**Digital Signal Processing**
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# Lecture 53:
# Magnitude Response (3)
# - Notch Filter

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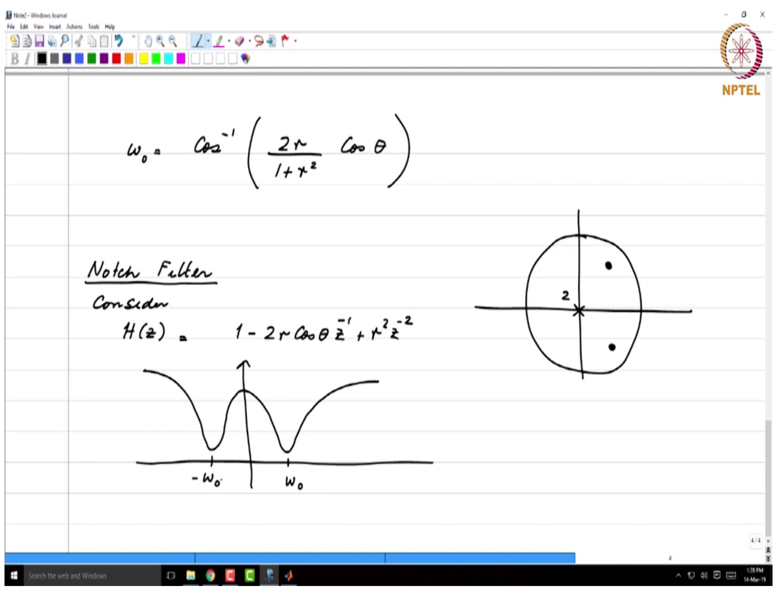


Now, let us look at related Frequency Response. This is the Notch Filter and the starting point of the notch filter is this. Now, let us consider $H(z) = 1 - 2r\cos(\theta)z^{-1} + r^2z^{-2}$. All we have done is we have inverted the previous example. We have taken the simple resonator and this is the transfer function which is 1 over the previous transfer function and hence poles become zeros and zeros become poles.

So, you had two poles here they have now become zeros and then the trivial second order zero has become second order trivial pole and the response looks like this. So, if you are on the unit circle, you have a certain magnitude response. You go across since you are approaching this zero, the response will dip, will reach a minimum and then again increase and have something like this. And it is very easy to see that the value of $\omega_0$ remains precisely the same. So, this is indeed $\cos^{-1}\left(\frac{1+r^2}{2r}\cos(\theta)\right)$.

Now, based on the name notch filter, you expect the filter to actually completely eliminate a certain frequency. So, for that to happen, you need to let $r = 1$.

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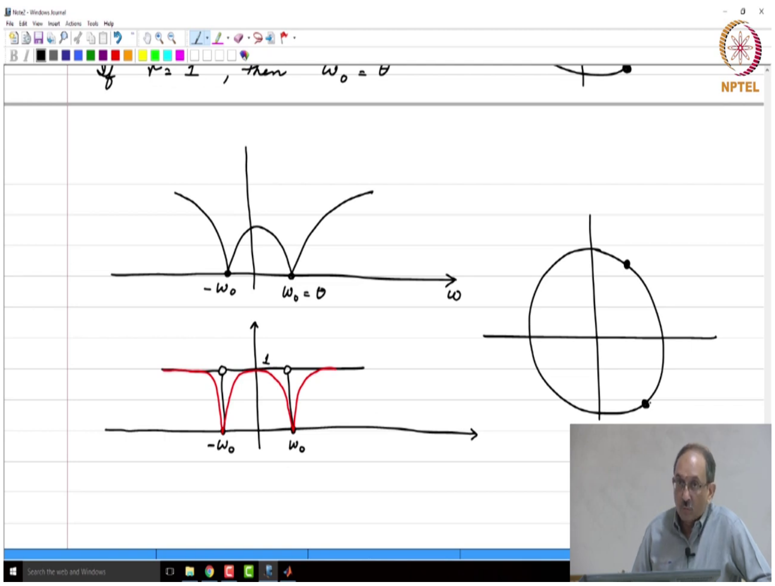

Therefore, if $r = 1$, then what can you tell about $\omega_0$? $\omega_0$ becomes $\omega_0$ becomes $\theta$. So, the picture corresponding to that is this. Now, the zeros are on the unit circle and the frequency response now has this shape and this $\omega_0$ is now $\theta$. And to such a system or to such a filter, if you applied an input which was $\cos(\omega_0 n)$, then what will be the output?

Output will be 0 because remember response to $\cos(\omega_0 n + \theta)$ that we saw, output will be amplitude scaled and phase shifted and the amplitude scaling will be the magnitude frequency response at that frequency. Here, at that frequency, the magnitude response is precisely 0. Therefore, if you apply $\cos(\omega_0 n + \theta)$ to this system, the output will be 0 and that is why this called as a notch filter because it notches out that particular frequency.

