Probability Foundations for Electrical Engineers
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Lecture - 62
One-To-One Transformations

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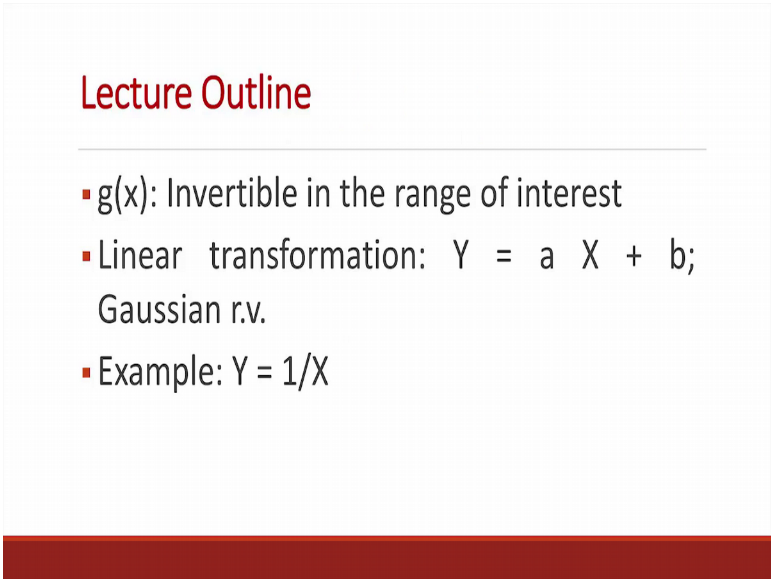

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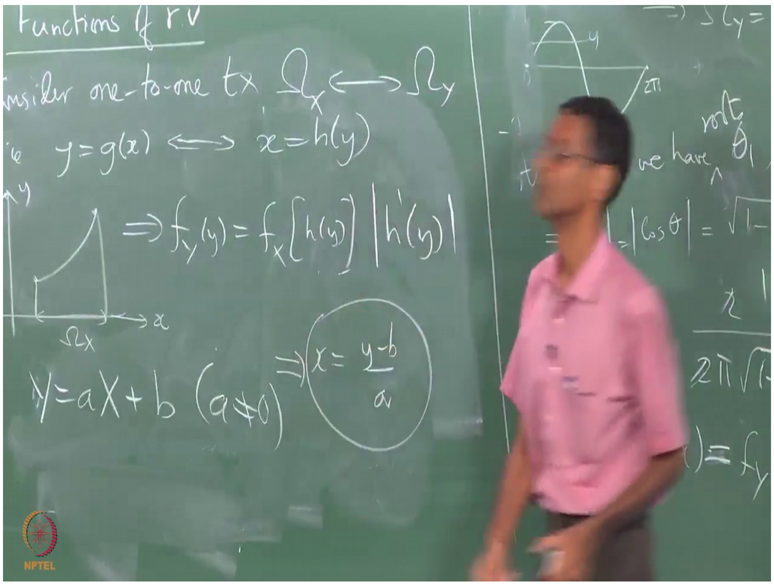

so supposing consider one to one tx between omega x to omega y. Now I can put the arrow both ways that is, ie I have y equal to g of x and I have x equal to h of y. I have both functions that this could be increasing or decreasing an increasing transformation is something like this one to one transformation, eg this is y me let me say this is omega x.

Of course, you can draw you can consider omega x to be the union of two disjoint into a set that also is fine you if you get its just said it has to be in the omega x that you take it should be one to one, outside if it is not one to one I do not care that is if it goes if it does something like this. For example, even y equal to x squared is one to one if x is positive that is what I mean right. So, supposing this, this way then you the formula reduces because there is only one root you know one to one transformation if you pick any y in omega y there is only one value of x y that could have come from and therefore, you can immediately write f y of y is f x of h of y into mod h prime of y. This can be the straightway written down without any if you know the other old formula I can immediately write down this.

Why can I write down this immediately? f x of x is evaluated at h of y that is there is no problem here there is only 1 root why do I get this one by mod g of or rather 1 by g of x is equal to h of y; g of x and h of y are reciprocals always reciprocals of each other when you have invertible transformations because one the one quantity is dx by dy and other quantity is dy by dx at the same point so obviously, they have to be reciprocals. If dx by dy is negative dy by dx also be negative that is if you have a decreasing transformation from going from x to y or again you have a decreasing transformation going from y to x remember.

