Multirate Digital Signal Processing
Prof. David Koilpillai
Department of Electrical Engineering
Indian Institute of Technology - Madras

Lecture – 02 (Part-1)
Sampling and Nyquist Criterion - Part 1

Let us begin lecture 2 of our course. Today's topics are, we will revisit a few aspects of the C/D versus A/D.

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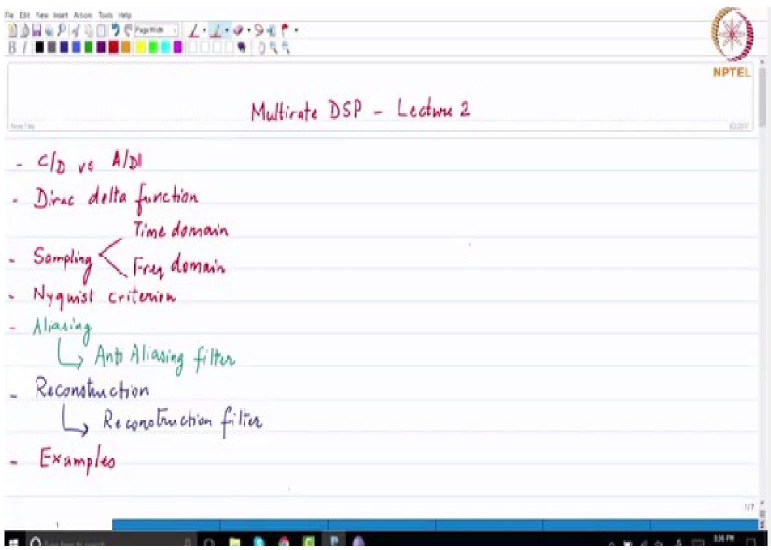

Again, this is a familiar topic but I thought I just wanted to mention one more time. Dirac delta, everyone has studied it probably in your networks and systems course. But worth revisiting the properties because that is important. We are going to be going back and forth from the analog domain to the discrete time domain, continuous time domain. Sampling, our discussion in today's class was going to focus primarily on the frequency domain.

We already looked at time domain sampling and time domain is uniform sampling. And we will also see how the Nyquist criterion comes from the frequency domain interpretation as well. So that is another part of today's discussion. What if Nyquist criterion is not satisfied? Then we have the problem of aliasing. How do you ensure that aliasing is not present? We use something called the anti-aliasing filter.

So why the anti aliasing filter comes and how multirate signal processing helps you work with the anti aliasing filter? That is a very key take away from today's lecture. And then of course, we are always interested in the final outcome. In DSP very often or entire processing starts in the discrete domain, you do the processing and then you do the output in the discrete time domain. You really do not worry about converting it back into an analog signal.

In our case, very often we do want to in the application that we are looking at, we are interested in the reconstruction process. So what is the reconstruction filter? What is an ideal reconstruction filter? approximations of ideal reconstruction filter and then also in the course of today's lecture a few interesting examples as well. So let us begin.

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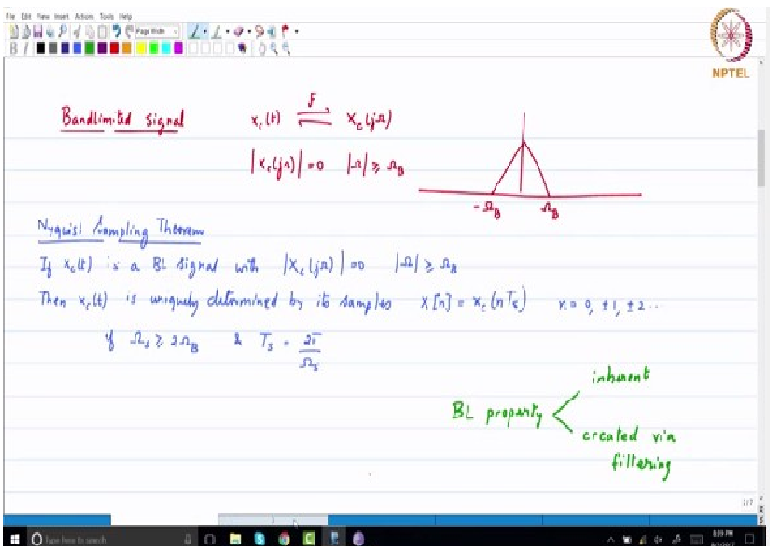

As always feel free to ask questions, as I mentioned in the last class, I will repeat the question because it has to be audible for the recording purposes. So our, by and large when we say that we want to sample a signal, the underlying assumption is that the signal is band limited by its inherent property. Alternatively, I have to make it band limited. So one or the two, either it is band limited or it has to be made band limited, then we apply the Nyquist theorem.