But, when you talk about notch filter, you also have an ideal reference. When we had low pass high pass and other typical filters, we knew what the ideal low pass filter was and then we had approximations to that. Now, what will be the characteristic of an ideal notch filter? Ideal notch filter if you want to notch out $\omega_0$, at those frequencies, the frequency response has to go to 0. But, for any other frequency, the response has to be, ideal notch filter response has to be 1. So, it notches out $\omega_0$, but for every other frequency, the ideal notch filters response will be 1.

This is how the ideal notch filters the response is, it goes to 0 at $+\omega_0$ and $-\omega_0$ and the response immediately comes back to 1 for every other frequency. And as you can see from this curve, this is far from ideal; is far from ideal because as you go away from $\omega_0$, the response does not rapidly come back to 1. Ideally should instantly come back to 1, at least you want it to come back to 1 rapidly and this is too slow.

So, what can you think of based on what you have studied so far to make the gain come back to 1 as quickly as you can make it happen that is, you want something like this.

Student: (Refer Time: 06:35).

Yeah, very good. You want the gain to come back to 1. So, you want to introduce poles now, all right. So, you have the notch filter and you want to introduce poles and where do you want introduce poles?

Student: (Refer Time: 07:07).

It cannot be outside because then if you are considering causality that would not work. So, it has to be inside. And then all right, same $\theta$ and at what radius? So, let us come, visit that question.

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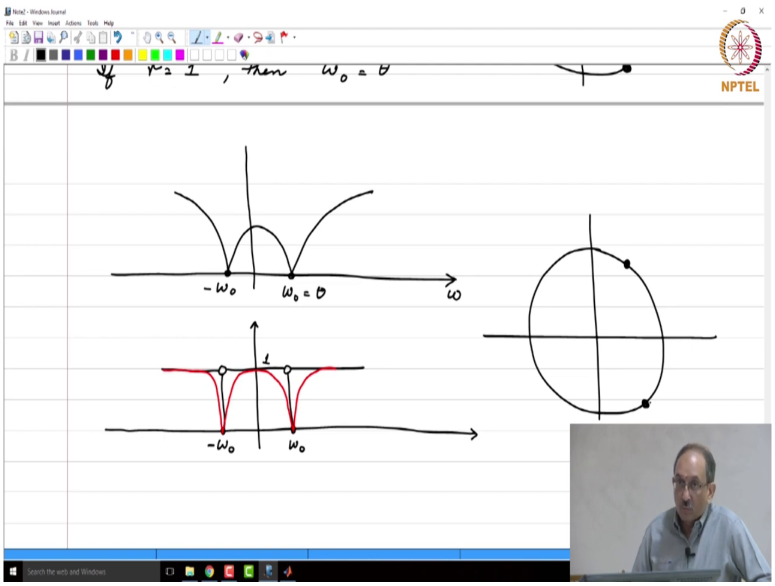

So, you want to place a pole at exactly the same angle and close to the zero. Now, in terms of transfer function, so the numerator remember as far as the noise filter goes, this is $1-2r\cos(\theta)z^{-1}+r^2z^{-2}$; for the notch filter $r=1$. Therefore, without any other modification, the transfer function is $1-2\cos(\theta)z^{-1}+z^{-2}$.

All I have done is, I have made $r=1$ which is what it should be for the notch filter. Therefore, $H(z)$ is $1-2(\ )$, $r=1$; therefore $2r\cos(\theta)$ becomes $2\cos(\theta)z^{-1}$ $r^2z^{-2}$, $r=1$ therefore, this is $z^{-2}$. But to make it better, you are going to introduce poles.

So, you are going to introduce poles at $re^{j\theta}$ and $re^{-j\theta}$ therefore, it is $H(z)=\dfrac{1-2\cos(\theta)z^{-1}+z^{-2}}{1-2r\cos(\theta)z^{-1}+r^2z^{-2}}$. And geometrically also this makes sense because if you are say at this point on the unit circle, then the distance between both pole and zero is roughly the same. So, when you are reasonably far away from this point wherever you are on the unit circle to a good approximation the distance to both poles and zeros is roughly the same and hence the ratio will be 1. Therefore, the magnitude response quickly rises to 1 and as you approach it gets pulled down because of the zero.

Now, in practice what should be the value of $r$?

Student: (Refer Time: 09:34).

As close to 1 as possible, very good. So, 0.9 is good, 0.95 is even better, 0.99 is very good better than 0.95. If 0.99 is good, then 0.999 is even better, right. Then your competitor will give you 0.9999. So, what is a good value?

Student: (Refer Time: 10:06).

Gain is 1 right, 0.99 you will get gain close to 1; 0.999 even better, right.

Student: (Refer Time: 10:20).

No, you want the angle to be exactly the same. There is no, does not make any sense to vary the angle, even if you vary the angle we are not talking about the radius right. No, for real accessibility, what you would realize is you will realize this difference equation. What you are having in mind is if you want to consider the impulse response and then if you want to convert the impulse response, then you have to truncate the impulse response to some limit but that is not how you implement IIR filters. IIR filters are always have infinite duration impulse response.

