# Digital Signal Processing
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## Lecture 24:
## Z-transform (3)
## -Absolute convergence criterion for the Z-transform

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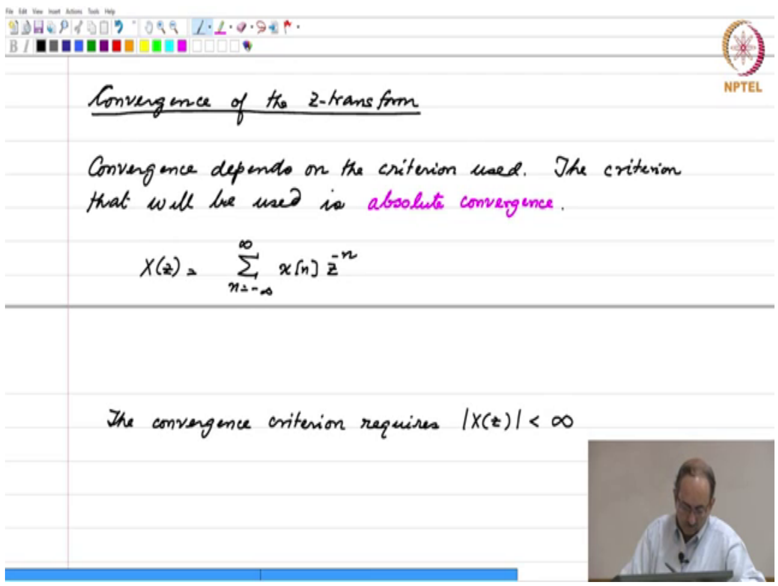


Now, let us talk about convergence. Because, you have some going from $-\infty$ to $\infty$, you need to worry about whether this quantity exists or not. There are many criteria for convergence, we will use a criterion that was used in the Laplace transform case. So, convergence actually depends on the criterion used, and we will use exactly the same criterion that was used in the Laplace case and the convergence criterion that was used in the Laplace transform case was; what was the criterion that you were using for the existence of the Laplace transform?

The criterion that I hope at least now it rings a bell and what is then some other convergence criterion that this terminology triggers, that you may have heard of?

Student: (Refer Time: 02:01).

In terms of integrals. Remember, this I am talking, I am recalling what you may have learnt in Laplace. The other criterion that could have been used is uniform convergence whereas, in the Laplace case, we used absolute convergence as the criterion. We will use absolute convergence criterion for the Z-transform case. There, the absolute convergence meant absolute integrability whereas, here absolute convergence would mean absolute summability.

Therefore, $X(z)$ is defined as $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$. The absolute convergence criterion demands that $|X(z)| < \infty$. So, this is the criterion for absolute convergence.

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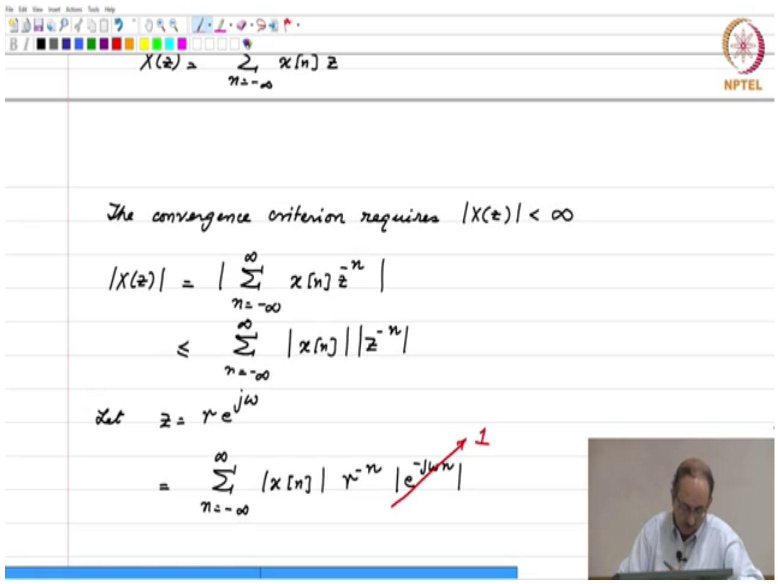