Nyquist theorem says that I must be > 2 times the band width of the signal. So keep that picture in mind that the band limiting process, so the band limited property either has to be inherent or it has to be sort of forced upon it. Or it has to be created via a filter, okay. That is going to be an important element in what we will discuss today, okay.

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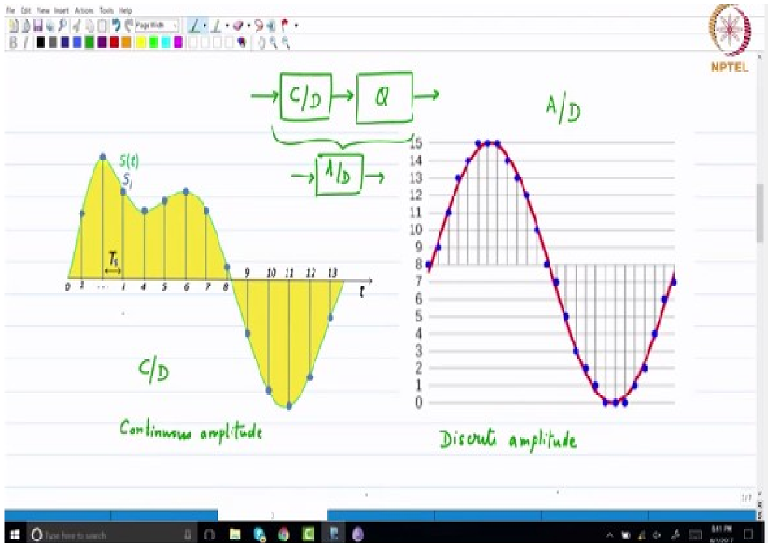

So then the next aspect of it is the differences between A/D and C/D system. So this is a C/D, continuous time to discrete time. This is an A/D. So can you link the A/D to the C/D, what is the part that takes you from here to here? A quantizer. So basically each of these samples instead of being represented with the infinite precision, do have finite levels. So in other words, you could think of a C/D block followed by a quantizer, gives you effectively the A/D operation, okay.

Okay. So this is the point that we wanted to mention. So keep in mind that here we do have a discrete, here we have continuous amplitude in the discrete domain. We do not have any loss of information due to quantization. Here it is discrete amplitudes. The minute you have discrete amplitudes, you have lost some information. So that is the very important element. So let me just spend a few minutes on this.

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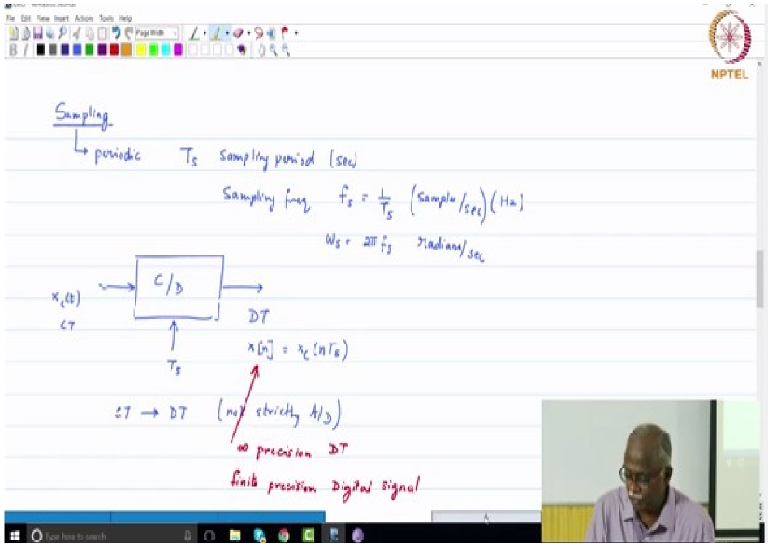

Again this figure is from the last lecture highlighting the fact that basically we are talking about uniform sampling. Throughout this course, the sampling is going to be uniform. Unless we specify we are talking about a C/D converter, not a quantization part. Therefore, that is an important element, we assume this infinite precision. Let me just mention one aspect that I think is important for us to keep in mind, okay.

