Digital Signal Processing

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Lecture 06:
Signal Symmetry, Elementary Signals (1)
The Unit Impulse

Keywords: elementary signals, unit impulse, sifting, dirac delta function

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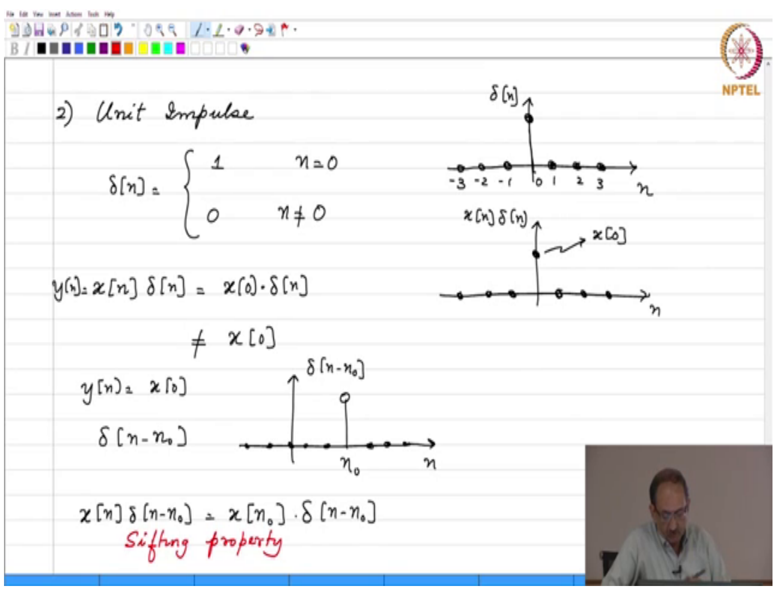

Now, let us look at the unit impulse. So, this is the simplest of signals that you can think of. So, this is using this notation similar to what was followed in the continuous-time case. So, this is

$$\delta[n] = \begin{cases} 1 & n = 0, \\ 0 & n \neq 0. \end{cases}$$

And the picture that is associated with this is like this; at $n = 0$, it is 1, at for all other points, it is 0. And this is the unit impulse sequence in the discrete-time case. And one of the things that happens when you multiply an arbitrary sequence $x[n]$ with $\delta[n]$; what it does is, you are going to multiply this $\delta[n]$ with an arbitrary sequence. At every other point namely, when $n \neq 0$, $\delta[n]$ is 0; therefore, the product is 0.

So, this is what you get at every other point. At $n = 0$, you will multiply $x[0]$ with 1, therefore, this sample value corresponds to $x[0]$. And therefore, really what happens here when you multiply $x[n]$ with $\delta[n]$ is

$$x[n]\delta[n] = x[0].\delta[n].$$

And one common mistake that can be done if you are not careful is this is not $x[0]$, alright. Because if you call this as $y[n] = x[n]\delta[n]$; $y[n] = x[0]$ means $y[n]$ is the constant sequence, whose values $x[0]$ for all n; whereas, what is happening here is you have to get the value $x[0]$ only at $n = 0$.

And not that value at other points; other points, the value has to be 0. To signify that, you need to multiply this by $\delta[n]$ which makes the product go to 0 at other values of n. Similarly, the shifted impulse is $\delta[n - n_0]$ and the picture associated with this is this. This is n_0. It is 0 everywhere else and here, I have shown the picture assuming n_0 to be positive. And similarly, by the same argument if you multiply this shifted impulse by an arbitrary sequence, it will pick out only the sample at $n = n_0$. So, this will be $x[n]\delta[n - n_0] = x[n_0].\delta[n - n_0]$, not just $x[n_0]$ by itself.

Because that would make this to be a constant sequence, whose values $x[n_0]$ for all n which is not something that we want, when we do this operation. And this is called as the sifting property; it shifts out that particular sample. So, this unit impulse is an extremely simple minded sequence. It is even simpler than the unit step sequence that we saw. Under normal circumstances, we should have introduced the discrete-time impulse first and then, the unit step sequence.

But now, our goal is to compare and contrast each elementary signal with its continuous-time counterpart. And from one extreme of being the simplest sequence, the counterpart we are going to encounter to be the continuous-time delta function which is really a complicated function. So now, let us revisit the continuous-time delta function. You might think this being a digital signal processing course, where we are dealing with only sequences.

Why should we care about the continuous variable impulse? The reason why we should is this will appear in the transform domain. So, the continuous variable impulse will appear in the transform domain. So, there is no escaping from this crazy function. So, now, let us try to get a feel for what that is.

