Probability Foundations for Electrical Engineers
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Lecture – 71
Examples: Symmetry

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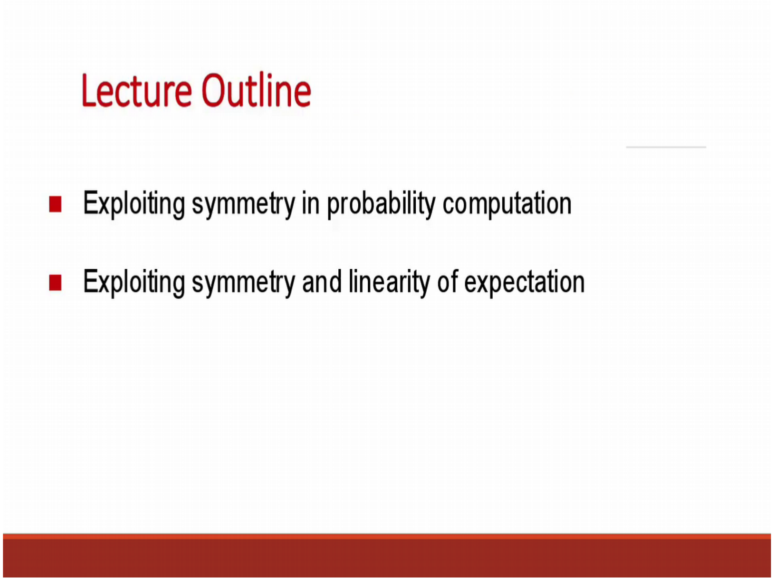

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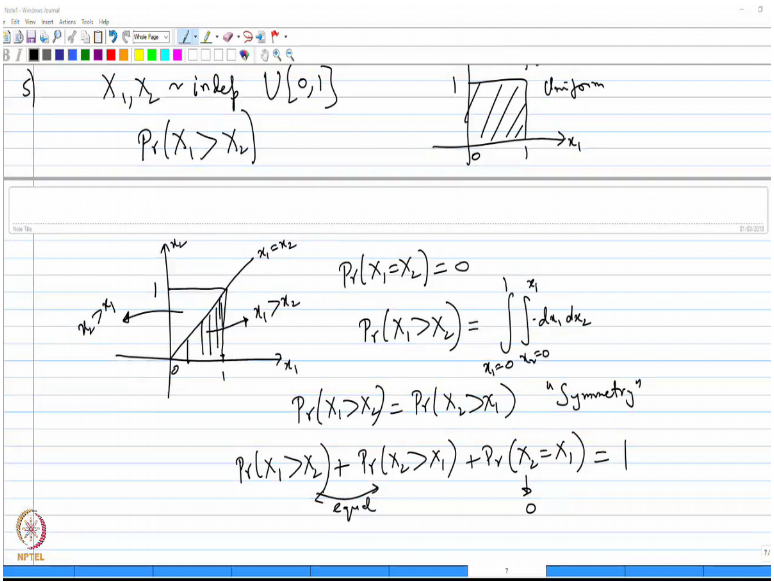

I want to give you one interesting little tidbit on how to exploit symmetry in calculations ok. So, the symmetry in calculations of joint PDFs and all that is very very important quite often a lot of work can be simplified if you do symmetry ok. So, let us say you have X 1 and X 2. So, usually symmetry will come from independence also. So, let us say these are independent uniform in 0 to 1 ok.

So, let us say somebody gives you this independent uniform 0 to 1. So, if you look at it the region of support, is this guy right I am sorry 0, 1, 1. This is the region of support uniform right X 1, X 2 ok. So, now, if you were to ask this question probability that X 1 is greater than X 2 ok. So, somebody might ask you those questions continuous distribution and you want to find probability that X 1 is greater than X 2.

So, the first thing you should do whenever; somebody asks you a question like this is to plot this region in the region of support right. So, this is what you would do this is X 1 this is X 2. So, X 1 equals X 2 is this line. So, everything on this side will have X 1 greater than X 2 right. So, couple of things to remember about these continuous distributions is probability that X 1 equals to X 2 is actually 0 ok.

So, think about why that has to be true you have infinite precision double two value distribution; and then two taking two different real numbers and you cannot have X 1 equal to X 2 with nonzero probability. So, this will be 0 probability and you have X 1 greater than X 2 and so, these two X 1 and X 2 are independent they are both the same whether you change X 1 or X 2 nothing is going to change. So, if you look at it very very carefully this region is X 1 greater than X 2, what about this region? This region is X 2 greater than X 1 ok.