So, let us examine $|X(z)|$. So, this is the absolute value of the summation over all $n$, $|X(z)| = \left|\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right|$ and then, since absolute value of this sum is always less than or equal to sum of the absolute values, you have $|X(z)| \leq \sum_{n=-\infty}^{\infty} \big|x[n]\big|.|z^{-n}|$. You will now let $z = re^{j\omega}$ and hence, this summation becomes $\sum_{n=-\infty}^{\infty} \big|x[n]\big|.r^{-n}$, $r$ being a non-negative quantity. All I have done is, I have replaced $z$ by $re^{j\omega}$ and this of course is; $|e^{-j\omega n}| = 1$ and does not play any further role. So, now we need to look this up.

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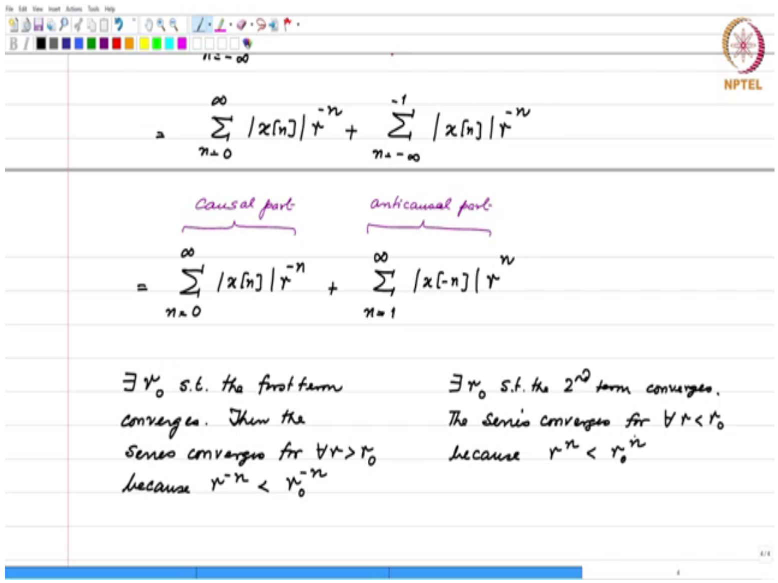


We will break this into two parts. So, this can be written as $\sum_{n=0}^{\infty} \big|x[n]\big| r^{-n} + \sum_{n=-\infty}^{-1} \big|x[n]\big| r^{-n}$. First term remains as it is and in the second term, we will replace $n$ by $-l$ and then we will relabel $l$ as $n$ once more. Therefore, this is nothing but $\sum_{n=1}^{\infty} \big|x[-n]\big| r^{n}$. And if you look at the way the summation has been split, what we have done is we have broken the sequence into two parts, this is the causal part namely the sequence with indices ranging from 0 to $\infty$, and this is the anti-causal part of the sequence, where recall we are using the term causality. Let us introduce in terms of systems, you are applying to a signal, signal is called causal if it is 0 for $n < 0$, an anti-causal if it is 0 for $n \geq 0$.

Now, let us look at each of these terms. So, let us look at causal part to see what that behaviour is. Suppose, let us assume that there exists a certain $r_0$ for which the first part exists, right. So, suppose there exists $r_0$, such that the first term converges. Let us assume this. Then, we want to ask the question what about values of $r$ greater than $r_0$. If you know that the first term converges for a certain $r_0$, what can you say about the convergence for all values of $r$ that are greater than $r_0$.

The sum will converge because if you consider a value, say $r > r_0$, then this term, the weight that you apply to the sequence decays more rapidly, right. Because, if you have $r > r_0$, then $r^{-n} < r_0^{-n}$. Therefore, the terms are weighted down more and more heavily. Therefore, if the series converges for $r_0$, then for all values of $r$ that are greater than $r_0$ the series necessarily will converge. And the theorem that is used to establish this is, intuitively you see that this is true, but the precise theorem that is used is the comparison test. So, series comparison test is used to establish the result.

