Digital Signal Processing

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Lecture 58:
Allpass Filter, Group Delay
-Allpass filter

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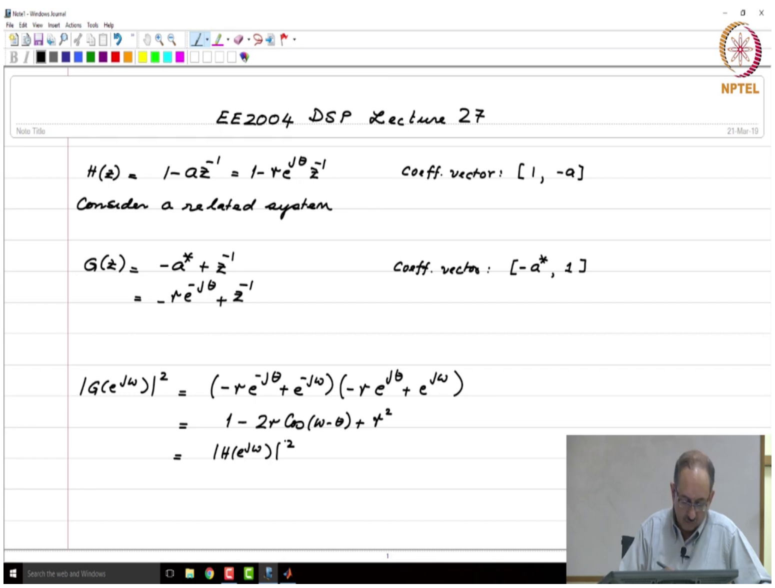

So, we are looking at phase response. And let us look at one more aspect. So, we have seen this simple single zero system before. Now, let us consider another related system and a in general is a complex number. Therefore, this is $H(z) = 1 - re^{j\theta}z^{-1}$. So, consider a related system, let us call this $G(z)$. And this is $-a^* + z^{-1}$.

If you look at $H(z)$, the coefficient vector is $[1, -a]$, all I have done is I have just taken all the coefficients and made them as a vector and clearly this is $[1, -a]$. And now let us look at the coefficient vector in this case, this is clearly $[-a^*, 1]$. So, what is happening here is, we have taken this coefficient vector and done two things; one is complex conjugate it, the other is time reverse it. So, we have time reversed and complex conjugated it and then if you did that, you will get this.

So, you can think of $G(z)$ as taking the coefficient vector of $H(z)$, complex conjugating it and time reversing it. And this clearly, because if a is $re^{j\theta}$, this is $-re^{-j\theta} + z^{-1}$. Now, let us look at the magnitude squared frequency response of this related system. So, $|G(e^{j\omega})|^2$ is nothing but $G(z)$ evaluated at $e^{j\omega}$ times its complex conjugate.

Therefore, this is $(-re^{-j\theta}+e^{-j\omega})$ times its complex conjugate. Therefore, this is $(-re^{j\theta}+e^{j\omega})$. And now if you simplify this, what do you get? Yeah, I think you need to push pen on paper and simplify it.

Student: $1-2r(\ )$.

$1-2r(\ )$.

Student: $\cos(\theta)$.

$1-2r\cos(\omega-\theta)+r^2$. And this is, is it something you have seen before?

Student: Yes.

So, this is exactly $|H(e^{j\omega})|^2$, all right.