This is an increasing transformation from x to y it will also if you plotted the other way it will also be an increasing transformation from y to x as y increases x will increase right. So, obvious if you go from here to here y is increasing x is increasing, if you do it do it the other way both will be decreasing. So, this mod takes care of the mod of g of x in the denominator d prime of x in the denominator. So, this very very simple formula allows us to apply it or to find the pdf for very important case which is the affine or linear let us call it linear transformation. The most important one to one transformation is the linear transformation what is the linear transformation y equal to ax plus b including I mean I am including; obviously, the b which is very important in our application, but, some of you might say no that is not linear strictly linear, but it is we call it linear right.

So, eg if I do y equal to ax I write in terms of random variables, but which means basically you are doing. So, a is not 0, if a is 0 it is not a transform it is a constant right. So, that means, if they have y is equal to b with probability one and so the y is not a random variable anymore in the strict sense of the y, but I mean this is very degenerate random variable any random variable which takes the same value is there actually a constant. So, you do not have to worry about that here. So, a is not 0. So, what is the relationship now what is h of y? So, this means that small x if I write will be small y minus b divided by a, this is the h of y. What is this transformation do? So, before that let us write the apply this to here, what do we get.

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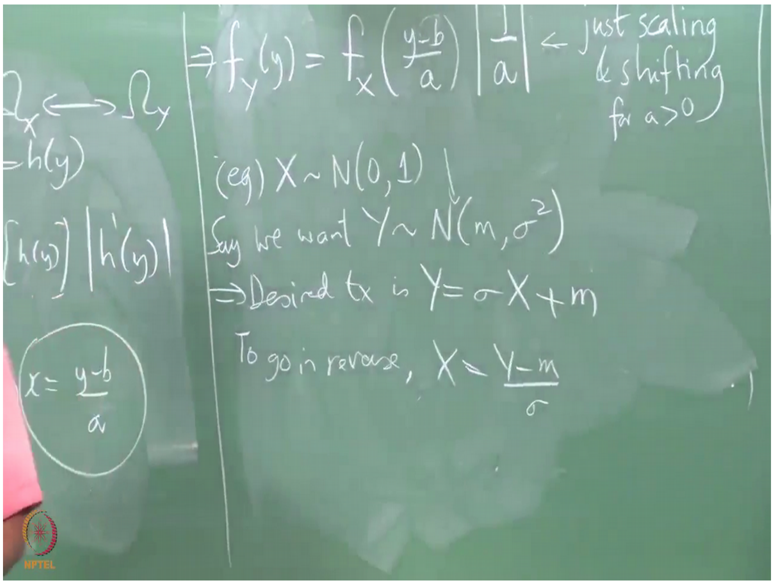

So, what is f y of what is the output density; output pdf is f x of y minus b divided by a there is no sign I mean there is a no mod here, but the mod sign will come here. So, what will it be here?

Student: Mod 1 by (Refer Time: 05:06).

1 by mod 1 by a. If typically for positive f this is very you know just 1 by 8. This allows you to go back and forth between what kinds of things. The most important transformation here which sub case of this is going back and forth between arbitrary m arbitrary sigma squared and 0 1 for the Gaussian case. So, if I say eg, I can go, I can assume either is the input in the other one is the output not necessarily always that it

should be x input and y output y can be the input x can be output also. So, if I want to go from 0 1 to some other m comma sigma square.

So, this let me before I get to that the, this is a shape preserving transformation what do I mean by that. What is the shape of f y compared to the shape of f x? It is exactly the same it is except for scaling and shifting. So, this is just scaling and shifting there is no other change to the shape as such and flipping of course, there is also flipping if a is negative. So, assuming that a is greater than 0 for a greater than 0 it is just scaling a shifting and scaling and shifting, if a is negative then there is a shift, but you know let us just test for understanding this I am assume a is positive.

So, the Gaussian case now coming to that supposing x is N 0 1 and we want y to be N m comma sigma squared what should you choose a and b to be this is. So, if a Gaussian transformation is the input any amount of scaling and shifting will only give you another Gaussian output, but it will have a different m and different sigma. So, if you want this from this you want to go here what should you do tell me. What should be a? What should be b? Think about it and tell me.

Student: (Refer Time: 07:39) a equal to sigma and b equal to (Refer Time: 07:41)

A equal to sigma and b equal to.

Student: m.

M correct. So, the desired transformation is Y equal to sigma X plus m go back. And think about this, this is not sigma squared its only sigma. The more the sigma the more the spread of y and that 1 by sigma will come in the pdf of y outside the exponential where does that go back and check the formula for the Gaussian pdf then what you get one by sigma root 2 pi 1 by sigma comes directly in the outside is not it and this m this comes in same as comes exactly as it is here.

