Digital Signal Processing

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Lecture 32: Properties of Z-transform (3) -Differentiation in the Z-domain

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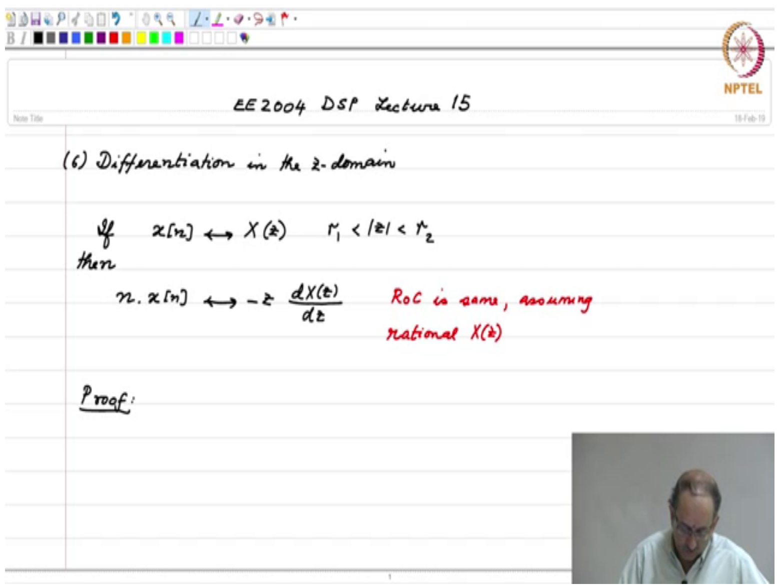

Let us get started. We are looking at Z-transform Properties. We have seen things like linearity, time delay, modulation and then time reversal and so on. So, we are now going to look at the next one. So, this is differentiation in the Z-domain. So, if $x[n]$ has Z-transform $X(z)$ with this RoC $r_1 < |z| < r_2$ then, $n.x[n] \longleftrightarrow -z\dfrac{dX(z)}{dz}$ and RoC is same assuming rational $X(z)$, which is the class that we are interested in. The proof is pretty straightforward.

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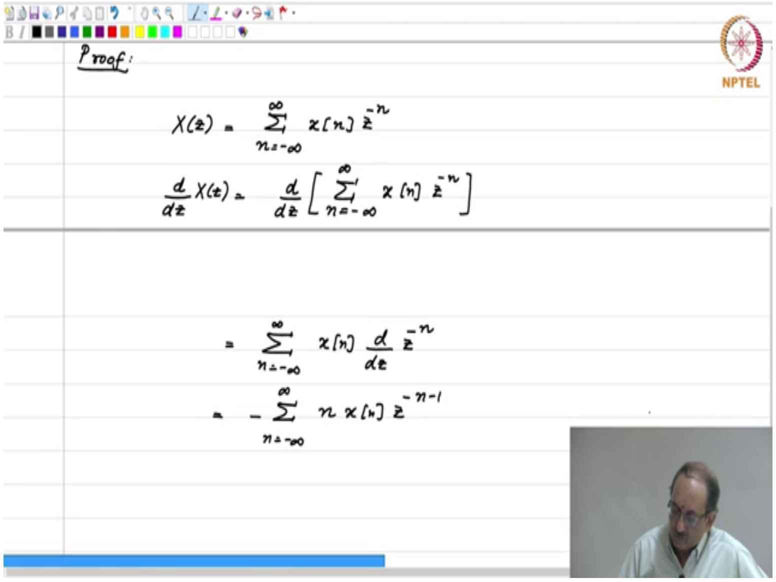

So, we have $X(z)$ by definition, $X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$ and then let us differentiate with respect to z. So, this becomes $\frac{d(\ )}{dz}$ of this quantity. And, usually when we reach this step, we happily interchange the two operations. Remember that $\frac{d(\ )}{dz}$ is a limiting operation. This also is a limiting operations and you cannot in general interchange these two operations.

However, if you had had in mathematics some discussion about a series, if the series is absolutely convergent then in those cases you can do term by term differentiation and term by term integration. So, here because we assume that the Z-transform does exist and because our criterion is absolute convergence, there are no issues taking the $\frac{d(\ )}{dz}$ inside. So, this is $\sum_{n=-\infty}^{\infty} x[n]\frac{d}{dz}(z^{-n})$ and this of course, is $-\sum_{n=-\infty}^{\infty} nx[n]z^{-n-1}$.

