Digital Signal Processing
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Lecture 44:
DTFT of Sequences not in l_1, Response to $\cos(\omega_0 n + \Phi)$
-DTFT of sequences that are not absolutely summable

So, let us continue with looking at inverse DTFT formula and some of the consequences. So, we are looking at some sequence and transform pairs which were not part of our attention or other things are not possible when you consider the Z-transform of subsequence.

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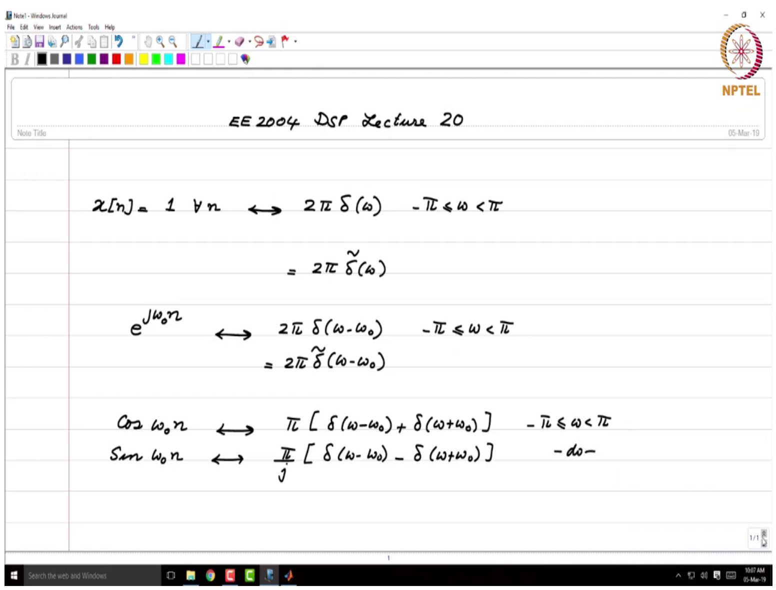

And one of the sequence was this and we can also rewrite this as $2\pi\tilde{\delta}(\omega)$, where if you recall we mentioned that the tilde notation is used for denoting things that are periodic. So, the DC sequence has an impulse train as the transform. And note that $x[n] = 1$, for all n does not possess Z-transform.

And this is analogous to $x(t) = 1$ having continuous time Fourier transform $2\pi\delta(\Omega)$ and $x(t) = 1$ for all t does not possess bilateral Laplace. And moment you have this, immediately you can get the transform of $e^{j\omega_0 n}$ using the modulation property and this of course, is $2\pi\delta(\omega - \omega_0)$.

And again, if you use the delta notation, you have to specify the range and this is the same as $2\pi\tilde{\delta}(\omega-\omega_0)$. And we had already seen the result here, when we are looking at time shift and when you are pointing out how the phase shift introduced in the transform domain for the DTFT can be viewed as a rotation in the complex plane, in that context we had seen this.

And now, we are actually formally deriving this relationship, this of course is $\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$. All you need to do is, you need to take $e^{j\omega_0 n}$, the transform of that and when you express $\cos(\omega_0)$ as $\frac{e^{j(\,)} + e^{-j(\,)}}{2}$, you will get the final answer. And similarly, $\sin(\omega_0 n) = \frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$. So, yes, question.

Student: (Refer Time: 03:35).

Yes. So, that is understood because then we were looking at $e^{j\omega_0 n}$, when you first introduced that we notice that if you replace omega naught by $\omega_0 + 2\pi$, you do not change anything. Therefore, you will get the same signal, if you view ω_0 as ω_0 mod 2π, right. So, in this context just to make sure I understand where this is coming from.