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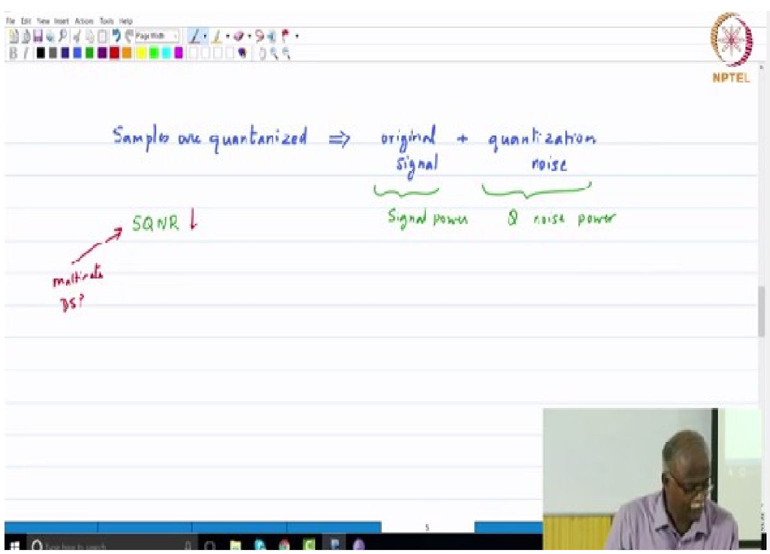

So when we have a quantized signal or a discretized signal, so if the samples are quantized, then we actually represent it in terms of the original signal + some impairment or noise. So it is the original signal + quantization noise. It is very important that the reason we keep emphasizing that we are talking about a C/D is that we really do not want to, at this point, focus on quantization noise.

Because if you talk about the quantization noise, then we can see that there is a signal power and there is a quantization noise power, okay. So the minute you start quantizing, you have to then also keep in mind there is a signal to quantization noise ratio. And the whole subject of A/D is to achieve as high an SQNR as possible. So that is a branch of study that has important elements by itself.

However, multirate DSP can play a part in reducing SQNR. So multirate DSP can reduce SQNR. That part we are interested in, okay. But by and large, since our focus is on the multirate aspects, except when we are studying about the enhancement of SQNR using the multirate processing. The rest of the time, we are, more or less, looking at a from a scenario where quantization noise is not the main aspect, okay.

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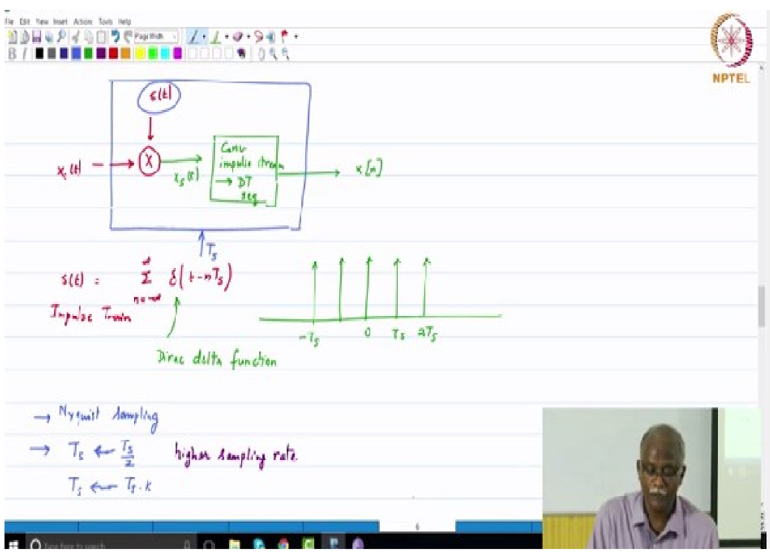

So given that picture, then we can now say that the C/D, that is what we are focussing on, is going to take the continuous time signal, multiplied by a train of impulses, as we talked about yesterday. The spacing between these impulse is going to determine how finely I will sample the continuous time signal. So here are some alternate schemes. So this is a scheme where I have a higher sampling rate because my sampling period has been reduced and of course, if I go the other direction, I will get the opposite result, lower sampling rate, okay. So this is the starting point.

