Multirate Digital Signal Processing
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Lecture – 39 (Part-1)
OFDM Applications – Quantization – Part1

Good morning, let us begin, the plan for today's lecture is to introduce some applications of multi-rate of signal processing, I mean understood and studied the tools over the last 38 lectures, the last 2 lectures that are today's and tomorrow's lectures will be focused on how do we use these tools and some of the novel ways in which we can think about it.

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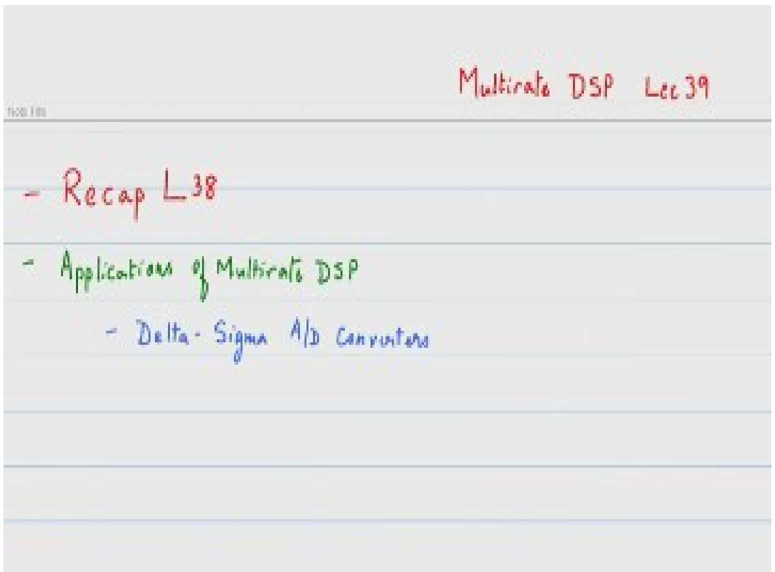

So, just quickly summarise what we have completed as for as lecture 38 is concerned.

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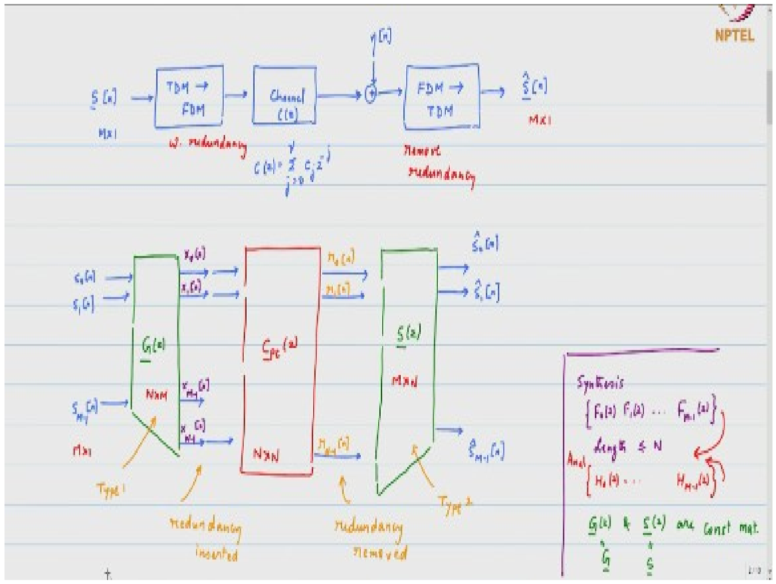

We wanted to transmit a wideband signal through a frequency selective fading channel that was the context and the way we said was that we would introduce redundancy at the transmitters, remove the redundancy at the receiver, 2 types of redundancies that we could introduce, zero padding or cyclic prefix.

