Digital Signal Processing

Prof. C.S. Ramalingam
Department Electrical Engineering
Indian Institute of Technology, Madras

Lecture 46: Causality and Stability, Response to Suddenly Applied Inputs, Frequency Response (1) - Causality and Stability

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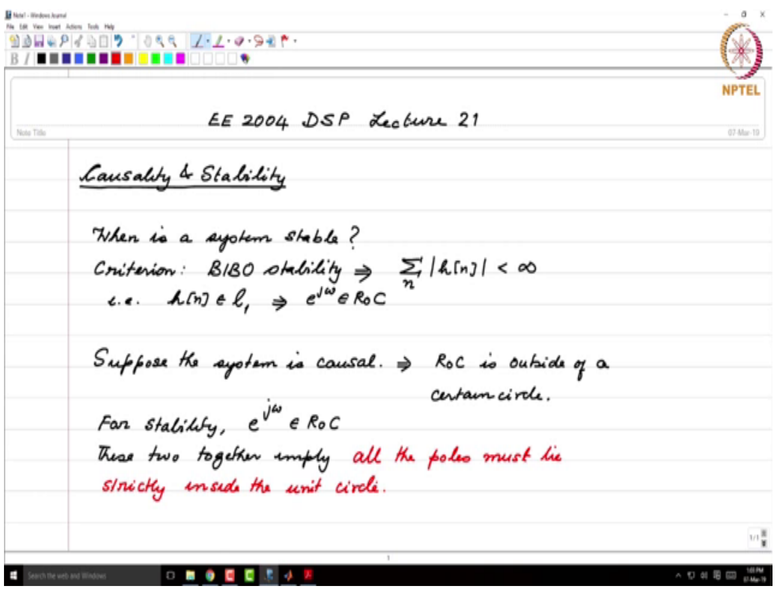

Causality and stability: after we are done looking at causality and stability, we will look at response to suddenly applied inputs. So, the question that we are interested is when is a system stable? To answer this question, we know that the criterion that we have been using is BIBO stability. So, the criterion for stability is BIBO stability. This implies that you need the impulse response summed up over all n to be less than ∞. So, that is $h[n]$ must belong to the class of l_1. We have already seen that this implies $e^{j\omega}$ must belong to the region of convergence. Now, typically systems in practice are causal.

So, suppose the system is causal so, this implies that RoC is outside of a certain circle. For stability, we require that $e^{j\omega}$ must belong to the RoC and then if you put these two facts together namely RoC is outside of a certain circle when the system is causal and for stability, this RoC must contain the unit circle. So, these two together imply all the poles must lie where, strictly inside the unit circle must lie strictly inside the unit circle. So, all the poles must lie strictly inside the unit circle.

So, this is analogous to what was happening in the continuous-time case. There if the signal is right sided, then the RoC is to the right of a certain vertical line and if the system is stable, the impulse response must be absolutely integrable; which means the $j\Omega$ axis must be part of the region of convergence. Therefore, if you combine those two facts together, namely RoC's to the right of a certain vertical line and the RoC must contain the $j\Omega$ axis for stability, then you infer that all poles must lie strictly in the left half plane.

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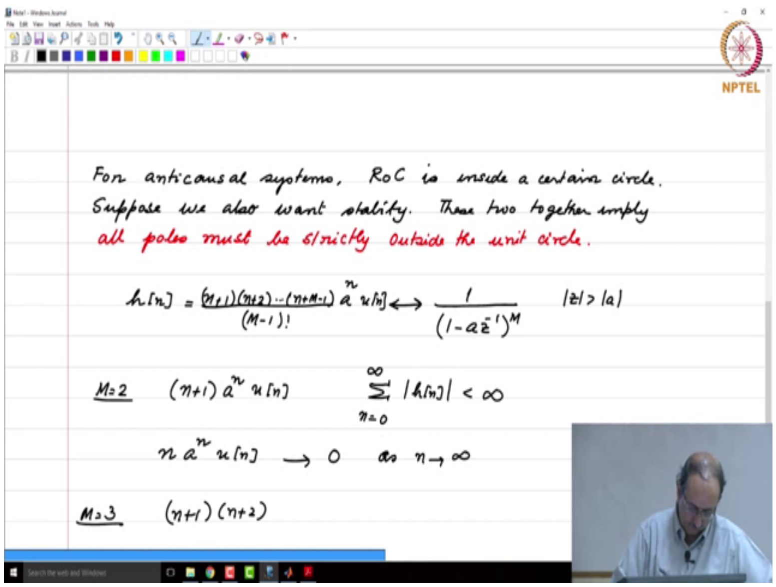