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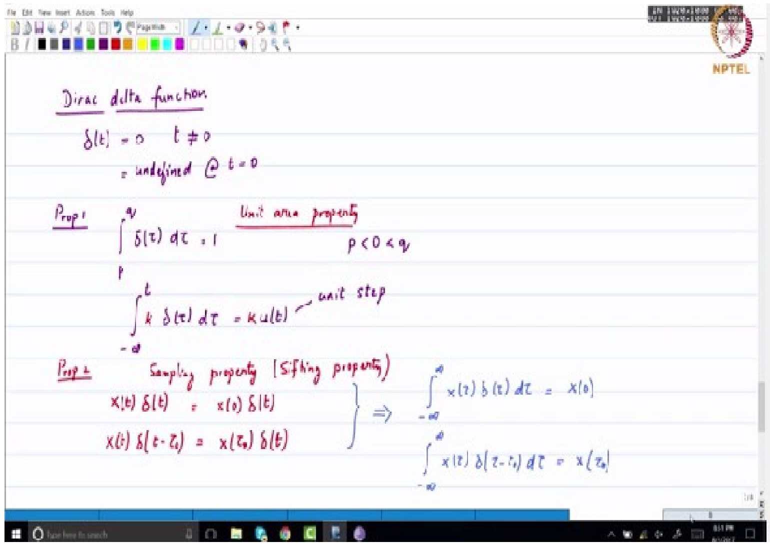

So let us spend a few minutes on the Dirac delta. Again this would be something that I believe most of you are familiar with but it is important for us to highlight the properties and therefore, we would like to. So Dirac delta, delta of t is a very unique function. Again each of the branches of study of who use the Dirac delta have their own perspective on it. There is a mathematics perspective.

There is a physics perspective. There is also the engineering perspective. I will give you the electrical engineering perspective. So we look at it as a function that is not, that is basically is =0 everywhere other than t=0, okay. It is =0 everywhere, but it is also undefined at t=0. So it is a function that has to be specified or defined by means of its properties. Because the definition itself says that there is something undefined about the function at t=0.

So the ways in which we define the Dirac delta are by series of properties and basically we will use 2 properties, the sampling property and the area property. So the area property or the property number 1 ; if I integrate p to q delta of tau d tau, that is I integrate the Dirac delta where p and q are on alternate sides of 0, of the origin, okay. So assuming that the period of integration is on either side of where the delta actually occurs.

Then this is equal to 1, okay. And there are several variants of this. One variant that I am sure you would have seen is -infinity to t delta of tau d tau, basically it is a same integral property. If I do this, then what we get is actually u of t, the unit step, okay. So for every t, you will get; which is t, where t is greater than 0, you will get a unit amplitude and therefore that becomes the unit step.

And interestingly, we also note that any scaling of this delta also scales the output. So it does have a useful property and that is very important for us. Because we are sampling a continuous time signal. It has got an infinite levels in terms of amplitude. So the delta function preserves the amplitude of the signal that it is sampling. So this is property 1. Property 2. So this is the unit area property.

The second one is the one that we are actually using that is the sampling property. Sampling property sometimes also referred to as the sifting property, but for us since we are actually talking about sampling, sampling is the term that we will use. Sampling property, okay. So X of t * delta of t, basically kills; X of t is some function, continuous time function, kills the function everywhere except at X of 0.

So this is =X of 0 * delta of t, okay. So it actually retains the Dirac delta property but it is scaled by the function at X of 0. And similarly, if you were to apply, multiply X of t with delta of t-tau 0, this would sample the function at X of tau 0. Again multiplied by delta of t, okay. This is a very useful property. Again, this may be very elementary because you have already studied it but this is important because sometimes when you study the sampling property, it is not stated in terms of this.

