is the word to use or not I want to specify it using a formula that is what I mean by that phrase out there. I means again I did not come prepare for correct term what exactly would you put there in place of that first for a closed form specification let us say if you do not like what I have written there. Of course, we love closed form formulas, you will not know f y of small y is some well defined function of y here. So, you want that function we do not want some implicit stuff run running and all that we cannot you know understand exactly right. So, if you want this in closed form you want.

So, let us look at some examples which will abundantly make this very very clear. The most n popular example is y equals x squared or g of x equals x squared.

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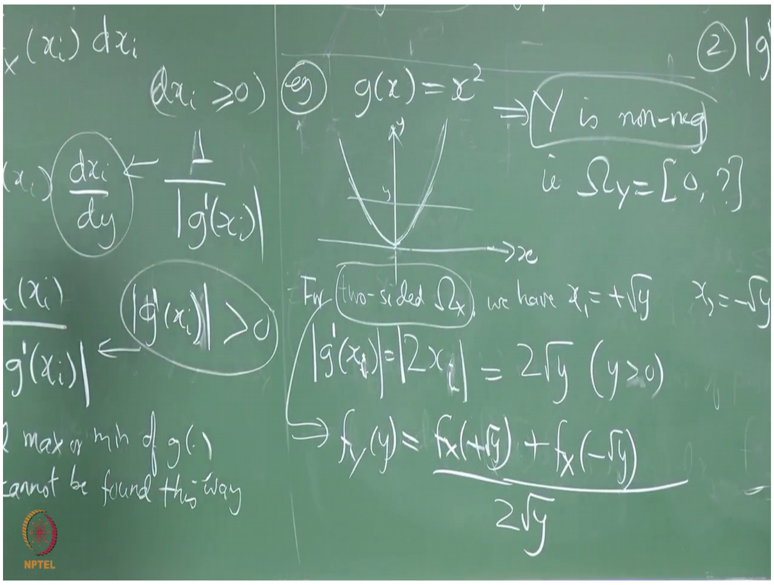

Squaring is one of them you know age old signal processing functions that we do starting with square law detectors which used to be taught in some communications class at some point, but I am not sure nowadays what whether its ever (Refer Time: 03:21) not. But basically squaring and actually what is done is obviously, I mean you are not going to square the individual numbers as such in a real life situation you might square an input signal which is some h of some f of t, let me call it x of t or x is not just a single sequence of numbers it could be a function of time. And you could obtain there a random variable by sampling that x of t at different points in time again that is very specific to electrical engineering not its very common in double e.

So, this is that you do not think of this is simply some as I said some I in isolated squaring of course, it also works for that, but in practical application that comes as a part of a squaring circuit anyway, but we are interested in this memory loss operation right. So, what happens if this squaring element is driven by samples of or realization of a random variable, and that of some variable capital X, so we will now we can drop exact picture rather than draw some funny you know up and down kind of thing. So, first of all as I said you have to determine omega y and omega y in this case is always what.

Student: Non negative.

Or y has to be a non negative random variable, ie omega y is the closed interval 0 assuming that x can take the value 0 which is all more often the case assuming that x is either starting from 0 or has a two sided density. So, this is this is assuming that and you do not care about the upper limit it can be anything.

So, this, we will take some small y out here and for this particular quadratic equation the roots are.

Student: 2.

You can write them in your sleep. There are 2, in general there are 2 roots. Remember if omega x is only on one side that is one x is also non-negative then you only have one root. So, assuming that x is takes on both positive and negative values you have 2 roots for two sided we have x 1 to be plus root y and x 2 be minus root y.

So, that is all the first part when the root and so there are many of these things you can do independently of this you can find x i is a function of y just by knowing g, you can also find is g prime of x i without knowing without knowing f x sorry. So, what is the, we are interested in the slopes at these two turns out the slopes by symmetry in this case have the same magnitude. So, g prime of x 1 is mod of that we are only interested in mod of that is what two x 1 magnitude be very clear about it x one can be positive or negative or I can I do not even have to write one here I can put I g prime of x is magnitude of two x i which is what is x i you know magnitude of x i is just root y root y remember is always a positive root of y.

Now, I am not going, I am not allowed to use that formula at this local minimum. So, I am only I have to consider only y which is positive strictly positive. So, what is the general formula now? Assuming that I have a two sided omega x. So, for the two sided omega x i know that f y of y is what is f x of plus root y i can write a general formula f x of plus root y divided by.

Student: 2 (Refer Time: 08:38).