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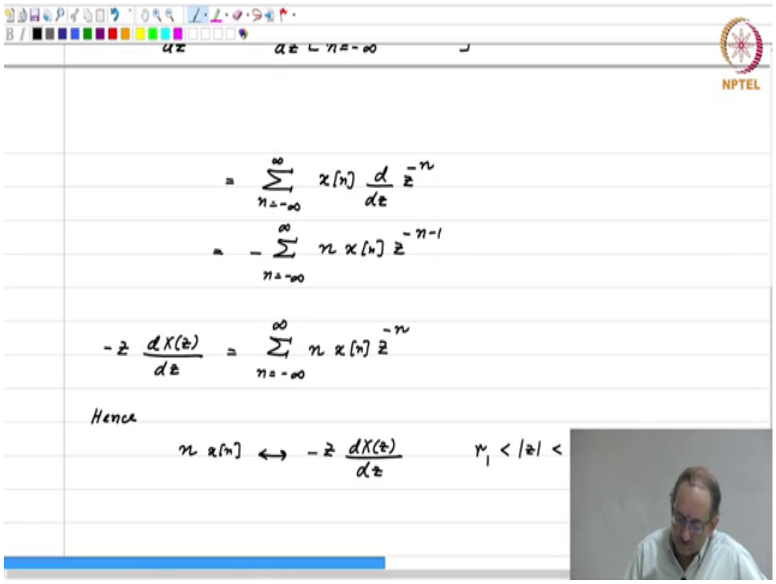

And, then if we multiply by $-z$ on both sides so, this becomes $-z\dfrac{dX(z)}{dz} \longleftrightarrow \displaystyle\sum_{n=-\infty}^{\infty} nx[n]z^{-n}$. And, this is precisely in the form of a Z-transform, except that instead of $x[n]$, we have $n.x[n]$. Therefore, this has to be the corresponding Z-transform. Therefore, we get $nx[n]$ having this as the Z-transform. And RoC, when you have rational $X(z)$, the RoC does not change. And, we will apply this to this particular example, we will start off with our usual $a^n u[n]$.

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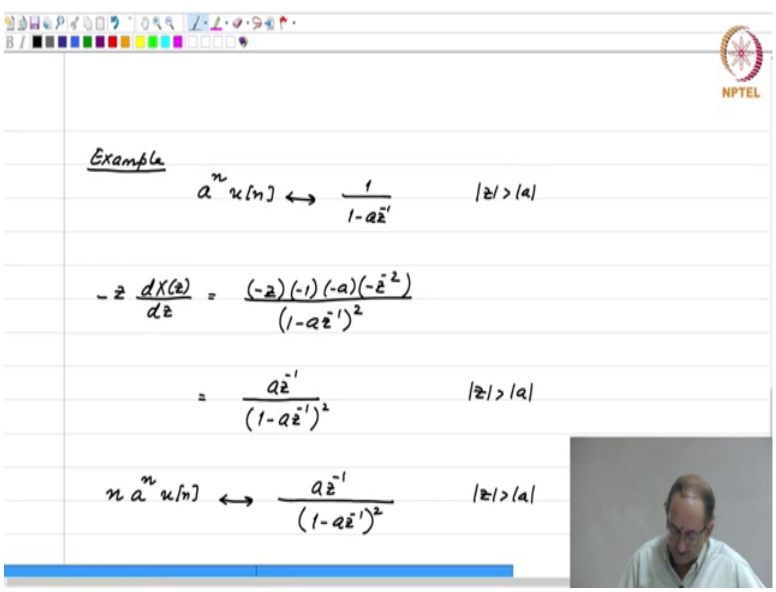

So, this is $a^n u[n] \longleftrightarrow \dfrac{1}{1-az^{-1}}$, $|z| > |a|$. And, then let us look at $-z\dfrac{dX(z)}{dz}$. So, this is $-z$. So, you need to differentiate this. So, this has -1, $\dfrac{1}{(1-az^{-1})^2}$. So, that becomes this, then you need to differentiate this and this becomes $(-z^{-2})$. All I have done is just simple differentiation. Therefore, this now becomes $\dfrac{az^{-1}}{(1-az^{-1})^2}$.