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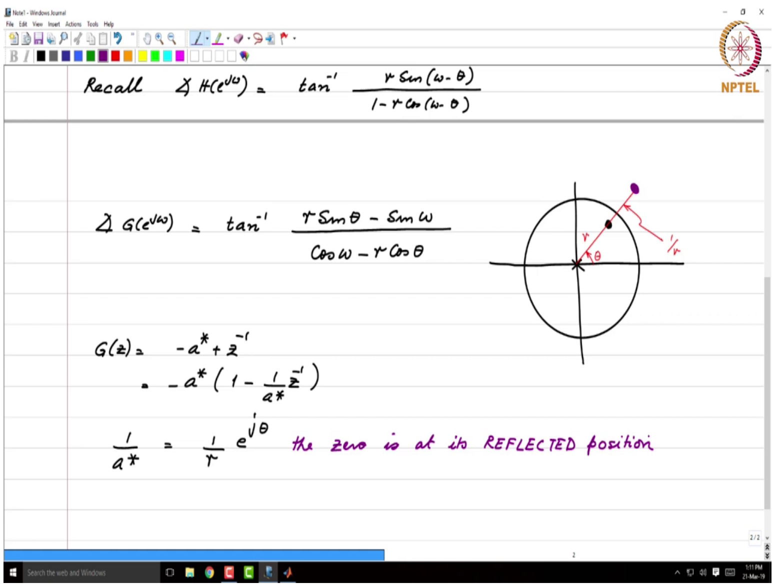

So, clearly the magnitude response is identical. And, if you recall, the phase response of the earlier system is $\tan^{-1}\dfrac{r\sin(\omega-\theta)}{1-r\cos(\omega-\theta)}$. So, this is the phase angle, so this is just recalling what we have seen earlier. Now, what about the angle of the related system? So, this is nothing, but tan inverse imaginary part by real part.

So, if you now look at this. So, the denominator will be $\cos(\omega)-(\ )$, this will be r.

Student: $\cos(\theta)$.

$\cos(\omega)-r\cos(\theta)$ and the numerator is the imaginary part so, this is nothing but $r\sin(\theta)-\sin(\omega)$. So, $\tan^{-1}\dfrac{r\sin(\theta)-\sin(\omega)}{\cos(\omega)-r\cos(\theta)}$ is the phase response. Clearly, the phase response is different; the magnitude response is identical, phase response is different. Let us also get a feel for what the pole-zero plot is. So, in the earlier case, for $H(z)$, you had a trivial pole and then you had a zero at $re^{j\theta}$.

So, this is r and the angle is θ. Note that, $G(z)=-a^*+z^{-1}$. So, this is the related system, what I have plotted here is the zero of the earlier system which is $H(z)$. And this can be written as $-a^*\Big(1-\dfrac{1}{a^*}z^{-1}\Big)$, simple rearrangement of the term. If I take $-a^*$ outside, I will get this and the zero therefore, is at $1/a^*$. The zero is clearly at?

Student: $1/a^*$.

$1/a^*$. And $1/a^*$ if a is $re^{j\theta}$, $1/a^* = 1/re^{-j\theta}$. And therefore, for $G(z)$, the zero is at $(1/r)e^{j\theta}$; that is it is exactly the same angle, but the zero location in strobing at r, it is at $1/r$. Therefore, I have the zero here therefore, this radius is $1/r$; clearly it is at the same angle.

And what we have done here is, the zero is at its reflected position, where the reflection happens about the unit circle. We are reflecting things with the unit circle as the reference. And hence, if we had $re^{j\theta}$, reflecting around the unit circle will give you $(1/r)e^{j\theta}$. So, this is what is happening here.

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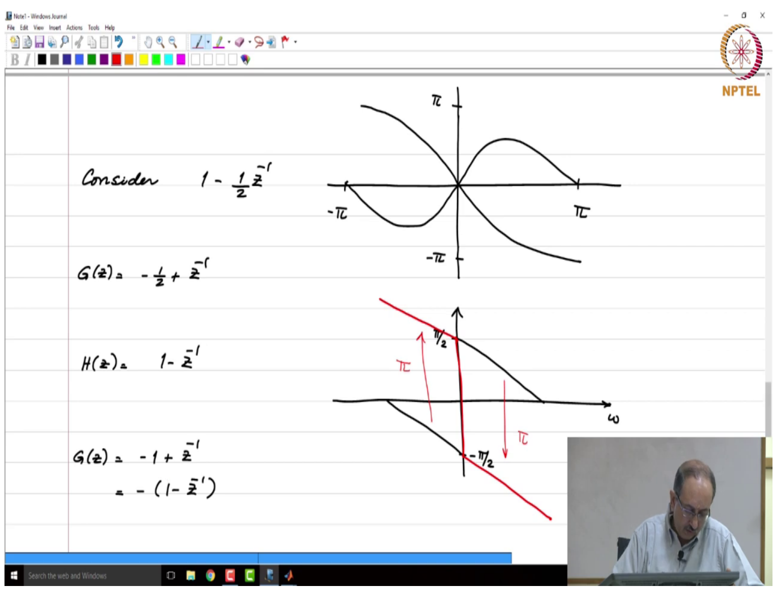