So, we can see then the series converges for all $r$, $r > r_0$, because $r^{-n} < r_0^{-n}$. Therefore, if a certain radius exists for which the series converges, then for all radii that are greater than this, the series is definitely bound to converge.

Now, very similar arguments can be made for the anti-causal part. Again, let us assume there exists an $r_0$ such that the second term converges. Then, what can you say about the convergence for all radii that are smaller than this $r_0$? The series will definitely converge because remember, now we are looking at $r^n$. Therefore, any radius that is smaller than $r_0$ will decay faster and hence by the comparison test, if the series converges for $r_0$, series will necessarily converge for all values of $r$ that are less than $r_0$ for the second term. So, the series converges for all $r < r_0$, because $r^n$ will be less than $r_0^n$.

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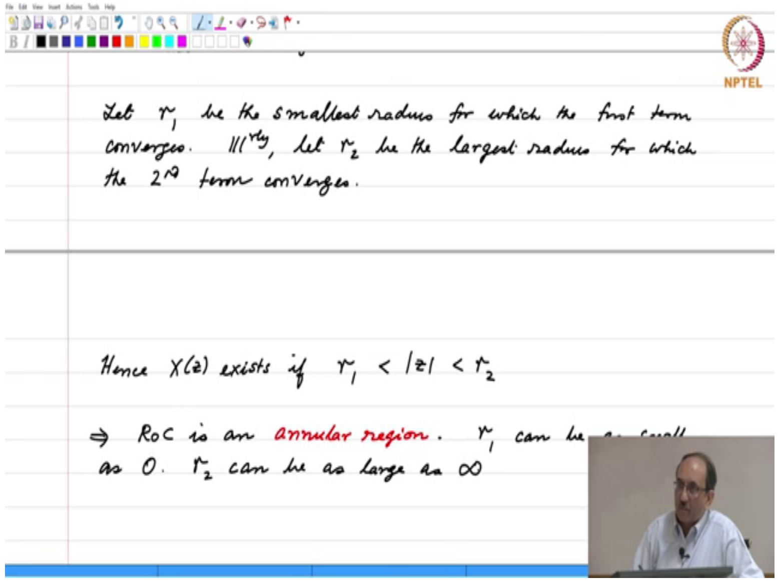


Now, let us assume, let $r_1$ be the smallest radius for which the first term converges. Similarly, let $r_2$ be the largest radius for which the second term converges. So, if there exist an $r_1$ for which the first term converges and if there exists an $r_2$ for which the second term converges, then the complete series which involves both the first term and the second term will converge when you have $r_1 < |z| < r_2$.

So, hence, $X(z)$ exists if you have $r_1 < |z| < r_2$. So, clearly this supposes that $r_2$ is larger than $r_1$, because you are looking at the intersection of two regions of convergence. For the first part, you require $|z| > r_1$, and for the second part you require $|z| < r_2$, therefore, these two regions must overlap which means $r_1 < r_2$. So, this implies that the region of convergence is an annular region and $r_1$ can be as small as 0, $r_2$ can be as large as $\infty$.

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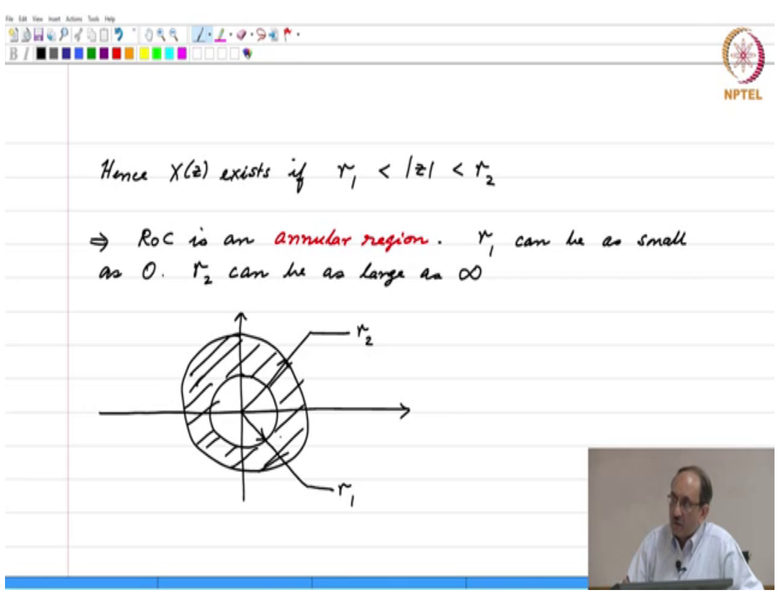