2 root again a common denominator plus f x of f x of minus root y the common denominator being

Student: 2 root.

2 root y. If you do not have a two sided omega x then you can say this is going to become 0 anyway or this will drop out and you will only have this. So, please be watch out for that do not apply this formula and then forget about this term I mean a I guess it will still work, but just be careful.

So, we can take specific examples like for example, supposing I want to take just one or two.

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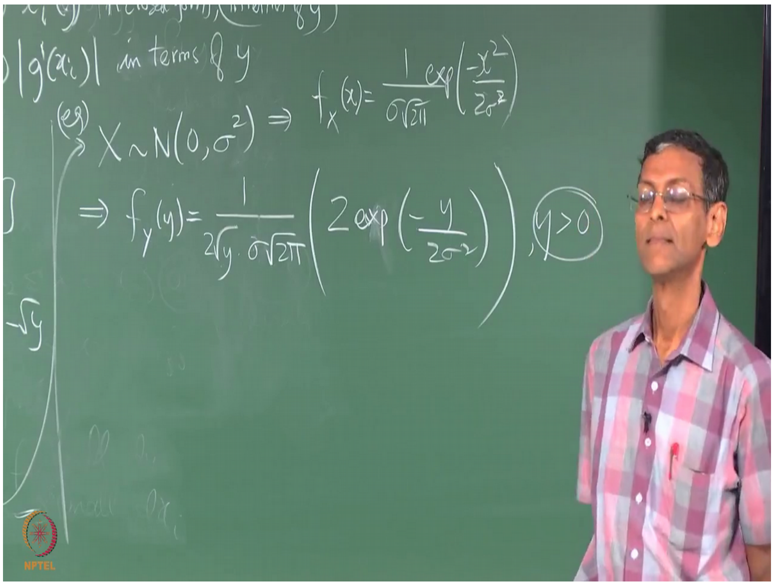

I will take x to be the 0 mean Gaussian right, here some arbitrary variance sigma squared for this density function input density that is I am squaring the Gaussian random variable 0 mean Gaussian random variable. So, what do I get turns out that f y has a. So, it is one by root y, 2 root y into sigma root 2 pi all of this will come here these two will be the same by the symmetry of the Gaussian the 0 mean Gaussian. What is f x we will write is to remind ourselves 1 by sigma root 2 pi exp of minus x square by 2 sigma square.

So, now I have x squared is y, is not it. So, what happens here these two are the same whether it is plus root y or minus root y. So, I will get two of something what will I get 2 of I get 2 of exp of minus y by 2 sigma square is everything in terms of y and obviously, y should be greater than 0.

Note that this process of this root y says that actually here that in this case the density function is undefined or y equal to 0. This is correct, you all agree with me. These two are the same have the here because of symmetry they are the same values. So, that is why this two comes and x square is y. So, I do not have to worry about this root y business, but does not matter whether a I square plus root y or minus root y i get y.

Supposing I had a nonzero mean Gaussian that is there that mean that there was a mean m then you could no longer claim that those two would be the same and you get a much messier thing to worry about, much much messier, because remember then you get an x minus m whole squared and you have to worry about x squared and x the x squared term x term are becomes a lot more complicated in this board is way too small to handle that or maybe I have to start over there and end over here right, which I do not want to do.

So, what? I can do some slight simplification these two will cancel. So, this together can becomes what some sigma root 2 pi y you can clearly take the y that root y and the root 2 pi can be put together and write it like that. And it turns out that this is a special density function which is not normal, which is a not a introduced in the first round because it is not commonly encountered, but it is important anyhow it is an example of a gamma density right.

There are two names for it one some people call it a chi squared density function with some degree of freedom.

Student: (Refer Time: 13:06).

I forget how many degrees of freedom they are attributed to this, they also call it a gamma pdf names, for this let me say.