For anti-causal systems what can you say about the RoC? RoC is inside a certain circle, right and now suppose we also want stability, then these two together imply all poles must be strictly where, outside the unit circle. So, again the counterpart for this is, in the continuous-time case, if the system is anti causal in terms of its impulse response, then RoC is to the left of a certain vertical line and you want the j$j\Omega$ axis to be part of the region of convergence for stability. Putting these two facts together, you conclude that all poles must lie strictly in the right half plane.

Therefore, when you say for stability, poles must lie in the left half plane in general, what you are also assuming is causality. So, similarly here in the discrete-time case, typically we say that for stable systems, you need all the poles to lie inside the unit circle. The underlying assumption is causality. And, in terms of discrete-time systems, suppose you had $\frac{1}{(1-az^{-1})^M}$, $|z| > |a|$, then if you recall this is nothing, but $\frac{(n+1)(n+2)\ldots(n+M-1)}{(M-1)!}a^n u[n]$. This is what the impulse response is for an n^{th} order pole.

Let us consider the case where $M = 2$ and 3 just to get a feel. So, if $M = 2$, this will be $(n+1)a^n u[n]$ and this system is stable if $|a| < 1$ therefore, this is less than ∞ where $h[n]$ corresponds to $(n+1)a^n u[n]$. So, we are looking at the behavior of $na^n u[n]$ and this decays to 0 as n tends to ∞. And, if you had $M = 3$, you have $\frac{(n+1)(n+2)}{2!}a^n u[n]$.

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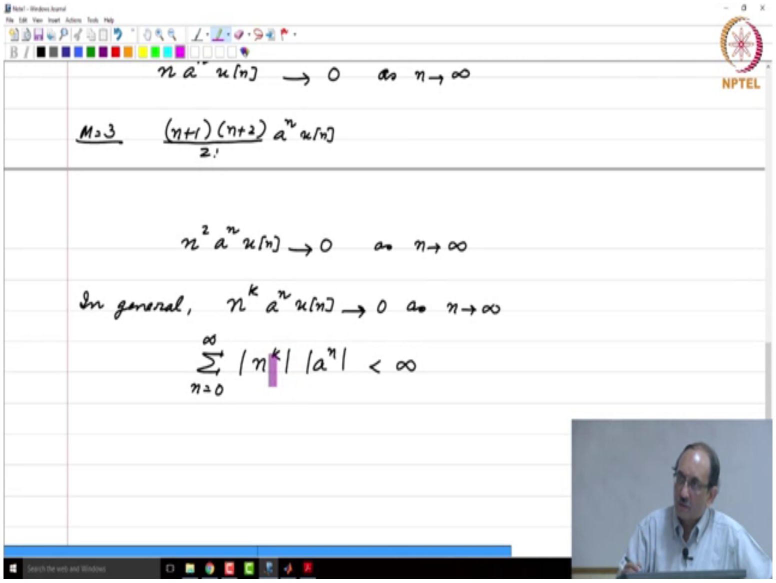

And you will be looking at in terms of time domain behavior, you will be looking at the behavior of $n^2a^nu[n]$, because if you multiply this out this will be $n^2 + 3n + 2$. So, if you look at $n^2a^nu[n]$ again this tends to 0 as n tends to ∞. So, in general $n^ka^nu[n]$ tends to 0 as n tends to ∞ and this also is true. So, $\sum_{n=0}^{\infty} |n^k||a^k| < \infty$. So, no matter what k is, exponential decay eventually wins over polynomial growth.

Therefore, the impulse response dies down and the impulse response also is absolutely summable. This you can also infer from the fact that the pole is inside the unit circle and hence the sequence is belonging to the class of l_1. Therefore, if the system is causal, your system has to have poles strictly inside the unit circle. If it is causal, if you want the system to be also stable and exactly the opposite is true if the system is strictly anti-causal you want all the poles to lie outside the unit circle.