Now to get a feel for this, let us consider a numerical example to illustrate this. So, suppose you had $1 - \dfrac{1}{2}z^{-1}$. So, this is similar to $1 - az^{-1}$ with a being $1/2$. And we have seen the phase response of this. So, if you plot the phase response, it will be something like this. This is how the phase response will be. And for this given system, the related $G(z) = -\dfrac{1}{2} + z^{-1}$.

And, if you plot the phase response of this, the response will be something like this, all right. And one way for getting a feel for this is remember, $\omega = 0$ corresponds to $z = 1$. Therefore, if you put $z = 1$, $G(z)$ turns out to be $1/2$ and the phase angle of the quantity $1/2$ is 0. Therefore, the phase response is 0 here.

And if you put $z = -1$ which corresponds to $\omega = \pi$, so this turns out to be $-3/2$; $-3/2$, the phase angle is π. Therefore, at this point, it will hit π and this will be $-\pi$. So, this is what the phase response will look like.

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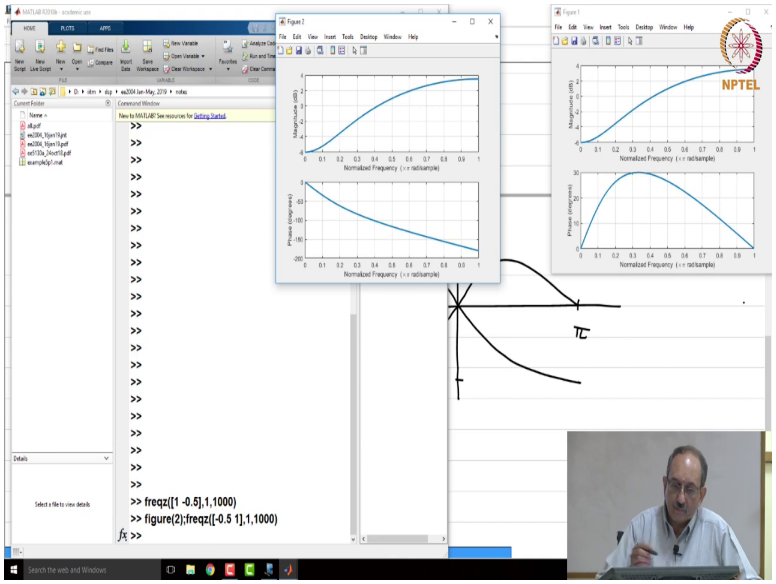

I can also show this in MATLAB. So, if I will use MATLAB's inbuilt frequency function and then illustrate this. So, the first system is [1 0.5] and then, the denominator is 1. I am going to evaluate this set a thousand points, $freakz$ by itself will plot both magnitude and phase. So, this is what the plot is.

So, this is the phase response and clearly you see the rough sketch I had drawn here. So, this is plotting from 0 to 0.5. Another thing you need to pay attention is I may have mentioned this before, MATLAB always maps π to 1 rather than π to $1/2$. So, this is from 0 to $1/2$ in the conventional notation and this is exactly what had drawn earlier; the rough sketch.

And if I now do this, $[-5\ 1]$ so, this is exactly $G(z)$. Let me plot it in a separate figure so they can compare both. So, now, if you look at this, so this is exactly what I had drawn here. Therefore, the phase response is quite different between $G(z)$ and $H(z)$ even though they share the same magnitude response.

