Digital Signal Processing

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Lecture 26:
Introduction to the DTFT, Properties of the Z - transform (1)
-DTFT definition
-DTFT existence and absolute summability

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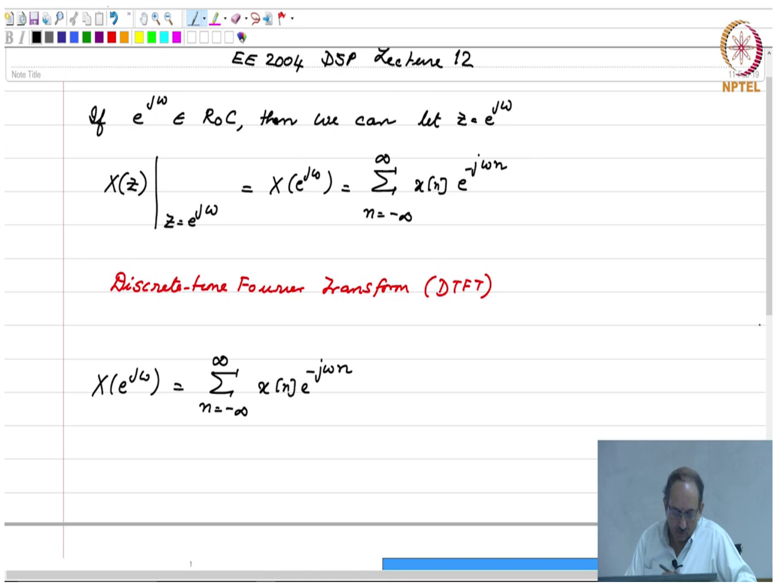

Let us get started for the day; we had looked at the properties of the region of convergence and let me recall one of the properties. One of the properties was if $e^{i\omega}$ belonged to the region of convergence, then we can let $z = e^{j\omega}$. In this case, we get $X(z)$ evaluated at $z = e^{j\omega}$ and this we denote it as $X(e^{j\omega})$ and this of course, is $\sum_{n=-\infty}^{\infty} x[n]z^{-n}$.

Now, we have let $z = e^{j\omega}$ therefore, this now becomes $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$. This was one of the properties and then if you recall, this was called as the Discrete-time Fourier Transform and abbreviated as DTFT analogous to the abbreviation we had used for the continuous-time case that was Continuous-time Fourier Transform (CTFT), this is the discrete-time counterpart of that. Before we get into more details of this, let us immediately deduce one consequence of $e^{j\omega}$ belonging to the region of convergence.

Therefore we have $X(e^{j\omega})$ to be $\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ and we can do this because the RoC contains the unit circle. Therefore, the DTFT exists and the criterion that we have been using is absolute convergence.

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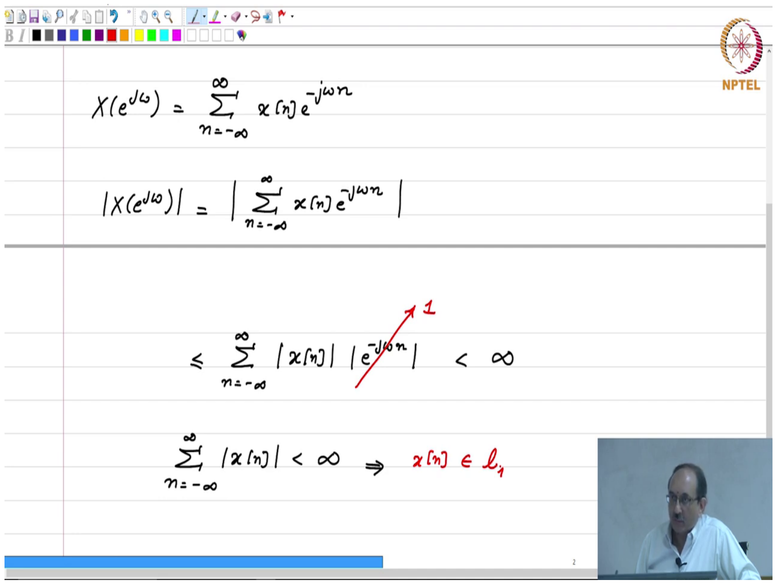