No additional thing need be said here, because even if ω_0 were not in the range 0 to 2π, it were outside this range, is the question then will not this impulse be located at that particular frequency. Is that the question that is it? Was that was the one that was making you ask? Then immediately, the periodicity will generate a complete train. Moment you realize that it has to be periodic even if ω_0 were not between 0 to 2π, if it were some other value, the impulse would indeed be located at that particular value, but then periodicity will again make an impulse show up in the range $-\pi$ to π, which will be equivalent to replacing ω_0 which is outside 0 to 2π in the range 0 to 2π. If you take mod, everything will be consistent, ok.

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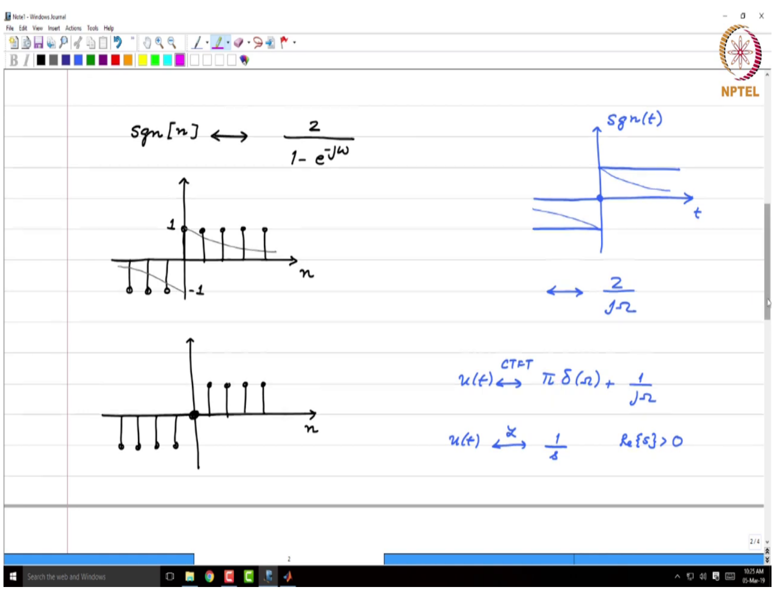

And now similar to what was done in the continuous time Fourier transform case, we will look at this particular sequence, the sign function, the signum function and we will show that this transform is $\frac{2}{1 - e^{-j\omega}}$. And the way in this sign function is defined is like this. So, it is 1 for $n \geq 0$, and -1 for $n < 0$. So, this is n, so this is 1, and this is -1. And the reason I am mentioning this, is in the continuous-time case, you had $sgn(t)$ and its definition was like this; so, this is t, this is $sgn(t)$.

And since this is an odd function, you can go ahead and define the value at $t = 0$ to be 0. So, this is 1 for $t > 0$ and -1 for $t < 0$. Whereas, the way this is defined here for the discrete-time case, at $n = 0$ is defined as 1. You can also define another variant of this, you can define the sign function to be like this, in the discrete-time case and then you can at the origin define the sign function to have the value 0. So, this now seems to be the counterpart for *sgn* has defined here. The continuous-time signum function that odd symmetric function, that counterpart can be thought of to be this.

But what we are going to do now is we are going to derive the transform of this. And once you derive the transform of this, it is easy to derive the transform of this with very simple manipulations. So, now, what we are going to do is, we are going to do exactly what was done when you derived the continuous time Fourier transform of $sgn(t)$. What did you do here? In $sgn(t)$, what did you do? No. What was it? That was the sequence of steps that were used to derive the CTFT of $sgn(t)$. So, the each one of you is saying a different answer, right. This is something that must you must have seen. If you had seen this, then all of you must be saying the same thing, roughly the same thing, whereas, you are giving completely different approaches here.

So, what was done was you have replaced this by this, and this of course is $e^{-at}u(t)$, where a is between 0 and 1 and this was $-e^{-at}u(-t)$ and then you let limit as a tending to 0. So, that $e^{-at}u(t)$ as a tended to 0 would tend to $u(t)$, all right. So, what we are going to do in the discrete-time case is exactly that. So, we are going to, I am going to draw the continuous-time envelope, but what I really mean is the discrete-time sequence whose envelope is this. So, I am going to do something like this and then I am going to take the limit of a parameter going to 1, all right. So, you will see that it all makes this once I write down the equation.