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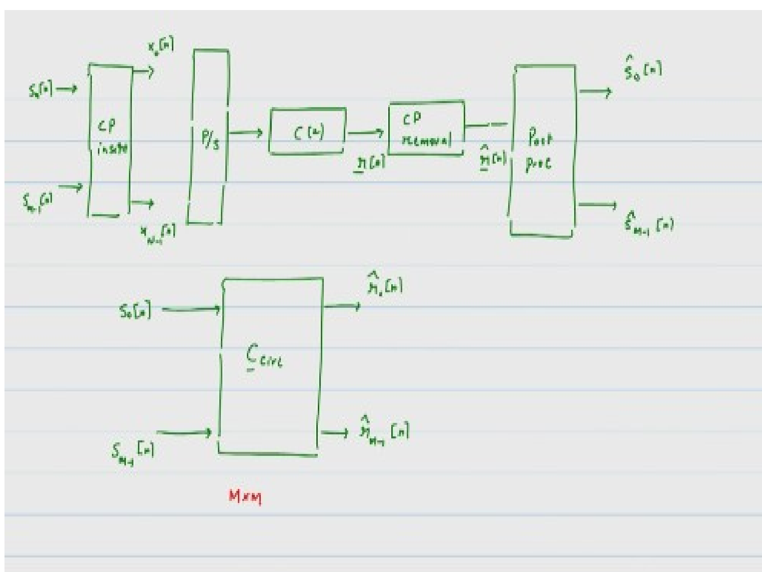

Once we introduced the cyclic prefix, the input output relationship got related by means of a square matrix which is a circulant matrix.

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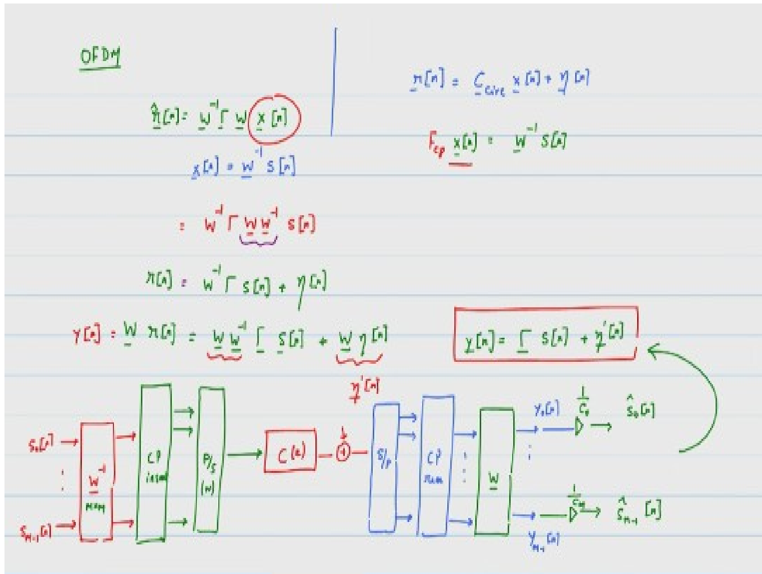

And we use the properties of a circulant matrix in terms of its diagonisability, using the DFT and IDFT matrices with the diagonal elements being the DFT coefficients of the channel; channel response, so basically what we could do at the receiver at the transmitter is to produce a signal which for which you have taken the IDFT.