Therefore, this means that this will be finite. So, this is nothing, but $|X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \right|$ and as we had noted before the absolute value of the sum is less than or equal to sum of the absolute value. So, $|X(e^{j\omega})| \leq \sum_{n=-\infty}^{\infty} \left|x[n]e^{-j\omega n}\right|$ and absolute value of the product is the product of the absolute values. Therefore, $|X(e^{j\omega})| \leq \sum_{n=-\infty}^{\infty} \left|x[n]\right| . \left|e^{-j\omega n}\right| < \infty$ and $\left|e^{-j\omega n}\right|$ of course, is 1.

Therefore, what we have is the above implies $\sum_{n=-\infty}^{\infty} \left|x[n]\right| < \infty$. Moment you have the fact that the unit circle belongs to the region of convergence, immediately that leads to this conclusion.

So, the implication of this is that $x[n]$ belongs to the class of absolutely summable sequences. Therefore, if the region of convergence includes the unit circle, then $x[n]$ belongs to the class of l_1. l_1 is of course, the class of absolutely summable sequences and this is the equivalent of what was happening in the continuous-time case. If the $j\Omega$ axis is part of the RoC, then the function $x(t)$ is absolutely integrable and it belongs to the space of L_1, the class of absolutely integrable functions.

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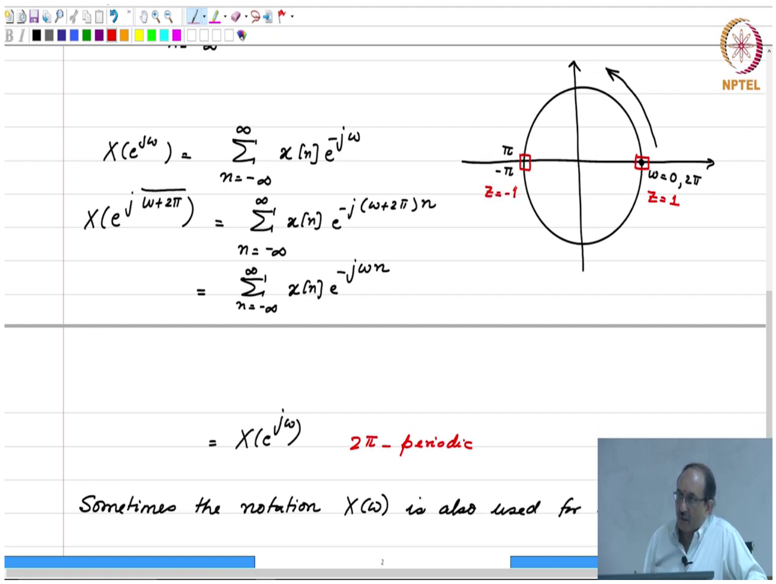

So, now let us look at the DTFT a little more. So, we have $X(e^{j\omega})$ by definition is $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$. And from this definition, if you replace ω by $\omega + 2\pi$, you get $x[n]e^{-j(\omega+2\pi)n}$. Wherever ω is there, you are going to replace ω by $\omega + 2\pi$ and this of course, is exactly the original DTFT. And, hence this is the same as $X(e^{j\omega})$. So, this is a function that is 2π periodic.

Remember, the continuous-time case, $x(t)$ had CTFT $X(e^{j\Omega})$ and that was aperiodic function in general. The continuous-time function was aperiodic; its transform also is aperiodic. Whereas, here in the discrete time case, you have in general a sequence $x[n]$ that is aperiodic, but its transform, the DTFT is periodic with period 2π. And, there is another way of seeing this. We are motivating the DTFT as the Z-transform evaluated around the unit circle assuming of course, the RoC contains the unit circle.