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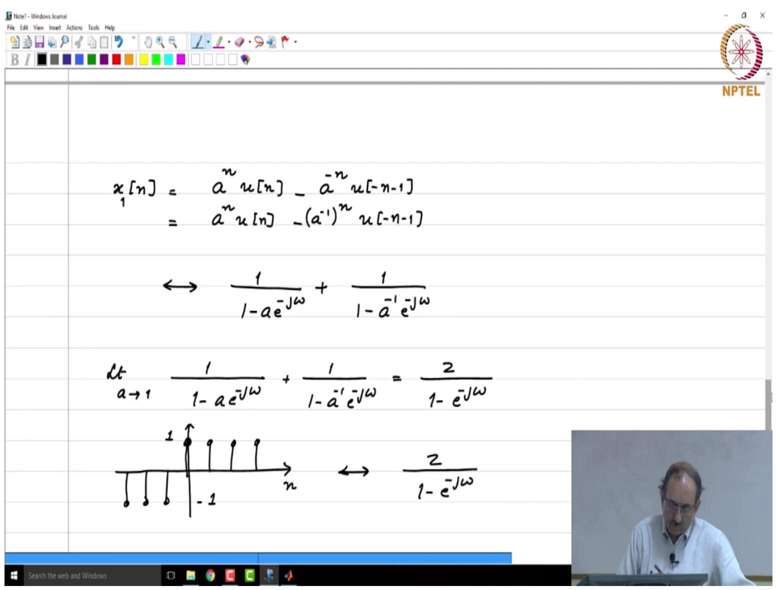

So, I will write $x_1[n]$ as $a^n u[n]$, where a is between 0 and 1. So, this would correspond to the exponential for $n \geq 0$ part, then for the negative indices, I am going to replace this with $-a^{-1}u[-n-1]$. So, this is indeed this sequence whose envelope I have drawn here. And, then if I now let a tend to 1, then I will get the *sgn* sequence as defined in this case.

So, this of course is $a^n u[n] - a^{-1}u[-n-1]$. And now we know the transform. So, this is $\dfrac{1}{1-ae^{-j\omega}} + \dfrac{1}{1-a^{-1}e^{-j\omega}}$. And now, we will take limit as a tending to 1. So, if you let a tend to 1 in each of these terms, you will get, this will tend to $\dfrac{1}{1-e^{-j\omega}}$, this will tend to again exactly the same term, $\dfrac{1}{1-e^{-j\omega}}$.

Therefore, in the limit $\dfrac{1}{1-ae^{-j\omega}}+\dfrac{1}{1-a^{-1}e^{-j\omega}}$. This of course, will be $\dfrac{2}{1-e^{-j\omega}}$. And hence, sgn as defined like this, so this is 1, this is -1, this is n, therefore, this sequence has transformed $\dfrac{2}{1-e^{-j\omega}}$. And, if you recall $sgn(t)$, its transform is continuous time Fourier transform of the sign function as shown here, its CTFT is? Yes.

Student: (Refer Time: 14:32).

You can almost guess right based on this picture. Very good; it is $\dfrac{1}{j\Omega}$. This after all is $\dfrac{1}{s+a}$ and this also you can, it will be $\dfrac{1}{s-a}$, right and then you will take limit as a tending to 0. Therefore, it will be 2 by rather than s, you would have worked this out in terms of the Fourier transform, it will be $\dfrac{1}{j\Omega+a}$ and then you are going to let a going to 0, so you will get two terms that each of them will be tending to $\dfrac{1}{j\Omega}$ therefore, you will get $\dfrac{2}{j\Omega}$ should remind you of this. And, the reason why you did $sgn(t)$ was because the very next function that you wanted to obtain the transform of was $u(t)$, right. So, we will follow exactly the same pattern here.