Let me put it here say we want this then the desired transformation is this. So, this y minus b y minus m by sigma in the exponent is exactly what is there in f y. So, you start with e power minus x squared by 2 just that and you go to e power minus y minus m whole squared by 2 sigma square, right. So, I think it should be obvious you just look at it for 5 for a minute you will see this no issue at all.

So, you can go back and forth. Supposing you start with some y which is which has some arbitrary sigma and some arbitrary m you want to get an N 0 1 out of it all you have to do is remove the mean y minus m and divide normalized by sigma. So, to go in reverse you do this both are important in practice right. So, this shows how both shifting and scaling comes, but in some other cases like for example, the exponential and so on you may not want to shift you may only want to scale, same principle. You can do, you can always have b equal to 0, right. So, I leave you to think about it what happens if you take the 0 exponential all those pdf is starts from 0, right.

So, if you want to preserve that you should not shift you should only scale. You can always get more weight or less with depending on the value of sigma sorry not sigma a. And you want to keep a greater than 0 to maintain polarity. If you take a less if you have want to have a negative exponential or a flipped version then you must take a to be negative it is not sub if you get if you take a to be positive it will only keep it positive in will give you a positive out. So, I think it is a simple enough that I do not want to spend more time on this linear transformations. Is anyone have a problem with this or a question.

So, what about y equal to 1 by x? Going away from this; what I am going to do is I will maybe I will erase this can I; earlier I when I started this whole discussion I said y equal to 1 by x can land you in trouble if you take x to be some discrete random variable with finite probability of taking the value 0. If you like for example, Bernoulli or something because 1 by x has undefined value and x is 0 and you do not want and undefined value you have finite probably as simple as that. But it turns out you can do a 1 by x transformation perfectly, I mean you can do it is a perfectly healthy transformation when x is continuous for even when inclusive value 0 like for example, x is 0 u is 0 1. So, if I do that by the time I erase this you should tell me the answer.

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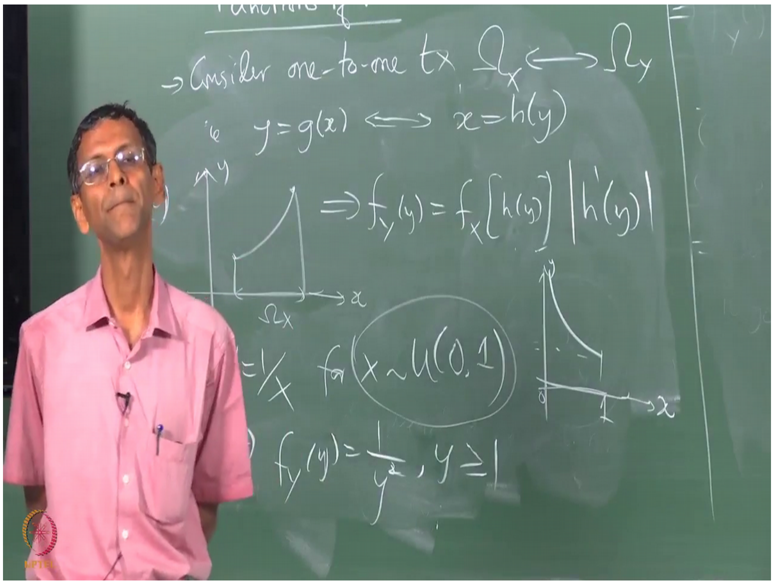

Reciprocal what is the answer? Is this; does this qualify as a one to one transformation? What is omega y? I am only looking at this interval 0 to 1 now, only 0 to 1.

Student: (Refer Time: 12:52) 1 to infinity.

1 to infinity; omega y is 1 to infinity. What is the pdf there? Very simple, just what happens you apply this formula, this is constant I do not even worry about this part because this is a f x is constant. What is this?

Student: Minus 1 (Refer Time: 13:11) 1 by (Refer Time: 13:12).

So, the minus will go away when you take the mod. So, this is an example of a decreasing transformation without worrying about the value of a or anything right. So, by itself is decreasing for that interval if you say, if you take x to be some negative portion then it is an increasing I mean then even then it is decreasing anyway. So, you can also try this it will you the fact that the transformation is a big jump from minus infinity to infinity if you include a two dimensional, I mean sorry two sided swing will not change the result in the same will not change the method. So, you can apply, you call this a one to one transformation even if x is let us say varying between minus one and one or whatever its perfectly doable.

But in this particular case for x u 0 0 1 you get f y to be 1 by y square, y greater than equal to 1. So, y takes unbounded values and I do not care about a infinity basically I do not care at all about x taking the value 0 because its happens to 0 probability in the limit and therefore, y will only take the infinitely large values in some in with 0 probability again and this is a perfectly healthy pdf.