Therefore, this is $\omega = 0$. So, you are obtaining the DTFT by taking $X(z)$ and then you are evaluating this around the unit circle. And, you will hit π which is the same as $-\pi$ and if you complete the circuit around the unit circle, you will also hit 2π. Therefore, the periodicity of the DTFT is also evident from this geometric interpretation where you are evaluating the Z-transform around the unit circle.

Therefore, if you keep going around the unit circle, it is clearly periodic with period 2π and we saw this algebraically here and this is the geometric interpretation of that fact.

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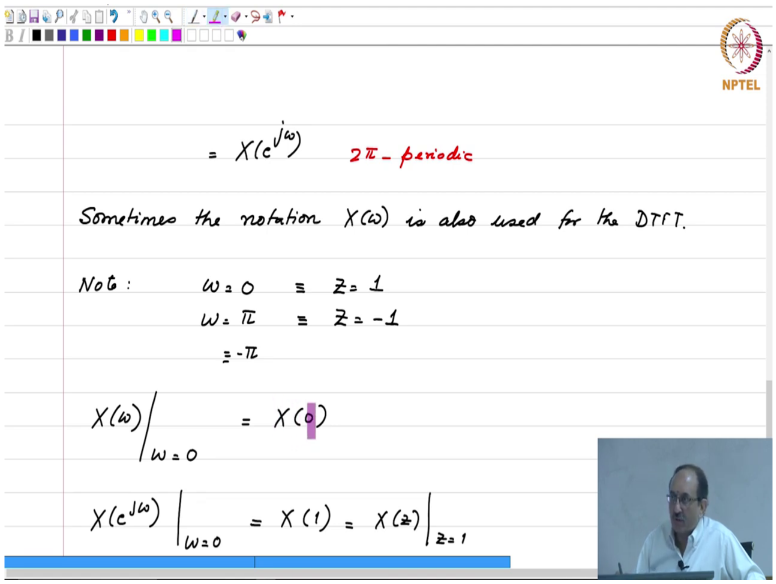

Sometimes the notation $X(\omega)$ is also used for the DTFT and note that $\omega = 0$ corresponds to $z = 1$, right. So,this is also the same as $z = 1$, this point and $\omega = \pi$ which is also the same as $\omega = -\pi$ corresponds to $z = -1$.

Therefore, this point which is $\omega = \pi$ which is the same as $\omega = -\pi$, this corresponds to $z = -1$. And, if you use the notation $X(\omega)$ for the DTFT and then if you evaluate this at $\omega = 0$, you are looking at notationally $X(\omega)\Big|_{\omega=0} = X(0)$. On the other hand, if you are using the notation $X(e^{j\omega})$ and then you are evaluating this at $\omega = 0$, notationally you are looking at $X(1)$ and this is the same as $X(z)$ evaluated at $z = 1$.

So, either notation is fine, but you have to be careful as to which notation you are using so that you are not tripped into thinking this $X(0)$ is $X(z)$ evaluated at $z = 0$, that is all. As long as you are clear as to what notation you are using, everything should be fine. The notation $X(e^{j\omega})$ has the advantage that it shows up the periodicity of the DTFT explicitly, because wherever ω is there, if you replace ω by $\omega + 2\pi$, you get exactly the same thing.

Therefore, the 2π periodicity of the DTFT is evident from $X(e^{j\omega})$ notation. And, if you use this particular one $X(\omega)$, this is also used in some context; sometimes it is inconvenient to write $X(e^{j\omega})$. So, even in the later development, we will use $X(\omega)$ notation when it is more convenient to write it out. And, all of this parallels what was happening in the continuous-time Fourier Transform case; you had the bilateral Laplace transform $X(s)$ and then if you replace s by $j\Omega$, you get $X(j\Omega)$ which was the notation used for CTFT. You also use the notation $X(\Omega)$ by itself to denote the CTFT. The $X(j\Omega)$ notation had the advantage that, it made explicit that it was obtained from the bilateral Laplace evaluating that along the $j\Omega$ axis.