It is actually stated in terms of the sampling property combined with the integral property. So basically this and this means the same thing. You may have seen it that the sampling property is defined as -infinity to infinity, X of tau del tau, d tau. And this is =, what is this? You would have studied this. This is X of 0. Because X of 0 delta of t, if I integrate from -infinity to delta of X of 0 as a constant that comes out integral -infinity to infinity of Dirac delta.

So again you may have seen it in this form, but we are interested in is the sampling property that the Dirac delta preserves. Of course, the variation of this one is also that -infinity to infinity X of tau delta of tau-tau 0 d tau=X of tau 0. So basically either way there is a certain property of the Dirac delta that we need and we are utilizing, okay.

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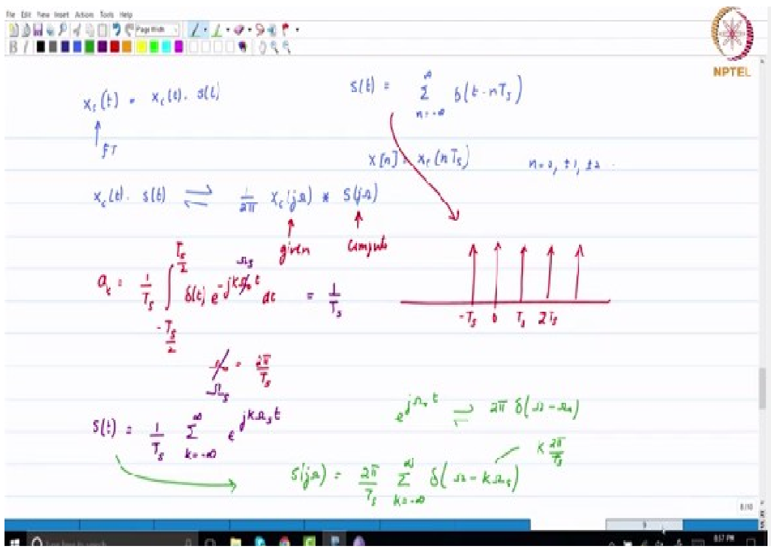

So now we will quickly develop the view of the sampling process. So let me just refresh your mind, Xc of t*S of t is the sampled signal. So that is the signal that we are going to be working with. The sampled signal Xs of t=Xc of t*S of t, where S of t is the periodic train of Dirac deltas. Summation n=-infinity to infinity delta of t-nTs, where Ts is the sampling period and so basically these Dirac deltas are spaced at that point, okay.

Now would like to get the representation, the time domain representation is straight forward. So basically what is happening is that the sampled signal, X of n=Xc of nTs where n is the number that is n=0,+-1, +-2. So time domain is very straight forward. So basically we are sampling the continuous time signal. But the frequency domain is also very insightful. So that is what we are going to be looking at.

So we would like to get the Fourier transform of this signal. So Xs of j omega. So basically we want to get the Fourier transform of Xc of t*S of t. It is multiplication in the time domain. So in the frequency domain, it is 1/2 pi the Fourier transform of Xc of j omega that is the spectrum of the continuous time signal which has been assumed to be either band limited to begin with or force to be band limited via filtering, convolution with S of j omega.

So the input spectrum, this is already given to us. This is already given and this is what we want to compute, and again this is a computation that I am sure you would have done in one of the earlier courses, networks and systems or one of the other courses but it is important for us. So we spend a few minutes just to get the relevant representation. So I want to get the Fourier transform of S of t.

So S of t is a signal which we have already drawn. It is a train of Dirac deltas which are spaced at Ts, 2Ts, -Ts and so forth, okay. And because of its periodic property, it has, it lends itself to Fourier series representation. So first step would be the computation of the Fourier series coefficient, 1/Ts integral -Ts/2 to Ts/2 delta of t e power -jk omega 0 t dt, where omega 0 is my fundamental frequency, that is 2 pi/Ts, that is my fundamental frequency and k*omega 0 becomes.

May be it actually even is okay to even change it at this point to omega S because this we have already defined to be the sampling frequency. So my fundamental frequency is actually my sampling frequency and so the Fourier series computation and all of the Fourier series coefficients come out to be 1/Ts. So S of t, in its Fourier series representation, all the Fourier series coefficients are 1/Ts, summation k=-infinity to infinity e power jk omega S*t.