So, if you want to do probability of X 1 greater than X 2, one method is to do a double integral go from X 1 from 0 to 1; X 2 from 0 to X 1 and do the integral you will get an answer not saying this is very hard you can do this quite quickly, but you can also observe that probability of X 1 greater than X 2 has to be equal to probability that X 2 is greater than X 1 this is where the symmetry comes in ok.

In the distribution there is nothing to distinguish between X 1 and X 2, if I just flip X 1 and X 2 around I call X 2 as X 1 and X 1 as X 2 nothing is changing right. So, that is the independence assumption is very important in the symmetry ok. So, the distribution is invariant to switching X 1 and X 2. So, here also if I switch X 1 and X 2, the probabilities will not really change it should be the same probability just the reason will change some nomenclature changes nothing else changes.

So, these two are equal and what will be the sum of these two probabilities probability of X 1 greater than X 2 plus probability of X 1 X 2 greater than X 1 plus probability that X 2 equals X 1 this needs to be equal to 1. So, this is like everything right. So, the entire region of support can be split into the region where X 1 is greater than X 2; X 2 is greater than X 1, then X 2 equals X 1. So, these probabilities have to add up to 1.

So, now this I saw as 0 these two are equal. So, what should be the value of each of these guys the value of.

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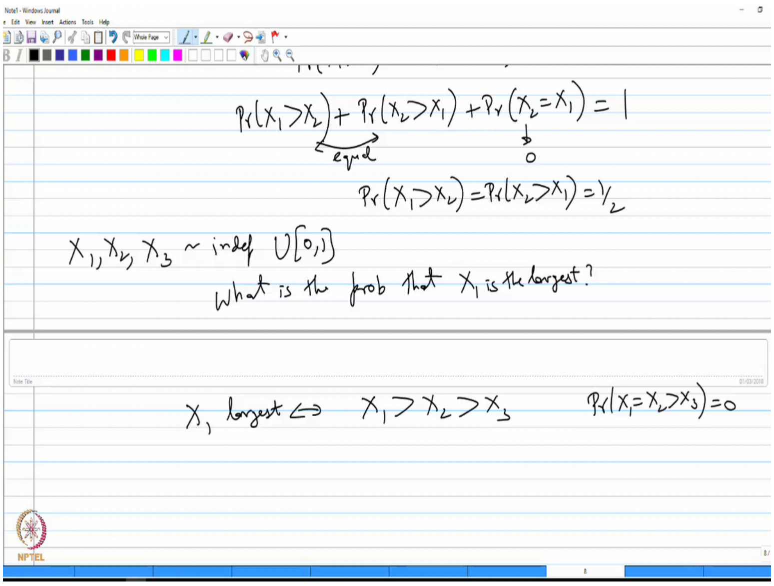

Each of these guys should be equal to half. Now, in this situation is quite easy to evaluate this integral and you can check that it is half, but things can get a little bit more confusing if I have three or more independent uniform distributions and then people will ask you for some probability like this what is the probability.

So, usually it is worded in English, they will ask; what is the probability that X 1 is the largest ok? So, now, so, what is the meaning of X 1 being largest? X 1 largest is equivalent to X 1 greater than X 2 greater than X 3. So, now, if you notice all the equality so, we can also have a situation where X 1 equals X 2 greater than X 3, but all those equality is I do not have to consider in a continuous independent distribution, because all those guys will have probability 0, the moment I put equality somewhere I will have probability 0 ok.

So, for instance if you say probability that X 1 equals X 2, but greater than X 3 this is 0. So, anywhere I put equality probability will equal to 0, that is the nice thing about independence and continuous distributions particularly a sort of. So, this is the region for X 1 greater than X 2 greater than X 3. So, of course, you can write a triple integral here it is not very it is not really that hard you can write a triple integral here and the joint distribution is most one right. So, you can imagine how you will write the triple integral here.

So, all these cases are between 0 and 1 ok. So, you write integral for X 2 from 0 to X 2, then X 2 from you know. So, you can write up triple integral I do not go into the details there it is a bit confusing and you can get messed up a little bit, but then one can use symmetry. So, how do you use symmetry? So, the way to use symmetry is if you look at the entire region, there is a region of support you split it into various things like this.

So, something should be greater or lesser right. So, you pick any point X 1, X 2, X 3 there will be a certain ordering and if look at all possible orderings all of them have to be the same, because I mean the probabilities of each of them should be the same.