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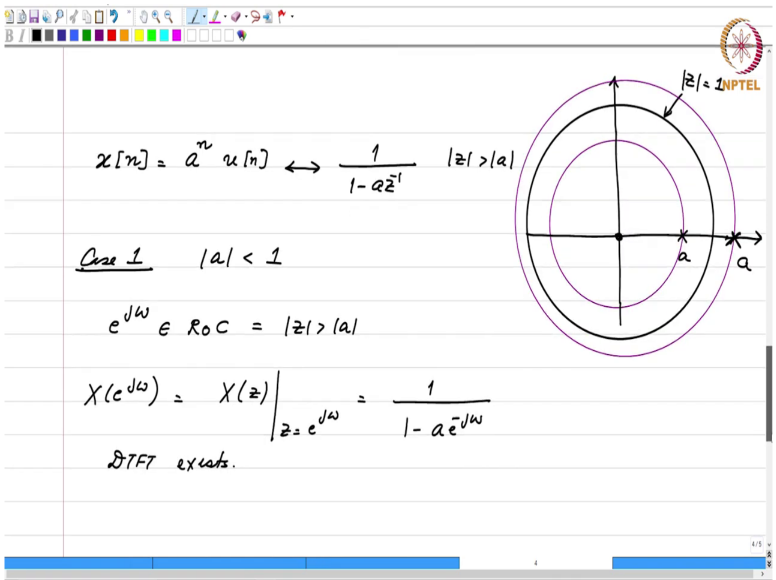

Now, let us look at $x[n] = a^n u[n]$ and we have seen this any number of times before and we will continue to beat this example to depth by the time we are done with this course. So, this is $\frac{1}{1 - az^{-1}}$. So, this is $|z| > |a|$ and then for simplicity, I am showing the pole to be real valued and positive. So, this is $z = a$, of course, there is one trivial zero at 0 and this is the circle $|z| = |a|$, all right.

Now, suppose we have $|a| < 1$, then this will be the unit circle. So, this is the unit circle here and the RoC is $|z| > |a|$ and hence $e^{j\omega}$ belongs to the RoC which is $|z| > |a|$. And, hence you can get $X(e^{j\omega})$ by wherever z is there, you can replace z by $e^{j\omega}$ and hence you will get this to be $X(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$.

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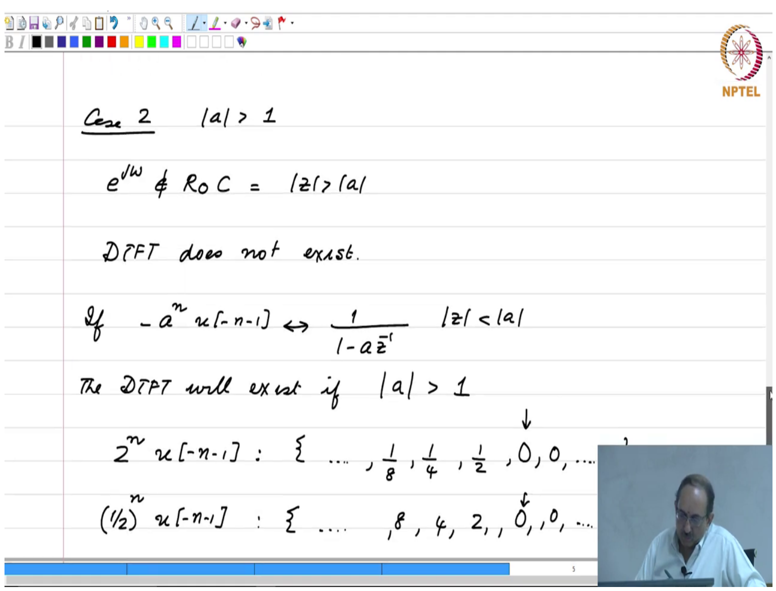

Therefore, in this case, the DTFT exists and you can immediately guess what the other case will be. The other case is $|a| > 1$. Therefore, if you now take this case and then make it like this. Now, in this case, this is the pole which is outside the unit circle and $e^{j\omega}$ now does not belong to the RoC which is $|z| > |a|$ and then this tells you that the DTFT does not exist.