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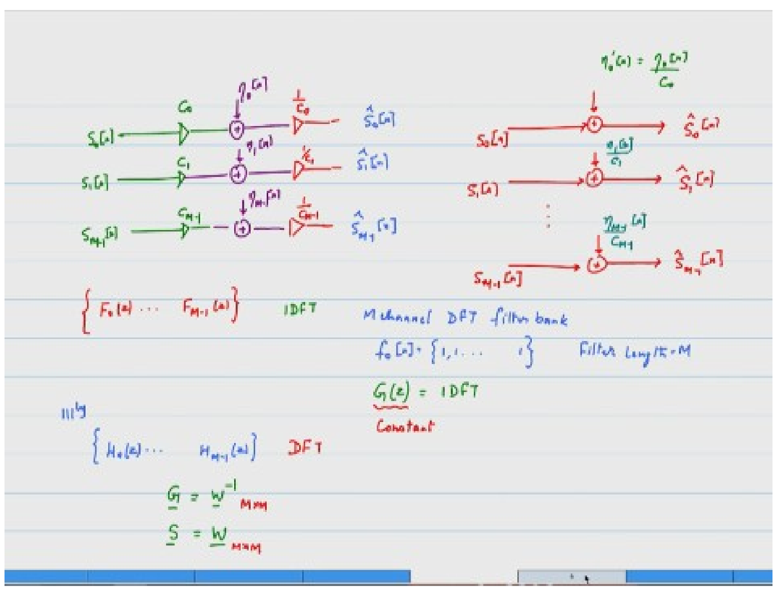

So, when you pass that through the channel, then what we get is a very elegant, very simple structure which is given by; it looks like a set of parallel channel though they were actually interleaved, they were transmitted through ISI channel what it looks like is a are of very clean parallel channel with different SNR's which are controlled by the coefficients themselves and this is actually doubly advantageous.

Because you can also do water filling based on these channel coefficients and then when you look at the other end, if you want to bring all of the signals together, then we can look at them as a scaling which will give you the resultant received signal, equivalently we can also think of it as parallel channels all with equal gain but with different noise terms, different noise variances means different SNR.

So, again you can think of it as scaling of the signal with all of them having the same noise variance or same signal power with the noise variances being modified, okay, so this is; this was the beauty of OFDM and it has become one of the most widely used transmission techniques and I believe it will be used in as we move forward as well.

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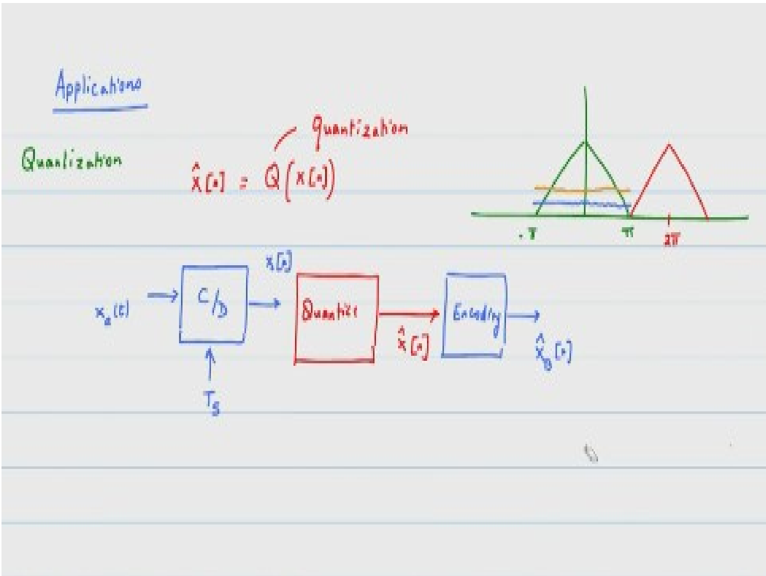

So, today we move into the applications part, where are the areas where multi-rate signal processing is being used and what are the ways in which we can can leverage, so the first application that I would like to take up and the first of 2 applications this in the context of signal quantisation, as you know in our practical implementations of any communication system, we would have to restrict the signal to a finite number of bits in terms of its representation.

So, anytime we have such a situation where we have a finite number of bits for representation, we have the quantisation of a signal, so x hat of n being the quantised signal which to which you pass a unquantised signal x of n, Q represents the quantisation operation, okay and the number of bits of operation, the scaling of the signals all of these are incorporated into the quantisation process.