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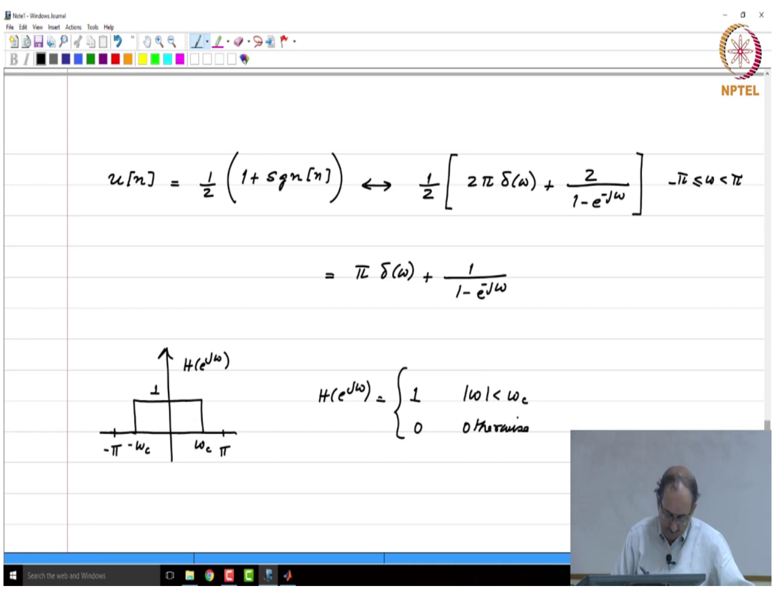

So, now we will look at $u[n]$. So, this is nothing but $1+sgn[n]$. If we did $1+sgn[n]$, this will be $2u[n]$, therefore, you really want half of this. Therefore, this now becomes $\frac{1}{2}(1+sgn[n])$ and the transform of 1 is $2\pi\delta(\omega)+(\ )$ this of course is $\dfrac{2}{1-e^{-j\omega}}$. And because $\delta(\omega)$ sitting here, we have to qualify this by specifying the range or if we had replaced $\delta(\omega)$ by $\tilde{\delta}(\omega)$ we need not have written this. So, this immediately becomes $\pi\delta(\omega)+\dfrac{1}{1-e^{-j\omega}}$.

And, if you recall the continuous time Fourier transform of $u(t)$ was $\pi\delta(\Omega)+\dfrac{1}{j\Omega}$, you can almost immediately see this because you can write $u(t)$ as $\dfrac{1}{2}(1+sgn(t))$. So, now, you see the similarity between that and this. So, now, you have the Fourier transform of $u[n]$. Note that you could not have obtained Fourier transform of $u[n]$ from the Z-transform of $u[n]$, because no, you cannot replace z by $e^{j\omega}$ because.

Student: RoC.

The RoC does not contain the unit circle. So, exactly the same thing that was happening here in the continuous-time case, $\frac{1}{s}$ and the region of convergence is? Student: (Refer Time: 18:44).

$Real\{s\} > 0$ therefore, you could not take the Laplace transform of $\frac{1}{s}$ and replace s by $j\Omega$, because the region of convergence does not contain the $j\Omega$ axis. Similar thing is going on here. Yes.

Student: (Refer Time: 19:00).

Yeah. So, that is why we are going through this route.

Student: (Refer Time: 19:11).

So, this particular thing, what is happening here is now, you have impulses showing up for $u(t)$, you have this whereas, if you look at the bilateral this is the CTFT whereas, $u(t)$ you consider just the Laplace, so this is $\frac{1}{s}$ and this of course is realistically greater than 0. If you replace s by $j\Omega$, you will only get this term, you will be missing this impulse.

So, for more discussion on this, you can look up Professor V G K Murthi's lecture, Networks and Systems on NPTEL. So, he has one specific lecture devoted to this where he takes a Laplace transform and then shows why for certain functions, you cannot replace s by $j\Omega$ and then for these cases, impulses will show up in the transform domain. And, it will be very beneficial to look at that particular lecture so, Networks and Systems by Professor V G K Murthi, NPTEL.