So, this is of course, is $|z| > |a|$ and this has to be the transform of $n.x[n]$. So, you have $na^n u[n]$. So, this has transformed $\dfrac{az^{-1}}{(1-az^{-1})^2}$. We need to get rid of this z^{-1}, for that we multiply by z. If you multiply the transform domain by z, you will replace wherever n is there n by $n+1$.

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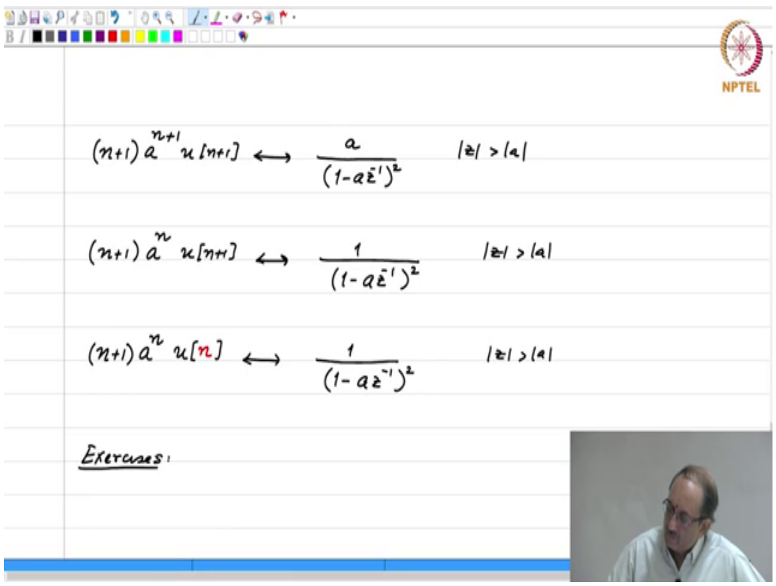

Therefore, this becomes $(n+1)a^{n+1}u[n+1]$ and this will be the transform of $\dfrac{a}{(1-az^{-1})^2}$. And, you can cancel one power of a on both sides. So, this becomes $(n+1)a^n u[n+1]$. This has transformed $\dfrac{1}{(1-az^{-1})^2}$, $|z| > |a|$. And, this in turn is $(n+1)a^n u[\ ]$, we will fill that in a minute. This can be written as $(n+1)a^n u[n]$ because?

Student: (Refer Time: 07:58).

Yes, very good. So, at $n = -1$, this is 0. Therefore, this starts off at $n = 0$ and to reflect that you have the expression that is usually written in this form. So, as exercises, $\dfrac{1}{(1-az^{-1})^2}$, the region of convergence is now $|z| < |a|$.

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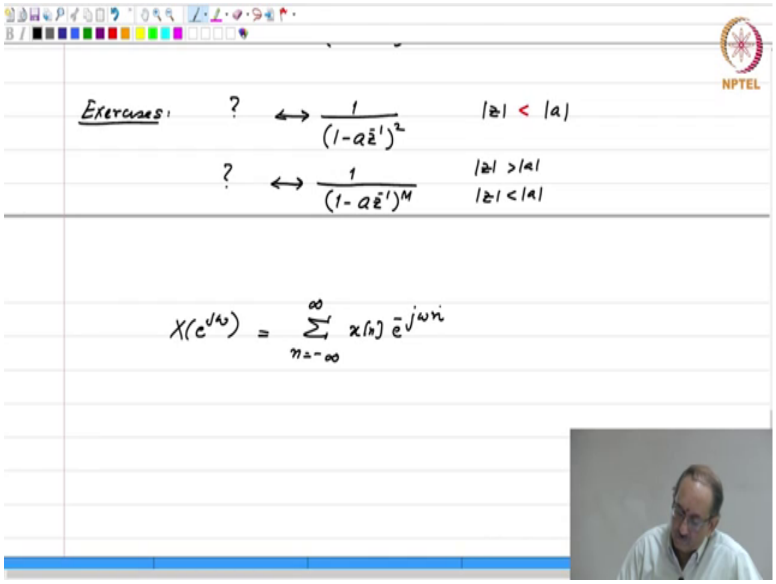

So, you need to figure this out. And, then $\dfrac{1}{(1-az^{-1})^M}$. Again two cases here, $|z|>|a|$ and $|z|<|a|$. For each of these cases, it is important that you derive this. And, the way you do this is, you apply the differentiation property to $\dfrac{1}{(1-az^{-1})^2}$. You apply the differentiation property to this transform.