Again, the notion that we have from our basic understanding is that the signal spectrum, if I were to look at the sampled signal spectrum, so the sampled signal spectrum at this point we are able to draw as – pi to pi, this is the sampled signal spectrum assuming I have an sample rate at Nyquist rate, this is the next copy and so on, so this is 2pi, right, yes, this is 2pi, okay, so this is the spectrum when there is no quantisation noise.

Then, we posed the question, how does quantisation noise enter and where do I have to keep track of it we say that okay, quantisation noise is something that we cannot avoid when we quantised, we go through, I am sure you have gone through the analysis of quantisation noise, we say that the quantisation noise has certain statistical properties, okay and we will go through them but by and large, the spectral properties of quantisation noise.

It is very random in nature, it is uncorrelated so therefore, it is spectrally white, so we think of quantisation noise has something that goes from – pi to pi, some wiggles basically, it is something that has a flat spectrum, almost flat spectrum, okay now depending upon how many bits of precision we have used, we will have more or less quantisation noise and if you have used less bits of precision, then what we would end up with is increased quantisation noise, okay.

It is still flat but it is increased in, now this is the problem that we encounter with quantisation and it is always a worry for us is that no, have I quantised a signal too much that I am not able to recover the or get enough of the information, so here is the visualisation of the process, I have a continuous time signal, I do not know whether I call xa of t or xc of t continuous time signal, the process that we have been studying so far is an ideal conversion from continuous time to discrete time.

We pass it through a C to D converter with a specified sampling period Ts, this gives us the discrete time representation x of n with infinite precision, then comes the quantisation block where we have to impose the condition for storage of a transmission, we have to quantise and the quantised signal is now, what we have to deal with which is x hat of n, okay and of course, the quantised signal is what we would encode into a form, it could be 2's complement our 1's complement whatever form you want to encode.

So, maybe you can think of another block either which is included with the quantiser or with the encoding operation so basically, we have obtained a B bit representation, which is a quantised version of the original signal, okay, so this is the framework, now we go back and now, the question that being posed is does multi-rate signal processing even help and if so, how and how do we leverage it, so that is the context of today's discussion, okay.

So, the first thing is that we must all be comfortable with what is the normal or conventional quantisation, I was checking with the faculty who teach DSP, they said sometimes it is taught, sometimes it may; we assumed that it is taught in communication; communication people say that okay, we think it assumes is taught in DSP, so in case, it has not been taught, this is the part that we are focusing on.

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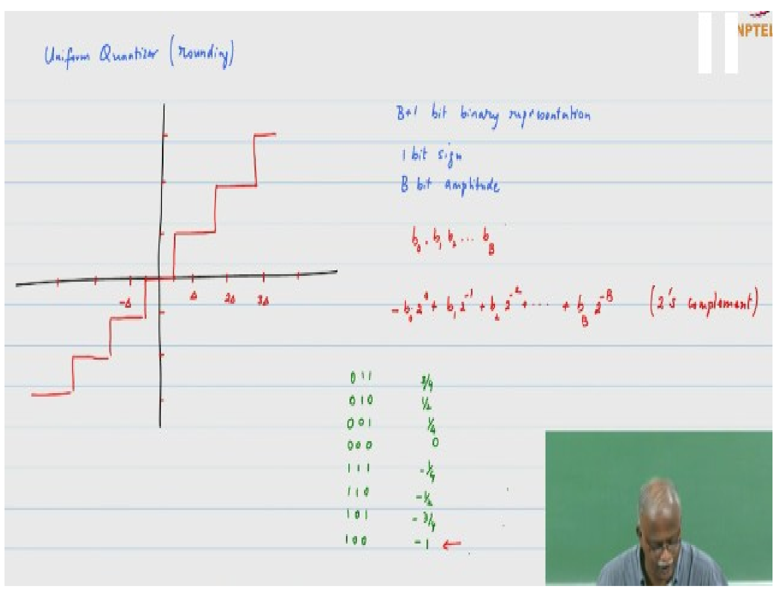

So, basically we would like to look at a uniform quantiser, this is the most common one, uniform quantised with rounding, okay we will assume that we have fairly large number of bits of representation, what is large; 10, 12, 16 bits of representation, so which means that you have sufficient number of levels to represent the signal, so very quickly what does the uniform quantiser to do it for every input signal, it produces a corresponding output signal which is a finite number of representation; bits of representation.