Now, just one more example, suppose $H(z) = 1 - z^{-1}$. Now, the zero is at $z = 1$ on the unit circle therefore, you can expect the phase response to be precisely linear. And you will have a jump of π at $\omega = 0$ therefore, we have seen this earlier. So, this is the phase response, you have a jump of π at $\omega = 0$ and everywhere else, the phase response is linear. So, this is the phase response of $H(z) = 1 - z^{-1}$.

Now, the corresponding $G(z)$, by the way this phase is $\pi/2$, this is $-\pi/2$. And $G(z) = -1 + z^{-1}$ and this can be written as $-(1 - z^{-1})$. The reason for writing this in this form is, if you write it in this form, you see there is a multiplication by -1. And therefore, the phase change will be π, all right. And if you plot the phase response of this, it would turn out to be this.

And what is basically happening is, this changes phase by an amount of $\ pi$. Therefore, this shifts up by a value of π and this has to shift down by π. Therefore, the curve in red corresponds to the phase response of $G(z)$. So, these examples illustrate that even though the magnitude response remains the same, the phase response is drastically different.

Let us continue with what we are observed now. Namely, if you have $H(z)$ and you form $G(z)$ by taking the coefficient vector of $H(z)$ and then complex conjugating and time reversing, the magnitude response does not change.

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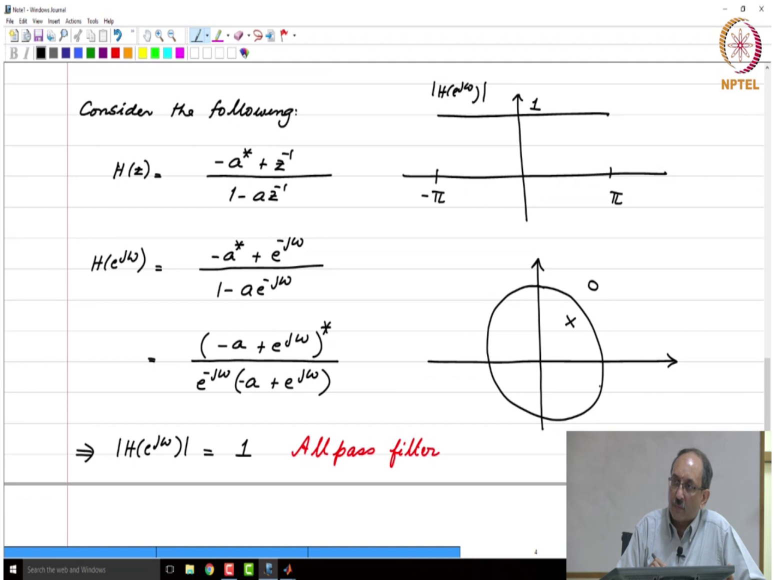

Now, let us consider the following system. So, $H(z) = \dfrac{-a^* + z^{-1}}{1 - az^{-1}}$. Therefore, $H(e^{j\omega}) = \dfrac{-a^* + e^{-j\omega}}{1 - ae^{-j\omega}}$. I can take $e^{-j\omega}$ outside in the denominator. Therefore, this becomes $-a + e^{j\omega}$, all right.

And here, I can write this as $(-a + e^{j\omega})*$. So, this immediately implies that $|H(e^{j\omega})| = 1$. You could have also inferred this from our previous argument. $(1 - az^{-1})$ and $-a^+ z^{-1}$ have the same magnitude and magnitude squared responses. Therefore, the ratio must be unity, so that is one way of seeing this, the other way of seeing this is by this manipulation.

And this is a very important kind of filter. If you plot the frequency response, if you plot the magnitude frequency response, it will all be 1 between π and π. And this is called as an Allpass filter, all right. Because it passes all frequencies and this can be generalized.