One thing that you must have learnt about the transform in the Laplace case, it is a complex function of a complex variable. What is one important property that this Laplace possesses, the same property is true for Z-transform. The Laplace transform is an analytic function, which means it satisfies the Cauchy Riemann equations. Is this familiar to you? That the function is analytic, it will satisfy the Cauchy Riemann equations, CR equations, is this? No? Its not being taught, ok. If the function is analytic, in the region of convergence, this is infinitely differentiable. Therefore, you can apply this any number of times.

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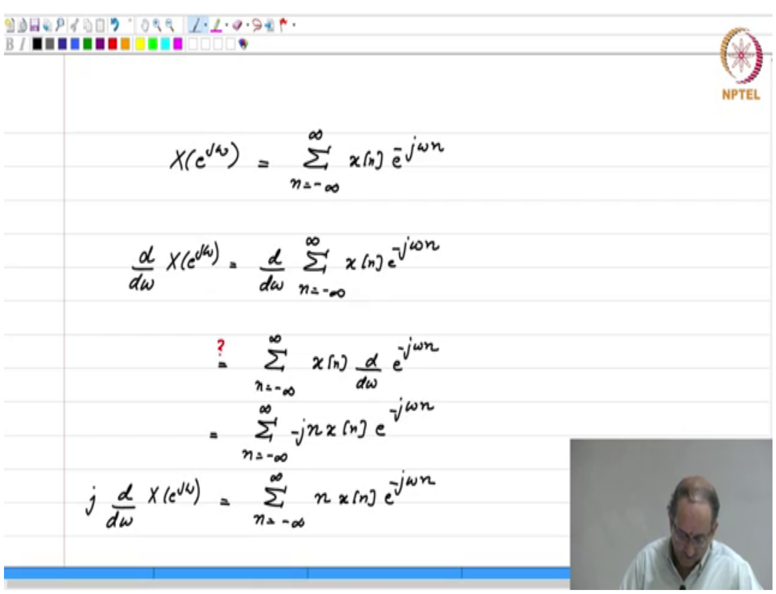

Let us look at the corresponding property for the DTFT. And now what we do is, we differentiate with respect to ω, because the independent variable is ω. Therefore, this is $\frac{d}{d\omega}X(e^{j\omega})$. Now, how about taking $\frac{d}{d\omega}$ inside? Is that ok? Or earlier, we did this right so, can we repeat what we did earlier?

Student: (Refer Time: 11:40).

If the RoC contains the unit circle then, it is absolutely convergent there and then you can. Student: (Refer Time: 11:46).

So, you in DTFT, you do assume absolute convergence?

Student: (Refer Time: 11:56).

o, that is what I am saying, once you say the DTFT exists, you are saying the sequence is absolutely summable, right. Suppose, we have a sequence like $x[n] = 1$, which is the DC sequence and later we will see that it has DTFT. So, this is analogous to $x(t) = 1$ having continuous and Fourier transform $2\pi\delta(\Omega)$.

So, clearly $x(t)$ is not absolutely integrable in that case and the sequence $x[n] = 1$ is not absolutely summable. Therefore, to satisfy the mathematician, we will put a question mark here and then you will take the derivative inside. But, then we will remind ourselves that some conditions have to be satisfied for this to be true. Therefore, this is $\sum_{n=-\infty}^{\infty} x[n]\frac{d}{d\omega}e^{-j\omega n}$ and therefore, this becomes $\sum_{n=-\infty}^{\infty} -jnx[n]e^{-j\omega n}$. And, hence $-j$ and you are right and therefore, if you take j to the other side, $j\frac{d}{d\omega}X(e^{j\omega})$ has this as the DTFT and hence the corresponding DTFT property is $nx[n] \longleftrightarrow j\frac{d}{d\omega}X(e^{j\omega})$, all right. The only wrinkle here is the interchanging of these two limiting operations where for the DTFT, we do not necessarily assume absolute sum ability.

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