So, the levels, if you were to look at it are marked off by some uniform step sizes, okay we will call it as delta and mark it uniformly on the x and y axis, so basically this is and we say that anything between, so this is delta; 2 delta 3 delta – delta, okay, so anything which is in this, sorry, anything which is in this range, we will denote between – delta/ 2 and delta/2, we will denote as 0, anything between delta/2, 1.5 delta is denoted by the next quantisation level.

So, then we go up to the next quantisation level and so on, okay, similarly we have the negative side as well and once you have drawn sufficient number of levels, you can leave it, so basically this is the quantisation; uniform quantisation that is happening all the bins are equal value and then you have quantised it to some levels, so this is signal where there are positive and negative signals permitted, we would like to have a representation in terms of the number of bits.

So, the standard convention that we use is we assumed that we have a B + 1 bit representation and I will clarify that in a minute, why we want to do that so, B + 1 bit binary representation, the +1 denotes the sign, okay, so it is actually a B bit quantisation in each for positive values and for the negative values, binary representation and again I am assuming that this is familiar to you just sort of giving you the key elements.

So, one bit for the sign and the B bits for the amplitude representation, okay that is what we have done and typically, we would like to represent these values in a range from -1 to 1 basically, we assumed that the signal is scaled by its maximum amplitude and if you were to think of the bits of representation, so it would be treated as B0 dot that means that is the one that denotes the sign, then we have B1, B2 all the way to Bb, upper case B, okay.

So, it will be –b0 times 2 to the power 0 that is 1 + b1 times 2 to the power of -1 +b2 2 times 2 to the power of -2, this is 2's complement representation, the final value is b subscript b 2 to the power of –b, this is 2's complement and you would have studied other forms of representation, all are fine since this is more of an analysis of the quantisation error, we do not really spent too much time on this element, okay.

So, 2’s complement representation uniform levels from a range of values, okay, so typically, what you will find is that if we have 1 level denoted as 0, then this is; this could be a typical representation, so if I have a 3 bit representation, so basically we are looking at a 3 bit representation, so 2; one of them will be the signal, the polarity of the signal, the second one, the remaining 2 bits are the levels.

So, this is how we could have a representation 0, 0, 0 representing the 0 value, zero’s representing the positive values, so I have level 0, 1, 1, 0 and 1,1 so if I were to do it on a scale on a +1 to -1 scale, this could be given the level 1/4, this will be 1/2 this would be 3/4 and then go to the negative side, the first value in 2’s complement 111 that would be -1/4 110-1/2, 101 is -3/4 and 100 is -1.

So, typically this is how you would see the quantisation and noticed that there is in terms of the negative values there is one extra value, okay you go slightly more on the negative side than on the positive side, again if you have enough number of levels, this is not an issue, so again it is a point to note that are invariably the quantisation will have one extra level on the negative side than on the positive side when we do the 2’s complement representation, okay.