Suppose, $|a| > 1$ and in this particular case, the DTFT does not exist because the ROC does not contain the unit circle, suppose you want the DTFT to exist in this case, what change is required?

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Yeah, exactly. So, in this case, if you want the DTFT to exist, you want the RoC to be $|z| < |a|$, right. Therefore, if we have $-a^n u[-n-1]$ and here the Z-transform of course is $\frac{1}{1-az^{-1}}$, the DTFT will exist if $|a| > 1$. Because, if $|a| > 1$, the pole will lie outside the unit circle and the RoC will be $|z| < |a|$. And, in this case, the unit circle will indeed belong to the RoC.

So, moment the unit circle belongs to a RoC, the DTFT will exist. Therefore, if you had an anti-causal exponential and if you want the DTFT to exist, the pole should be outside the unit circle whereas, if the exponential were causal, you want the pole to be strictly inside the unit circle so that the RoC will now contain the unit circle. And again, if you look at the sequence, for example, if you looked at $2^n u[-n-1]$, right. So, this sequence is, if this were the origin, all positive indices are 0.

Now, you need to start substituting values of n that are negative. Therefore, when $n = -1$, this is 1/2; when $n = -2$, this is 1/4, 1/8 and so on. So, this is how the sequence is, all right. And, clearly this belongs to l_1.

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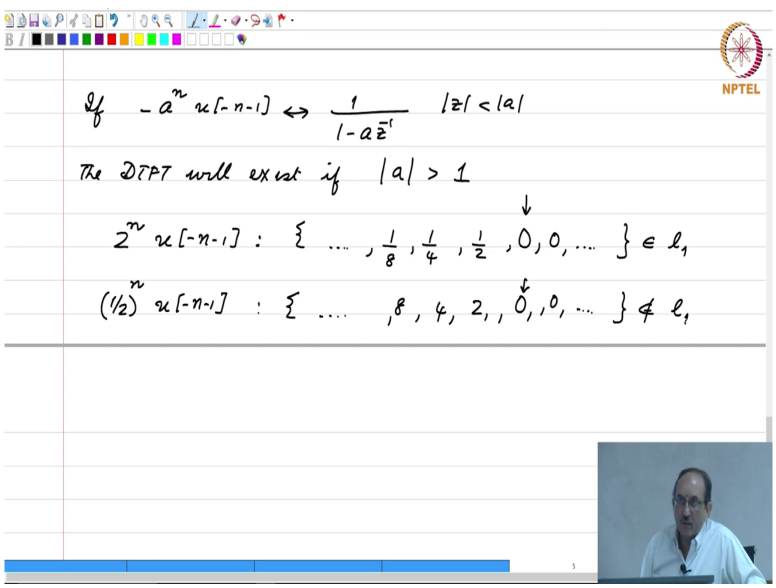

On the other hand, if you had $\left(\frac{1}{2}\right)^n u[-n-1]$, when you put $n = -1$, this becomes 2, 4, 8 and so on. So, clearly this is not an absolutely summable sequence. So, this is what is basically happening.

So, if the pole is outside the unit circle, the sequence is anti-causal. When you put negative values of n, the sequence really will die down and become absolutely summable, as simple as that. You can also have an intuition as to why the DTFT exists in this case for cases where the pole is outside for anti-causal sequences.

So, we will revisit the DTFT later.