Now, since we have the inverse DTFT definition, so this is $H(e^{j\omega})$ and this is 1, so $H(e^{j\omega})$ is 1 for $|\omega < \omega_c$ and 0 otherwise. By otherwise I mean in the interval $-\pi$ to π because this after all is periodic with period 2π. So, this is your ideal low pass filter. And it is very easy to obtain the impulse response. So, this is $h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$.

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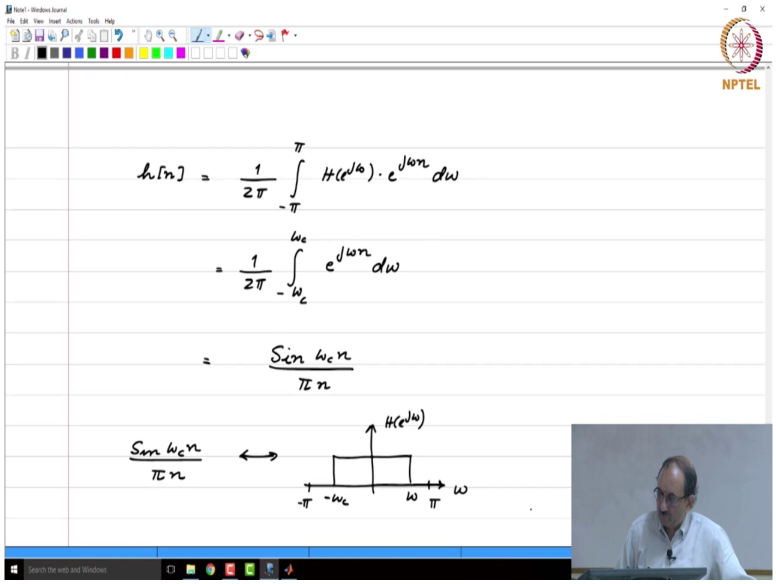

And in this particular case, this simplifies to $\dfrac{1}{2\pi}\displaystyle\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega$ and this is immediately seen to be $\dfrac{\sin(\omega_c n)}{\pi n}$. And if you recall, we had used this sequence when we were demonstrating the use of one of the properties that is if you sum up the sequence over all n, it is exactly the same as the value of the transform evaluated at $\omega = 0$, right. And if you look at this case, the value of the transform at $\omega = 0$ is 1. Therefore, this impulse response sum to the power all n gives you 1. In that context, we have seen this particular example; and we had also used this example in Parseval's relationship.

So, this transform pair was just mentioned without proof and now we are seeing the proof here. Therefore, $\dfrac{\sin(\omega_c n)}{\pi n}$ is your ideal low pass filter with cut off ω_c. By the way, this impulse response, if you the sum up all our n of this is 1, which is the illustration of the property that is sum of power time domain sequence, you evaluate the Fourier transform at $\omega = 0$. So, that is clear. But, what about this being the impulse response? What about sum over all n of magnitude of $h[n]$? Very good. Do others see this as well?