And, the way they are implemented is in the time domain you will implement the difference equation. So, you will write this as $y[n]$ equals; all these terms $x[n] - 2x[n-1]\cos(\theta) + x[n-2]$ and then you will have $2r\cos(\theta)y[n-1] - r^2x[n-2]$. So, this is a difference equation you will implement in practice. So, it does not depend on how quickly or how slowly the impulse response decays. So, that is not an issue.

So, in practice, what value of $r$ you choose is dictated by finite precision effects. Remember never forget that you are going to use finite number of bits to represent these coefficients and closer $r$ is to 1, if you want to represent it more and more accurately you will need lot more bits. And the more bits you have, the more expensive the system is going to become. So, how close $r$ can be to 1 depends on how many bits you have to play with depends upon the cost of implementation.

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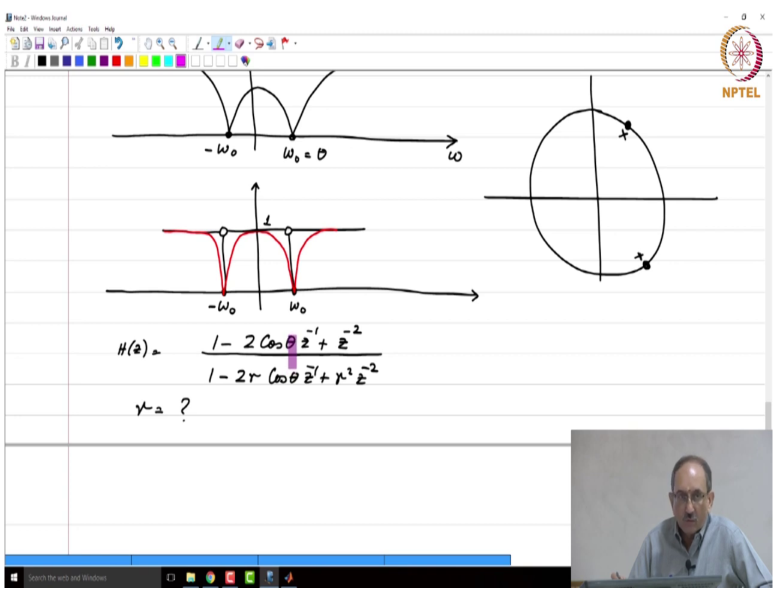


Quickly to give you a feel for this, let us look at frequency response without the poles and with the poles. So, let us make $r$ to be; $r = 1$. We will worry about $r$ when you look at the improved notch filter. So, theta, let it be $pi/4$. So, now, let us plot the simple notch filters frequency response. So, this is, you need only the numerator polynomial. So, this is $1 - 2\cos(\theta) + r^2z^{-2}$, $r = 1$. So, this is the numerator and then denominator is 1 and then you want evaluate this at 2000 points and let us evaluate this for the whole frequency range between $-\pi$ to $\pi$. So, this is the frequency response of a simple notch filter and now let us plot this.

So, this is command called fft shift. What this does is, there are, right now we have evaluated the response between 0 to $2\pi$. What this will do is it will help you to plot the same response between $-\pi$ and $\pi$. So, that is all this command does, and now let us plot the frequency response. So, this is between $-999$ to 1000. I have to divide by 2000, so that now the $x$-axis is between $-1/2$ and $+1/2$. I am normalizing, instead of plotting between $-\pi$ and $\pi$, I am plotting it between $-1/2$ and $+1/2$ and then let me plot the absolute value of the frequency response.

So, this is how the response looks like and theta is $\pi/4$. So, $(\pi/4)/(2\pi)$ is $1/8$; $1/8 = 0.125$. So, the zero occurs precisely at 0.125. So, this occurs exactly at 0.125. So, this is the simple notch filter.

Student: (Refer Time: 15:00).

Say that again.

Student: (Refer Time: 15:04).

Yeah, that is ok, as far as we are concerned, we are mainly interested in notching out. So, the fact that the gain is much more than 1 brings out one more case against the simple notch filter in that it is far from ideal. So, the gain being much greater than 1 is a case against it, for its departure from ideality. Of course, at $+0.125$ and $-0.125$, the gain is exactly 0.

Now, let us make the pole to be 0.9. So, $H1$, so the numerator stays as it is; the denominator is now $1 - 2\cos(\theta) + r^2z^{-2}$. So, this is the denominator. So, denominator has now been it is not just 1, it is now the pole set $re^{j\theta}$ and $re^{-j\theta}$, where $r = 0.9$. Now, let us make this to be 0.99. So, this is $H2$ and $H2$ is $fftshift(H2)$. So, now, let me plot all three. So, this is what we have already seen. On top of that I will also plot $H1$ and $H2$ and remember $H1$ is 0.9, $H2$ is 0.99.

So, the red one is 0.9 and the orange is 0.99. So, clearly you see the huge improvement in the frequency response, right. So, this is 0.99 is so much closer to the ideal notch filter. The notch exactly occurs at the same point as it should be but when you go away from the notch, the response becomes so much better when you place the pole closer and closer. So, this very nicely illustrates the effect of adding the pole to boost the gain when you go away from the notch.