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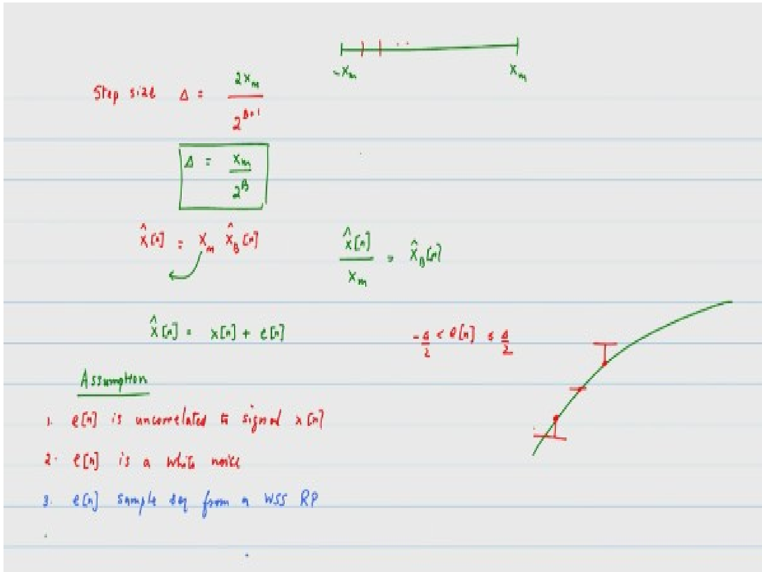

So, what is the step size; the step size that we have worked with that is the delta, step size delta, there are 2 to the power of b +1 bits, right that those are the number of levels that we can get and I go from a positive maximum value to a negative maximum value, so xm is the extreme range of my signal, so if my signal lies in this range xm to -xm and I want to quantise this into the number of bins that are available to me.

So, this would become 2 times xm that is the range that I have to quantise and I have 2 power b +1, so this is the; so this delta is often related to or denoted as x of m divided by 2 to the power of b and so this is a and but you know where it comes from because of the b + 1 bit representation, so the quantised signal x hat of n which has been quantised with the b + 1 bit representation can be actually recovered or you can be related to the maximum value xm, right that is the peak value.

And the quantised signal has been represented in the range; once you have scaled your signal by xm, then you can quantise it in the range -1 to +1 that is xb hat of m or if you want to think of it as a scaled signal, take x of m to the other side, so you take your discrete time signal scale it by the maximum magnitude and then quantise it because now it is in the range +1 to -1 that is x b hat of n, okay.

So, I think the quantisation framework is clear, it is a uniform quantiser, it is a rounding quantiser, so basically once you go past the half level, you are going to be quantised to the upper level that is where the rounding element comes, so here is a general statement that the errors that you will see in your quantised signal, if you were to write your quantised signal as the unquantised signal x of n + e of n.

Some error sequence which is; we can say that typically, e of n has a range of values, it typically goes from –delta/ 2 to less than delta/2 provided you have not reach the sort of the upper level, the extreme levels, if you have reach. If you have cross that then it is clipping but in the range of the quantiser, this is typically satisfied so, this is a good way to look at the signal, okay so some statistical assumptions about e of n which helps us in the quantisation or in the analysis of the quantisation is that e of n is an uncorrelated sequence.

It is basically from sample to sample, it is an uncorrelated and it is a very reasonable assumption. Because if you were to look at the signal that is varying like this at a given point in time, this is the sample value, the quantised level could be here, okay, the next instant of time the sample value that could actually be on a quantised level itself, the next instant of time it could be sampled here but it could be quantised to a higher level, so basically, you can see that from sample to sample, there is no necessity or there is no underlying correlation.

So, we make the following statements and assumptions that first of all that the error sequence e of n may be positive, negative 0 and it is uncorrelated to the signal, is uncorrelated to the signal, it helps, it is a very helpful assumption, it is uncorrelated to the signal, okay that is the first signal x of n, the second element; e of n is a white noise process that means, it is uncorrelated from sample to sample, okay and you can also think of this as a sample of a wide sense stationary process.

That means, whatever things like mean, variance and other parameters are not dependent on time they are constant, so that is another aspect, so e of n, we consider it as a sample sequence from a wide sense stationary random process, a sample sequence from a wide sense stationary random process RP; random process, okay and the last assumption that the quantised value; quantised value of e of n is uniform that is the probability is uniform.