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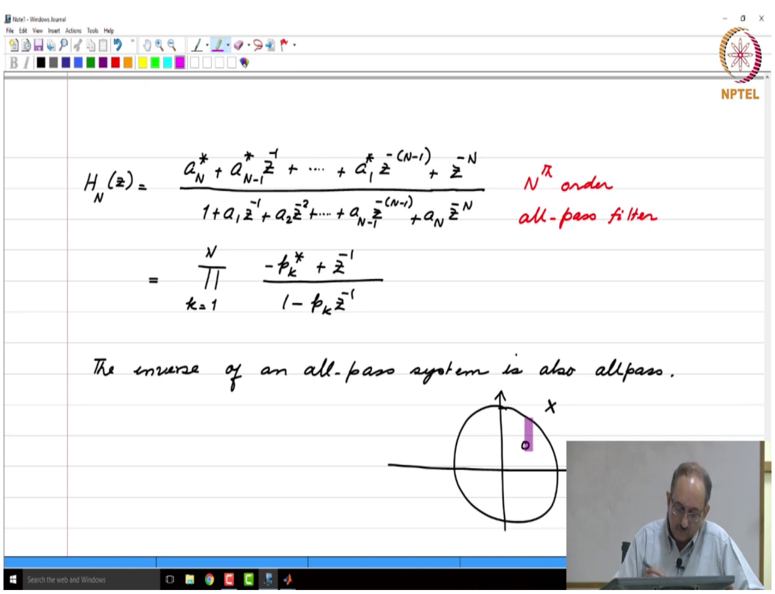

So, let me call $H_N(z)$ as the N^{th} order allpass system. So, denominator I have $1 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_{N-1} z^{-(N-1)} + a_N z^{-N}$. And let us apply the idea that we pointed out for the first order case, there what we said was the related system takes the original system, forms the coefficient vector, time reverses it and complex conjugates the vector.

So, the denominator vector here is now $[1\ a_1\ a_2\ \ldots a_N]$. So, we will take this coefficient vector, time reverse it and complex conjugate it. And therefore, if you form the numerator as $a_N^* + a_{N-1}^* z^{-1} + \ldots + a_1^* z^{-(N-1)} + z^{-N}$, this is an N^{th} order allpass filter. And its an important exercise for you to show starting from this, that this indeed gives you unit magnitude response.

This is a simple exercise you can do based on what have you learnt so far. And any N^{th} order allpass filter can be written as product of N first order allpass sections. Therefore, this can be written as $\prod_{k=1}^{N} \frac{-p_k^* + z^{-1}}{1 - p_k z^{-1}}$. So, each section is allpass, but it is first order and if you cascade N such first order allpass systems, you will get this N^{th} order allpass system as given here.

Notice that, the inverse of an allpass system is also allpass, because the magnitude response is unity; the inverse system also will have unit magnitude. Let us go and revisit this first order section to get a feel for what is going on. So, it has a pole at $z = a$, the zero is at $1/a^*$ therefore, if the pole is at a here then the zero must be exactly at its reflected position.

Therefore, if you have $re^{j\theta}$ being the location of the pole, then $(1/r)e^{j\theta}$ must be the location of the zero. Therefore, if you had an allpass system, for every pole, there must be a zero at the reflected position. Yes.

Student: Sir what is reflected (Refer Time: 24:53)?

So, reflection around the unit circle.

Student: Distances will not be equal.

Yeah, the distances will not be equal. So, this is different from the reflection that you are used to saying reflecting something about the y axis, there the distances will be maintained. Whereas, here what we mean by reflection is if you had $re^{j\theta}$, the reflected root about the unit circle is $(1/r)e^{j\theta}$. So, here we are not meaning the distances have to be maintained.

So, this is the simple first order all pass section, the pole-zero configuration. And note that if this were allpass, its inverse system also will be allpass. And in that particular case, you will have a pole here and a zero here because, if you take the earlier system and then if this were $H(z)$, if you considered $1/H(z)$, poles will become zero; zeros will become poles.

Therefore, this is how the pole-zero plot will look for a first order allpass which is the inverse of the earlier system. The point to note is if you want allpass systems to be both causal and stable, your only choice is this. Because, its only this choice that gives you poles that are strictly inside.