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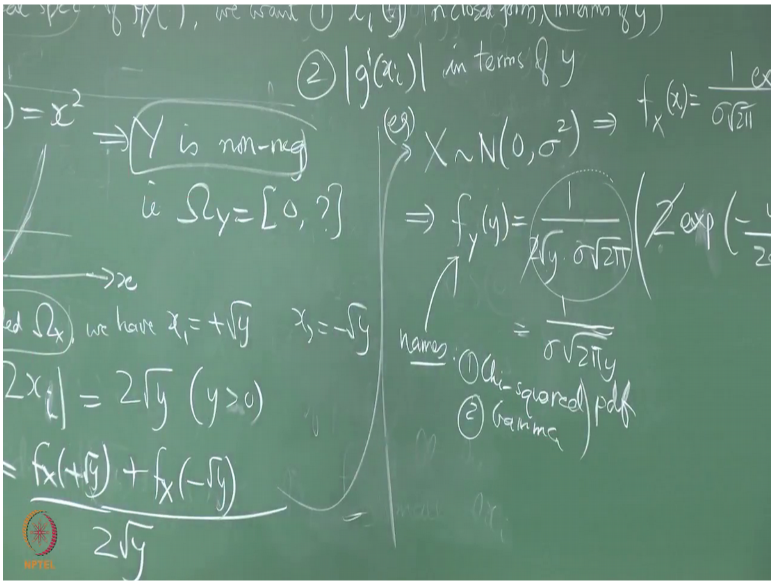

And these gamma, the gamma pdf is actually not just one pdf it is actually a parameterized by something. You please go and look it up look up what a gamma pdf is you will find that it is a family of pdfs this is just one parameterization of that gamma pdf.

In general it uses a gamma function you know have you people study the gamma function. So, that gamma function will prominently figure in the pdf. So, this is for one value of that gamma that parameter.

Student: (Refer Time: 14:02).

Gamma of half is what, what is gamma of half?

Student: Toot pi (Refer Time: 14:06).

So, that is that root pi. So, that you get this particular pdf you substitute half that gamma function this is ok. So, this is a two sided pdf. If I now take a one sided pdf and now I want to take the Rayleigh as an example, I can of course, take the exponential also, but I have there is a reason for me to take the Rayleigh and for that I think I have to go here.

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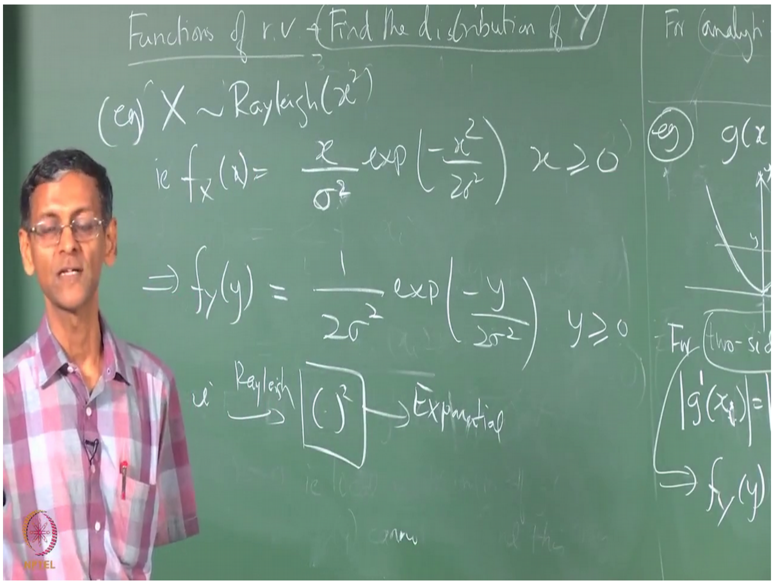

If it is Rayleigh with the parameter sigma squared, ie f x x is what, x by sigma squared or is it 2 sigma squared. It is checked by sigma squared only exp of minus x squared by 2 sigma squared, but this is true only for x bigger than 0. So, once I did x is non negative what will you get can you do it in tell me. What is the f y?

Student: (Refer Time: 15:33).

There is an x here outside. So, that x will become square root of y.

Student: (Refer Time: 15:42).

So, what happens?

Student: (Refer Time: 15:45) exponential.

You get exponential you will get 1 by 2 sigma square.

Student: (Refer Time: 15:52).

That this 2 is coming from this 2 this root y will cancel with this x this sigma square comes here of exp of what minus y.

Student: By 2 sigma.

By 2 sigma square. See for yourself this is a valid pdf and now I can write y greater than equal to 0 because the offending value of 0 has got cancelled out. So, I do not mind I did, this thing was not had the infinite value at y equal to 0. So, I did not want to set y equal to 0 out here, but still this is a valid pdf, this is one more of those pdfs which shoots off to infinity while remaining valid.

Here I do not have this exponential pdf there is only discontinuity y equal to 0, there is no problem about the ends of the discontinuity is 0, you can say it is 0 for y less than 0 and its whatever you I can assign it the value 1 by 2 sigma square or one other lambda that lambda is basically 1 by 2 sigma square.