Basically all of them are complex exponentials. All of them have got the same weight. Just recall that the Fourier transform of e power j omega 0 t, Fourier transform of this is 2 pi*delta, Dirac delta of omega-omega 0. So basically that tells me that the S of j omega will be 2 pi/Ts, 2 pi coming from the Fourier transform of the complex exponentials, summation k=-infinity to infinity delta of omega-k omega S, okay.

Or in other words, this can also be written as k*2 pi/Ts. So again the S of j omega also is a series of exponentials, I am sure these are well known results, okay.

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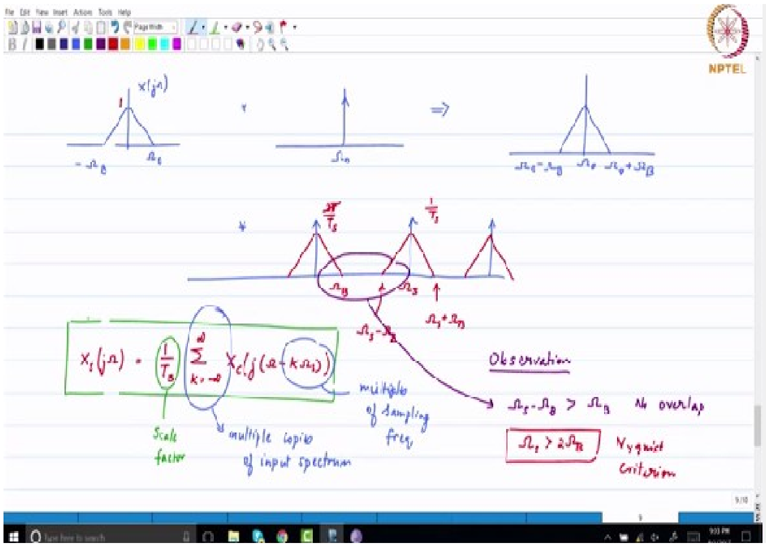

So the convolution that we need to have, so the property of convolution, if this is my input spectrum, X of j omega, convolved with a Dirac delta at omega 0, this gives me the following result. It basically shifts the spectrum to the centre frequency of the Dirac delta. So if this is omega B, -omega B, then after convolution, it becomes omega 0+omega B, omega 0-omega B, okay, and the centre frequency, okay.

So that is the basic convolution property. So expand it. Now to convolve with not a single impulse but a train of impulses, each of them; amplitude of the impulse is : 2 pi/Ts, okay. So these would be at an amplitude of 2 pi/Ts. When I do the convolution, there is a 1/2 pi. So the 2 pi gets removed. So what I am left with is the scale factor of 1/Ts. So 1/Ts is what is present and what will happen is each of these will get a copy of the X of j omega, okay.

So this is omega S, this is omega B. This point is omega S-omega B. This point is omega S + omega B, exactly using the basic principles that we have talked about. So I hope you are clear that the input spectrum if it had an amplitude of 1, through the process of sampling, would have come that. And mathematically, the expression is S of j omega, this is sampled signal, is the product of the 2.

So basically it is the, sorry Xs of j omega. Xs of j omega=1/Ts summation k=-infinity to infinity Xc of j omega-k omega S. This is after the convolution. All the shifted copies of the spectrum, a scaled factor of 1/Ts which is already visible from the graph. So I would like to just highlight 3 elements from this expression, these are useful for us in our understanding and comfort level with the whole sampling process.

First of all, keep in mind that there is a scale factor. Should not ignore it. Should not omit it. This is something that we have to be careful because we are interested in reconstruction, okay. The second element, very important, basically we have produced multiple copies of the input spectrum. So in fact an infinite number of copies. So I will just write it as multiple copies of input spectrum.

Through the sampling process, this is inevitable. This is of the input signal or I will just write it as input spectrum. And these input spectral copies are separated by multiples of the sampling frequency, and all of these are very important in our study because we are going to be talking about multiple sampling rates but at the same time, the ability to reconstruct. So therefore, multiples of the sampling frequency, okay.