So, that is the critical idea here. So, what are the various different orderings possible you can have X 1 greater than X 2 greater than X 3.

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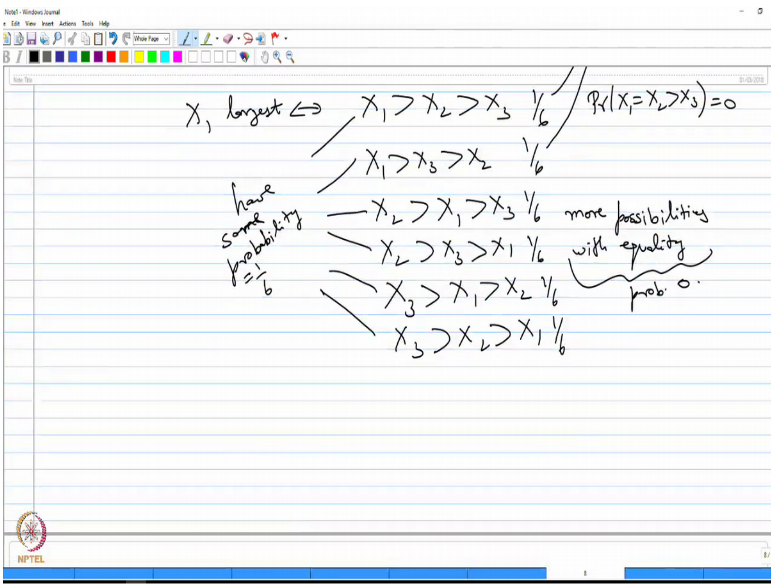

Or you can have X 1 greater than X 3 greater than X 2. So, remember it is critical that any equality I do not have to consider if you have to consider the equality is things will become more more murky ok. So, this is a possibility the other possibility is X 2 greater than X 1 greater than X 3. The other possibility is X 2 greater than X 3 greater than X 1 the other possibility is X 3 greater than X 1 greater than X 2 the other possibility is X 3 greater than X 2 greater than X 1 ok.

So, there are 6 different possibilities here and then additionally some more equality possibilities more possibilities with equality, but all of them will have probability 0 all of these guys have probability 0 and the point is each of these guys have to have the same probability this is the crucial thing to understand ok. They all have the same probability the reason is, you know I mean there is nothing to distinguish between 1, 2 and 3 I just flip 1, 2, and 3 around I should get the exact same distribution, because it is independent and it is uniform in the same interval ok.

So, all of these guys have the same probability and they have to add up to 1. So, what will be the same probability that is equal to 1 by 6 each of these cases probability 1 by 6. So, what is the probability that X 1 is the largest in 2 of these cases X 1 is the largest. So, this will work out to 1 by 6 plus 1 by 6 which is 1 by 3 rights. So, these two cases will land up to X 1 being the largest. So, you can also ask any other question; for instance, what is the probability that X 1 is in the middle 5. X 1 is sandwiched between X 2 and X 3.

So, you see that is. So, again two cases again will be probability 1 by 3 you can ask for a specific ordering. So, any ordering when you have these kinds of independent distributions it is easy to calculate ok. So, this is one use of symmetry the other use of symmetry is also a calculation of expectation; now we also saw one of the critical properties of expectation; was expectation is linear right a linear operation over addition ok.

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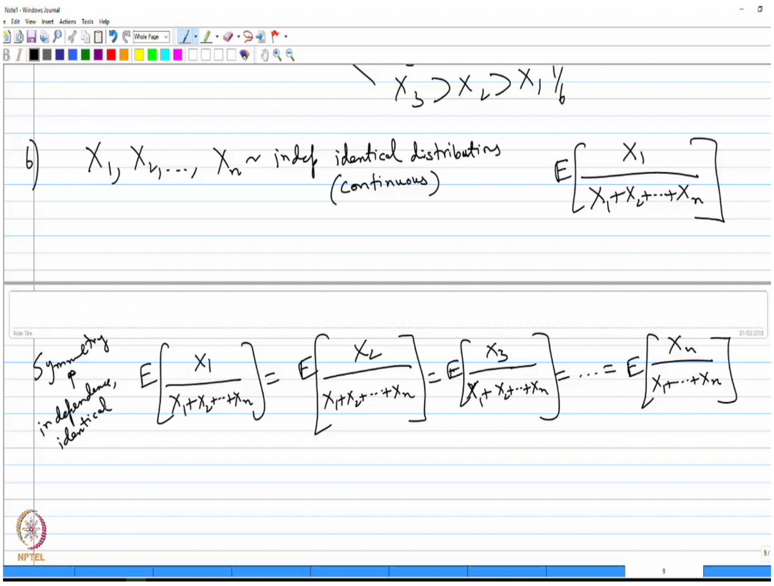

So, supposing this is a very interesting question. So, you have n independent identical distributions somebody tells you and they are also continuous ok.