Therefore, the region of convergence as far as Z-transform is concerned is in general an annular region with radii $r_1$ and $r_2$ and this is the region where in general the transform exists. And the reason why you have the region of convergence to be an annular region is because the convergence criterion that we have used is the absolute convergence criterion, and the absolute convergence criterion boils down to the parameter $r$. This is the reason why the region of convergence is an annular region.

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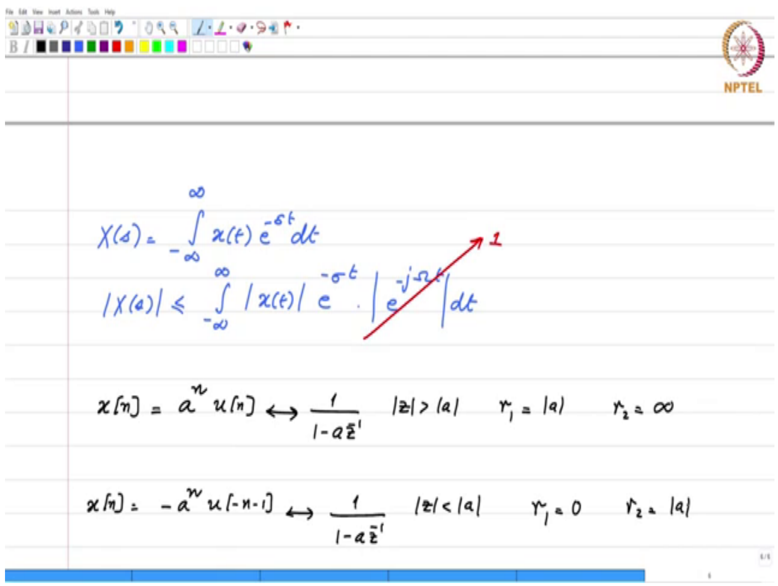


So, if you recall your Laplace transform case, there you had $X(s)$. So, this was integral $X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$. Again, there you used absolute convergence. Therefore, you required this. So, $|X(s)| \leq \int_{-\infty}^{\infty} |x(t)|e^{-\sigma t}.\left|e^{-j\Omega t}\right|dt$. You had to take mod, $\left|e^{-j\Omega t}\right|$ goes off, therefore, the convergence of the integral was dictated by the parameter $\sigma$ and $\sigma$ is the real part of $s$.

That is why, there the convergence happened between vertical lines, because the parameter governing convergence was the real part of $s$ whereas, here the parameter influencing convergence is the radius $r$, that is why what is happening in the Z-transform case is between two circles. What was happening between two vertical lines earlier is now happening between two concentric circles.

And we saw this example, $x[n] = a^n u[n]$. So, this was $X(z) = \dfrac{1}{1 - az^{-1}}$. So, this was $|z| > |a|$. In this case, $r_1$ happens to be not $a$, but $|a|$. Remember, $a$ in general is a complex number, whereas $r_2$ is $\infty$. Similarly, if you had $x[n] = -a^n u[-n-1]$, again you had $X(z) = \dfrac{1}{1 - az^{-1}}$, same algebraic expression. But in this case, it is $|z| < |a|$ and here $r_1 = 0$, $r_2 = |a|$. So, just to illustrate the fact that $r_1$ can be as small as 0 and $r_2$ can be as large as $\infty$, we have already seen examples.