The PDF of e of n is uniform, okay these are the assumptions so in other words, the PDF being uniform says that the error goes from – delta/2 to delta/2, the probability is that all of these values are equally probable, so I have a height of 1 over delta that is my PDF; probability distribution function of the error signal, so this is the probability of e of n of e, okay. So, basically this is the framework, this is what you would have used to analyse.

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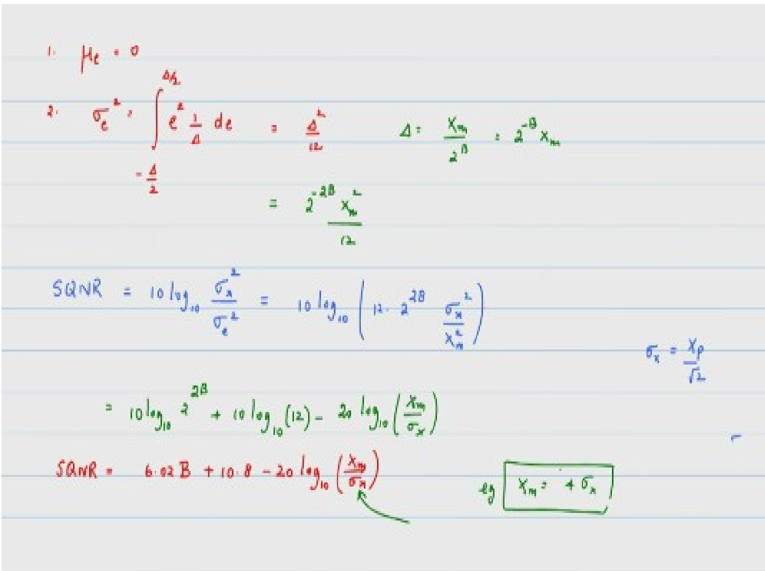

So, using this framework, please verify that these are the results that you would have shown or derived the mean value is 0, uniform both positive and negative values straightforward to show, second one is the variance of the error, okay this one says I go from – delta/2 to delta/2, the variance means I have to do e squared, PDF is 1 over delta times de, this comes out to be delta squared/12, okay.

So, this is also result that is well known delta itself we have said is can be written as x of m divided by 2 to the power b or you can write it as 2 power -b times x of m, so this can actually be written as 2 power -2b x of m squared divided by 12, okay, so here comes the first major equation or the outcome of this analysis, which is very useful tool for both communications and signal processing engineers, signal to quantisation noise ratio.

This is not channel noise, this is quantisation noise, okay so on top of it, the channel may add but right now at the very representation stage itself there is some noise being introduced, so signal to quantisation noise ratio defined very similar to what you would do for the signal to noise ratio basically, it is 10 log based 10 of the signal power sigma x squared divided by sigma e square, okay sigma x squared representing the signal before that is the signal x of n.

And sigma e squared and if you go ahead and substitute for this from the previous expression, this becomes 10 long 10 of 12 times 2 power 2b times sigma x squared divided by xm squared, sigma x squared/ xm squared, okay, so now what is xm; the maximum value that it can achieve, sigma x squared is the variance of the or the power of the signal right, sigma x, so that is a very, very important element just to that you are feeling comfortable with it.

For a sinusoid, if the peak of the sinusoid is x of p, what is the RMS value; sigma x will be 1/root 2, so there is a relationship between the extreme value and the RMS value and that is what we are capturing here, so now if you were to write down this equation, this would be 10 times log based 10 2 power 2b +10 times log base 10 of 12 -20 times log base 10 of xm/ sigma x, okay I have deliberately inverted the ratio and written it, okay.

So, simplify this equation, what you should get is the following, it should be 6.02 times b +10.8 the other term -20 log base 10 of x of m/ sigma x, okay so the key question is how have you; how are you going to design your quantiser, you have to tell what are the range of your quantiser and of course, how many bits you are going to do, so if I choose my range to be xm and I will chosen the b +1 bits as the representation.