So, let us say we will assume it is continuous I am not really sure if this continuous is needed, but for now let us just take continuous it does not really matter. So, one of the questions that people would ask is this expected value of X 1 divided by X 1 plus X 2 plus X n ok. So, you I mean for instance continuity is sometimes needed to avoid some dangerous situations of all of them going to 0 etcetera.

So, we will assume these things exist in some sense ok. So, when you do this when you do this calculation, I mean I have not told you anything more I have not told you anything about ah; how the; what the distribution is this that? And etcetera, but then one can use symmetry and the fact that they are independent. We have some symmetry that comes from the fact they are independent and the linearity of expectation to evaluate this expectation ok.

So, you might be surprised by how that will work out the crucial point observe is expected value of X 1 divided by X 1 plus X 2 plus. So, on to X n is actually the same as expected value of X 2 divided by X 1 plus X 2 plus X n; which is actually the same as let us say X 3 by X 1 plus X 2 plus X n and it is actually dot dot dot dot same as expected value of X n divided by X 1 plus X n ok.

So, this comes from symmetry which in turn comes from the independence identical nature of the distribution ok. So, this is the first observation you make and then the next observation you make is if you add up all these expectations; what will you get?

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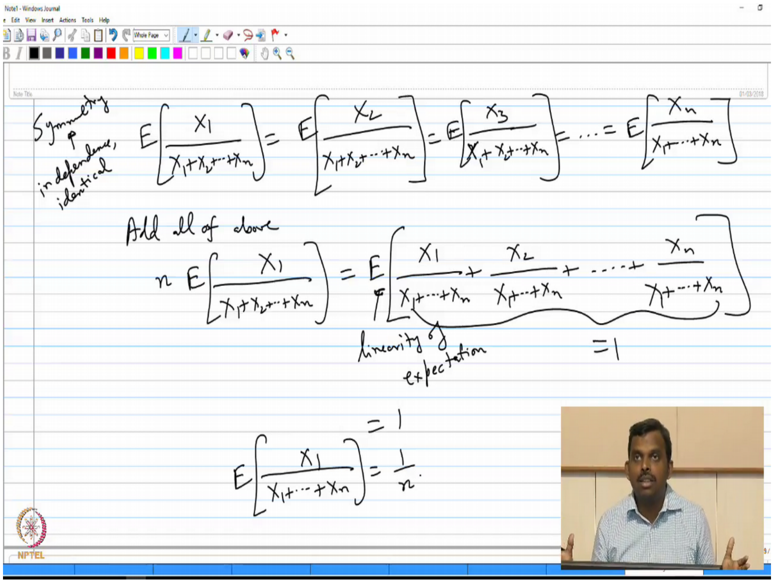

If you add all, of the above do you get they are all equal. So, you will get n times expected value of X 1 by X 1 plus X 2 plus X n right they are all equal if I add all of them I will get n times this, but you can also add them the other way around right. So, what is this? This is just expected value of X 1 by X n plus X 2 by X n plus dot dot dot X n by X n X n by X 1 plus X n ok.

So, you can see why this is true right. So, this is by linearity of expectation distributes over addition. So, if I add them n times I get this now, but what is this guy inside this what is this guy this guy is just one and what is the expected value of 1 it is just 1. So, what is the expected value of X 1 divided by X 2 plus X n. This evaluates to 1 by n ok. So, without doing any major integration you have got to the answer.

So, quite often while integration is an obvious way of getting to the answer, if you use symmetry independence linearity of expectation you can avoid a lot of hard work and get some very simple and elegant answers to calculations in what. So, a lot of probability is simplified by this ok.

So, I will stop with this as far as this lecture is concerned this lecture showed you how to deal with expectations, how to do these calculations of expectation when you have continuous distributions one distribution one variable two random variables, how to look at region of support etcetera and how to do how to exploit the symmetry of the given situation.

In particularly, when you have independence and identical distributions how to exploit symmetry to simplify your calculation and quickly get to the answer ok. So, I will stop here for now.

Thank you very much.