If you want this also to be stable, the system has to be anti-causal. And in practice, you want causal and stable filters and hence the allpass filter will have poles strictly inside the unit circle. And for every pole inside the unit circle, you will have a reflected zero.

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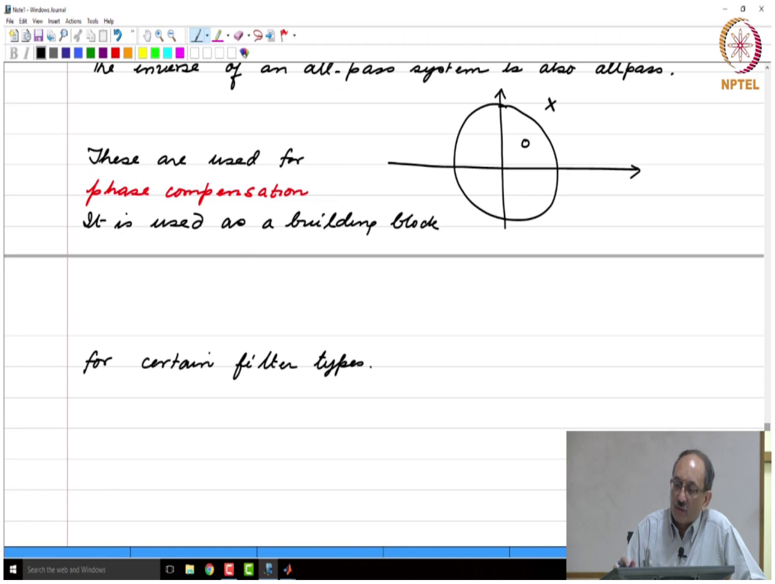

These are very important building blocks in practice, these are used for phase compensation. And the term itself kind of gives you a hint as to what this is going to do. If you cascade an allpass system with another system, the overall magnitude response is not going to change. Because, the magnitude response of the allpass is unity, all it can do is change overall phase response. The original phase response will get added to the allpass systems phase response.

And later when we talk about group delay, you will learn that certain phase responses are desirable. And if your original system has a phase response that is not what you want, you can alter the overall systems phase response by cascading that system with this allpass. And you have to design the allpass such that the overall phase response is decidable and that is what phase compensation is.

So, it is one of the uses for allpass is that. The other important use for all pass is, it is used as a building block; for building more complicated filters for certain filter types. And to give you a feel for what this means, if we had an all-pole filter, that can always be expressed as a sum of two allpass sections. An all-pole filter can always be represented as a parallel combination of allpass sections.

And why is it that you want to do that? Again to get a feel for why that might be desirable is that, remember, in practice whenever you have coefficients when you are implementing it, you are going to quantize them. So, when you quantize, you no longer have the original system, but only an approximation. And if you look at this, typically we will be interested in real valued impulse responses, which means all the coefficients will be real valued.

And you can always build a real valued system as cascade of first and second order building blocks. Because, we had two complex conjugate poles, you can combine them and form a single second order system with real valued coefficients. Therefore, in practice, if you now consider all these coefficients to be real valued. So, p_k^* will really be p_k for a first order section and for a second order system, you will have a second order denominator and the second order numerator but the coefficients will be time reversed.

Therefore, the numerator coefficients and the denominator coefficients are not independent. You can realize them using just one set of coefficients, because the same set of coefficients will be used in the numerator as well. Therefore, when you quantize, you are not quantizing independently the numerator coefficients and the denominator coefficients.

Therefore, this is where the quantization advantage comes about when you realize these all-pole filters as made up of allpass building blocks. So, these are just a couple of examples of uses of allpass and there is a full paper by P. P. Vaidyanathan and a couple of other authors. The title of the paper is Allpass sections: a versatile building block.

Google book, if you just type Vaidyanathan allpass and versatile, low and behold you will get that paper as the first hit and you can look that up. So, this is just to give you a feel for what all pass sections can do for you.