This is the equation that tells me what my signal to quantisation noise ratio is, okay now usually for most signals, we can sort of specify, okay how do I set these up these limits, so you can say that for example, you can say that I will chose xm to be = 4 times sigma x basically, 4 times sigma on this side, 4 times sigma on that side, so that is the range that I want to cover, so if you take some assumption like this, you have bound to capture most of your input signals.

Of course, there could be occasionally an input sample that has got that goes outside but by and large, you have captured it, you can do 3 sigma, 5 sigma whatever it is, keep in mind that the larger that you want to keep x of m, it is going to reduce your SQNR right because that means that with a finite number of bits that means your quantisation error is going to be more because you have a larger range to cover.

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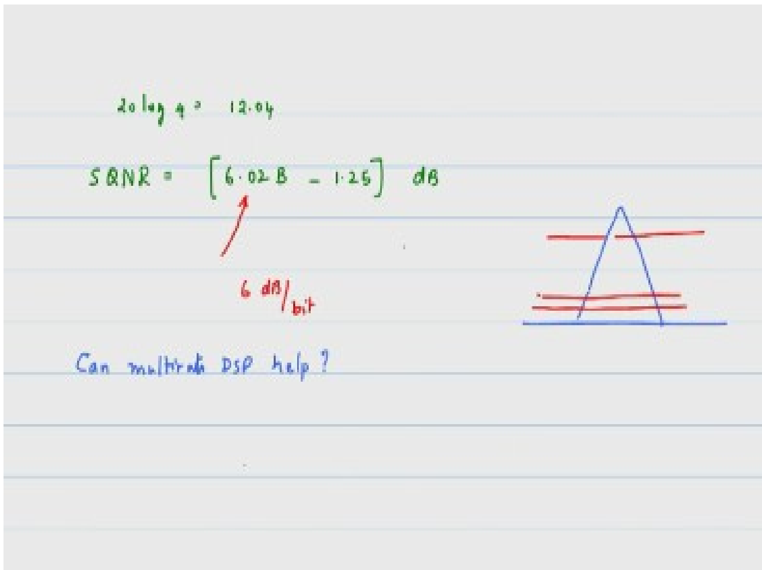

So, this is an important choice, now one of the common choices is 4 sigma, therefore, if you make that assumption, then we have 20 log of 4 which comes to be 12.04, so if you go back and substitute your SQNR, now has the following 6.02b -1.25, this in dB, okay, so very cleanly you have now linked your SQNR to the following result, okay, so this is where you get that thumb rule which says 6 dB per bit, okay.

Many times you will see people saying, oh, how many, what is the quantisation noise you want; quantisation noise, if you say I want to get 96 dB, there will say 16 bit representation, how did they get it, how did they without even computing everything because they say okay, if I have 16 bits of representation, the SQNR will be of the order of 16 times 16, 16 times 6 and that is the 96 dB, okay.

This is where that 6 dB per bit rule comes and so if you want to go from 64 dB SQNR to 96 dB SQNR, you know exactly what you have to do in terms of the; so this is the quantisation noise and issues, so again up to now nothing knew nothing different, so the key question is; can multi-rate techniques make any difference, multi-rate DSP help or can it do something and surprising result which says that if I use multi-rate DSP, I can reduce the number of bits of representation.

And I can still maintain the SQNR, okay, now you may say to what extent can you reduce the number of bits of representation; the answer comes out be, I can go down to 1 bit representation, so 1 bit representation if you want to think of it in terms of the conventional approach, if this is your signal spectrum, remember we drew those lines, okay this is the quantisation noise if you at 16 bits, this it will go up here.

One bit means you are pretty much you are swamping your signal with noise, so almost nothing you can do with the but so with the conventional method, if you go down to 1 bit quantisation nothing can help but with multi-rate the claim is that it can go down to 1 bit, so here is the first step; the first step is over sampling, okay, so let me just give you the flavour of it, does a since this is more of an application talk.