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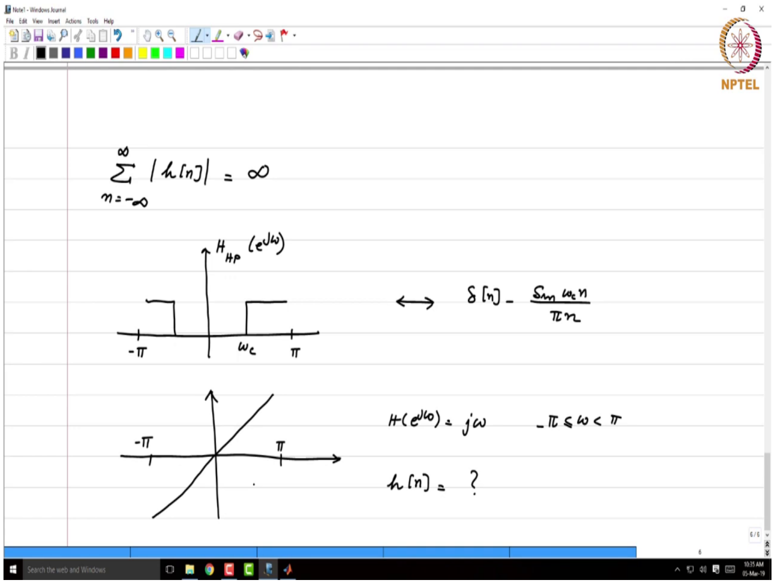

So, for this particular case, $h[n] = \frac{\sin(\omega_c n)}{\pi n}$, this value is; yeah, this is a basic result that you should have known. So, this is what push the value. He said the right answer, but; oh so you arrived at the answer by thinking it was constant times $1/n$, then I am not sure the proof that you are outlining is fully correct. You need to look this up. Anyway so, this is in case you have not seen this, this is not absolutely summable. So, this is not absolutely summable, therefore, the ideal low pass filter is not a BIBO stable system, all right, and this is no different from what is happening in the continuous-time case.

The ideal low pass filter in the continuous time case, its impulse response was? Just replace ω_c by Ω_c and then instead of the highest frequency being π, the highest frequency will be ∞. Therefore, if you had the ideal low pass filter in the continuous-time case, its impulse response was what? Should be $\frac{\sin(\Omega_c t)}{\pi t}$, all right. The ideal low pass filters impulse response in the continuous-time case is $\frac{\sin(\Omega_c t)}{\pi t}$ and that impulse response is not absolutely intergrable. Therefore, even in the continuous-time case, the ideal LPF is not a BIBO stable system. So, this is the counterpart of that.

So, just I am making this observation in this context; that is all. And once you have the impulse response of the ideal low pass filter, its counterpart, the ideal high pass filter with the same cut off. So, this impulse response is what? Very good. You can think of this ideal high pass filter as 1 minus the frequency respond to the low pass filter. Therefore, if the frequency response is 1 minus the frequency response of the low pass filter, the corresponding impulse response will be $\delta[n] - \frac{\sin(\Omega_c t)}{\pi t}$, that is all. So, this is let me use the suffix, H_{HP}, so this is to indicate it is high pass. And if I want it to be consistent I should have written this as $H_{LP}(e^{j\omega})$. And as an exercise, I want you to find the impulse response of an ideal differentiator.

In the continuous-time case, the ideal differentiator the frequency response is, the frequency response of an ideal differentiator in the continuous-time case is $j\Omega$, right. So, there it is j omega over all of Ω, all of Ω means Ω going from $-\infty$ to $+\infty$. Here in the discrete-time case, the frequency response of the ideal differentiator is still $H(e^{j\omega}) = j\omega$, except now it is restricted to the range $-\pi$ to $+\pi$. Therefore, so, this is your ideal differentiator's frequency response, so this is $H(e^{j\omega})$ being $j\omega$ between $-\pi$ and $+\pi$.

So, I want you to find $h[n]$. So, it is a very simple integral, you have to use udv formula.

Student: (Refer Time: 29:44).

Ok; so, for example, if you, $cos(\omega_0 n)$ to a differentiator, what do you expect? So, it is in that sense. The other way of doing this which will require some thinking is, you can use the derivative property. What I want you to do is get $h[n]$ using the straightforward approach, it will be after all $\frac{1}{2\pi}\int_{-\pi}^{\pi} j\omega e^{j\omega n} d\omega$, because $H(e^{j\omega}) = j\omega$ and that you can very easily derive using the udv rule. So, you will know what the final answer is as far as the impulse response goes. See if you are able to get the same answer using the differentiation property. It will require some thinking.