So one interesting observation which we can draw from this diagram. So observation. Now what should be the condition that the images do not overlap? So basically look at this portion of the spectrum, omega S-omega B is the lower edge or the trailing edge and the leading edge is omega B. If this is greater than; let us keep it strictly greater than just for; is greater than omega B, then there is no overlap.

Am I right? And each of those; now you may say why not you take it for any other of those gaps. You can take it for any gap. Eventually you will find that the condition that it boils down to is that sigma S should be greater than 2 times omega B, right. And this is nothing but the Nyquist criterion. So the Nyquist criterion actually can be viewed; just from the sampling process, it becomes very evident.

Now you may say why not greater than or equal to? Okay. If you say that it has to be equal to, okay. Let me just give you the answer for that as well. That is what actually Nyquist criterion says. Anyone knows why in most of the practical DSP cases we just say, okay over satisfy the

Nyquist criterion, do not stay exactly at Nyquist. Okay one at a time.

The Nyquist is not at the midpoint, the midpoint of the exact image. Okay, that is one, you have to be very careful about what is the value of the spectrum at that omega B, okay. Any other answer? The filter design, okay! That is a very important element.

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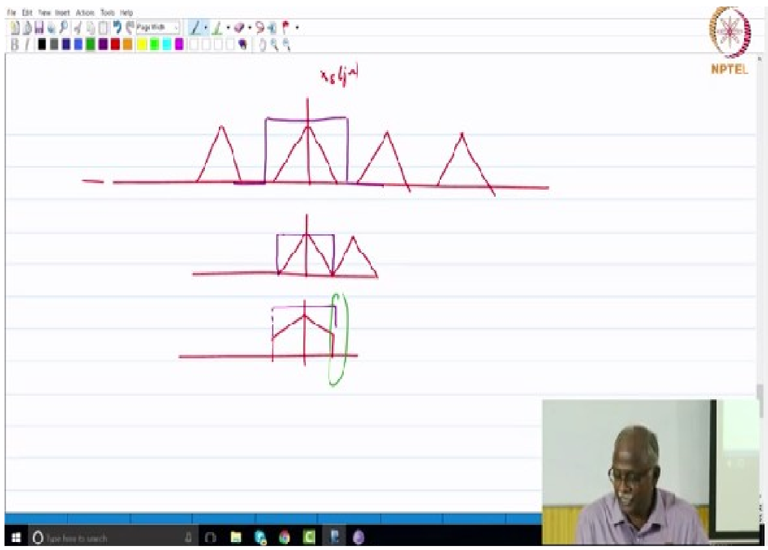

Because now even from this particular discussion, we can move into the next aspect of our, what we wanted to cover is the reconstruction process. So the sampling process has created the following. Sampling process has created these images. Now let us say that we have completed whatever task we wanted to do and now we want to reconstruct the signal. So this is the sampled signal.

This is Xs of j omega. Now the reconstruction process basically requires you to get rid of these images. So the reconstruction filter is actually a low pass filter which removes all the unwanted images, keeps only the central image. Now the aspect that was mentioned is that if you had exactly Nyquist criterion, then what you will need is a filter which is an ideal filter, okay. A brick wall filter, okay.

And in neither in digital nor in analog are these realizable filters. So and of course if it was not 0 at this point, okay, that was another point that was mentioned, okay. If it was not 0, then it adds another dimension to the problem, because if you had a situation where you had something like this, then it is even more tricky when you want to do the reconstruction process. Because you want to make sure that your reconstruction filter does not remove any of the information of the signal.

Now when this is the scenario and your filter is also drawn in this fashion, okay, what is the frequency response of the filter at the band edge? Suppose to be 0 because it is a cut-off filter. It has to cut-off. So basically you create some issues about the reconstruction at the band edge. So for one reason or the other, you would want to be careful with the Nyquist criterion. So I would say that by and large, it is safer for us to over satisfy the Nyquist criterion, satisfied in a manner that it is easy for us to work with.

Okay, let us take a couple of examples which I believe are helpful and instructive and we will then look at some interesting applications as well. So everyone comfortable with the statement about the reconstruction process? Okay, everyone is comfortable with that? Okay. And also with the statement about the requirement of the continuous time filter.