So, we have this important property now Rayleigh upon squaring becomes exponential. So, you take Rayleigh I can put Rayleigh square exponential. And what is important wrote in such cases the reverse transformation will also be equally valid. So, what is the reverse transformation here root x or root y, this all, these transformations work both ways. Why should they not, right.

So, it works both ways this way also it works. So, any transformation path that you derive even there for that matter works both ways excepting there you cannot the problem is there if you take the square root you only get positive value. So, you do not get the negative values of the Gaussian. So, what you will get there will be the flip, I mean a Gaussian we will talk about it maybe tomorrow maybe I will finish it off today we get the full wave rectified version of the Gaussian if you take the square root of that. What is the full wave rectified version? See since we started late I am; unfortunately I have to you know go until at least 9 52. So, let me. So, I want to finish off the full wave rectifier also along with this right. So, after all it is something which is very similar to squaring.

So, this is, here there is no issue because both of these are non negative, but here unfortunately if you take the square root you can never, you do not know all there is no way of telling that you should put in a sometimes you should put a minus sign at the output and sometimes used for plus sign you cannot do that right. So, root of y or root of x always only takes the positive square root. So, if you see the function root x y you show, your, this is the something is being burned into you then it always means only the plus square root.

So, therefore, think the plus square root of a gamma, what was it give you? It gives you it turns out the full wave rectified version of Gaussian.

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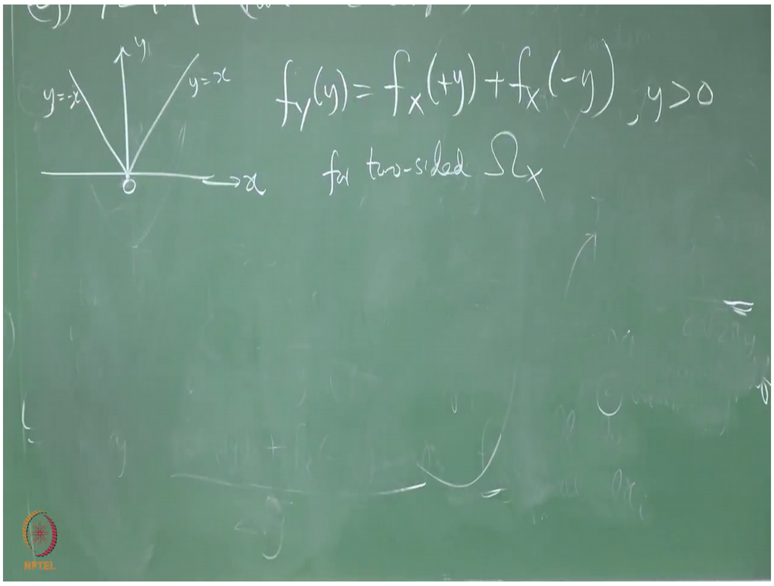

So, in general what is a full wave rectified, version of any random variable that is y equal to mod x. Ideal full wave rectifier, why is it called a full wave rectifier because again x is some x of t in general not just samples individual samples and you agree when you write x's some x of t that if you do mod of x of t is an ideal for I suppose nobody will find fault with that right.

So, here you have this I am not for lack of time I am just doing not me drawing it that precisely, y equal to x here and y equal to minus x here. So, once again here at this point I do not have a unique slope at y equal to 0 because I do not have a unique slope at y equal to 0 again the formula cannot be applied exactly because I do not know what value to write for g prime of x i, but again it does not matter its only one point. So, I can look at I can write the density function for all y not equal to 0 that is enough, what value I write for the density and one point y equal to 0 makes absolutely no difference because density (Refer Time: 21:05) we are all always integrated to find probabilities. When the process of integration again something which maybe is worth emphasizing now isolated points do not make a difference in when you integrate and actually that is a deep measure theoretic statement or whatever some real analysis statement if you take a match cause they will you know go hammer and tongs at this.

Student: Right.

And isolated points are the set of measure 0 and so on and so forth and so they treat, they make a big deal out of this kind of thing right. But here we just say that we do not care about. So, what is f y? So, it turns out that its very simple, it is as simple as that assuming for two sided, for why do I get this very simple statement. Of course, I have to say y greater than 0. How did I get this?

Student: (Refer Time: 22:09).

The derivatives are always unity in magnitude. So, if I push the Gaussian through I am not going to write all that, but it becomes clear what you will get for two sided symmetric, two sided like the Gaussian. If x itself is one sided then f y of y will be equal to f x of y, if f x if sorry if x is for negative then the full wave rectified has no change at all and input will be equal to output. So, I think I want to finish off this.