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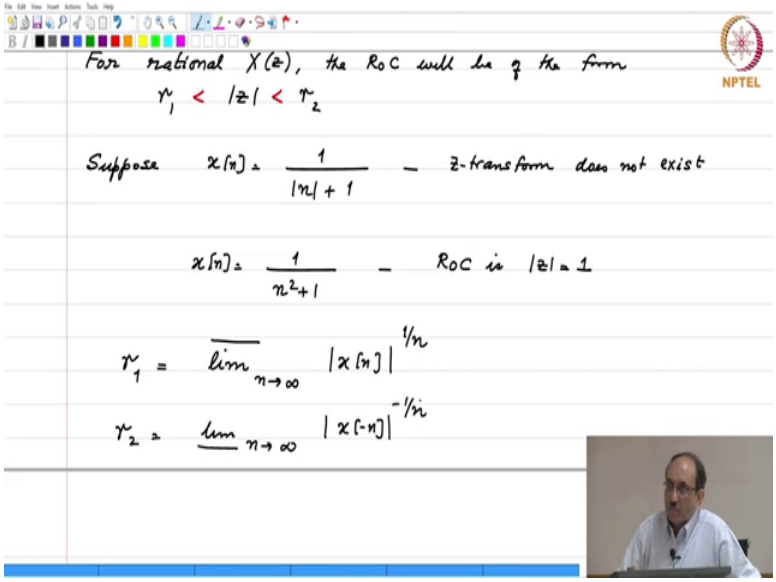


Just to give some more insight into this RoC, for rational $X(z)$, recall that rational means $X(z)$ is of the form $\dfrac{B(z)}{A(z)}$, ratio of polynomials. For rational $X(z)$, the RoC will be of the form, $r_1 < |z| < r_2$. For rational transfer functions, this is how the RoC will be, and the RoC will be bounded by poles.

So, this is exactly the same as what was happening the Laplace transform case. In the Laplace transform case also, you would have had $\sigma_1 < Real\{S\} < \sigma_2$. There also you had the RoC bounded by poles. And suppose, $x[n] = \dfrac{1}{|n|+1}$. So, this we just note the fact that, if you want to take a guess about the Z-transform of this, what would be your guess? Yeah, very good, it does not exist and the reason why it does not exist is?

Student: (Refer Time: 20:25).

Ok. Suppose, I had $x[n] = \dfrac{1}{|n|+1}$ for $n \geq 0$. Remember, now this exists for all $n$ from $-\infty$ to $\infty$. This is a two sided sequence. Your answer is correct, but I am trying to understand more as to what made you conclude that. Suppose, I had $x[n] = \dfrac{1}{|n|+1}$ for $n \geq 0$, then if I had asked you about the Z-transform, what would be the answer? Ok. We will look at similar examples later very soon and maybe you will see the reasoning that we are going to make in those cases will apply here also.

Student: (Refer Time: 21:36).

No, that is not right. So, here, the Z-transform does not exist. We will not, we will make a remark later about sequences like this. And, suppose you had a sequence like this $x[n] = \dfrac{1}{n^2+1}$, the RoC is $|z| = 1$. So, this exists only on the unit circle.

So, we will not further get into details of sequences like these. Just want to make you aware that there are cases like $x[n] = \dfrac{1}{|n|+1}$ for which Z-transform does not exist, and there are sequences like $x[n] = \dfrac{1}{n^2+1}$ in which the annular region has degenerated into a circle where $r_1 = r_2$, that is all. And, if you ask the mathematicians to define $r_1$, they will define $r_1$ like $r_1 = \overline{\lim}_{n\to\infty}\big|x[n]\big|^{(1/n)}$ and $\overline{\lim}$ stands for?

Student: lim sup.

lim sup. So, this stands for lim sup and $r_2$ will be $r_1 = \underline{\lim}_{n\to\infty}\big|x[-n]\big|^{(-1/n)}$. So, these are the definitions of the two radii and this is the Cauchy-Hadamard result.

Again, we will not use any of this. If you did not know what lim sup and lim inf were, you need not be concerned whether you will be asked to drop on that knowledge to solve problems. We are considering only the class of rational transfer functions, and they are the simplest class and moment you evaluate the poles and zeros, you can always be guaranteed that the RoC will be bounded by poles.

tudent: (Refer Time: 24:22).

Question; yes.

Student: (Refer Time: 24:24).

Very good; yeah; yes; whereas for rational transform functions, the two radii will be distinct. So, I just want you to be aware of these formula but our problems are much simpler that we will not have to use any of these.