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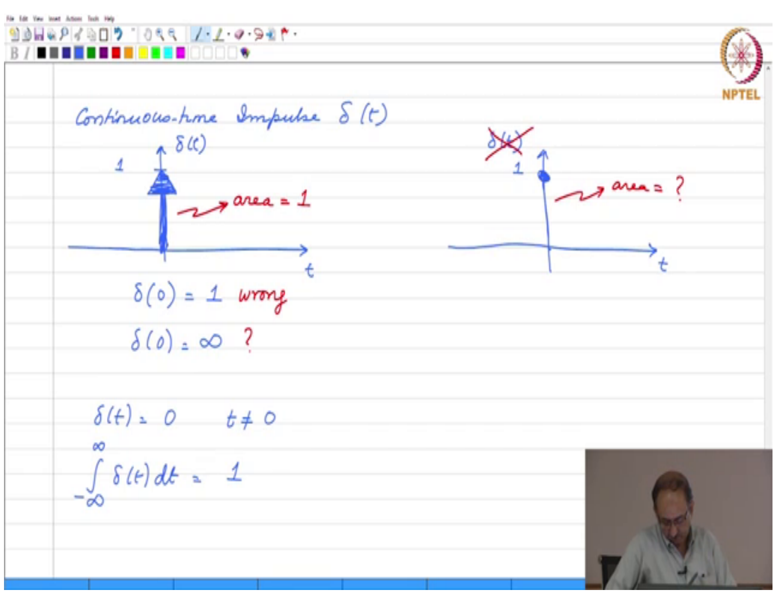

So, this is the continuous-time impulse and this is called as the delta function $\delta(t)$ and this also called as the Dirac delta function. After it was introduced by Dirac in quantum mechanics, but this idea of impulse was known even before Dirac's time. So, it is not that he was the one who invented this. He came up with this definition, used it and got the correct results, that was completely against what the mathematicians were used to at that point in time.

So, since he got the right results, they could not argue against it. But they could not explain it based on whatever knowledge they had. So, the theory of $\delta(t)$ was built much later. So, let us first get a feel for this. So, this is t; this is $\delta(t)$ and the unit delta function is plotted like this in terms of picture. Its height whatever that means, is 1 and what we really mean by this is we imply that the area is 1. For the unit impulse, the area is 1.

This is not the delta function, here is a function that takes on the value 1 at $t = 0$. Therefore, coming back to the impulse, $\delta(0) = 1$ is wrong. Now, as far as this function is concerned, the one on the right; what is its area?

Student: 0.

0; I mean a few people are piping up 0, somewhat hesitantly, ok. So, area is 0. No more voices are getting added, very good alright. So, the area is indeed 0. So, $\delta(0)$ being 1 is wrong. Then, what is $\delta(0)$?

Student: Not defined.

Not defined ok; any other choices?

Student: Infinity.

Infinity; infinity is a good number, alright. 0 also is a very good number by the way, except when it appears on your test paper, it is a profound number. So, $\delta(0) = \infty$. We are not sure whether this is indeed true, but what we are used to in terms of delta function is this. That is, the definition that you will find most common in engineering textbooks is $\delta(t) = 0$ for?

Student: $t \neq 0$.

$$\delta(t) = 0, \qquad t \neq 0,$$
$$\int_{-\infty}^{\infty} \delta(t).dt = 1$$

So, this is a unit impulse and the picture that we associate with this in engineering textbooks typically is this. We have a sequence of pulses.

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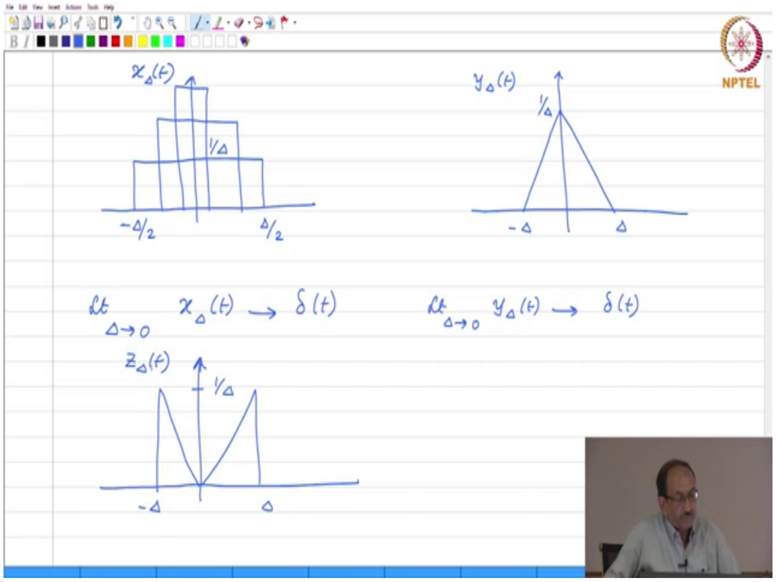

So, this is goes between $-\frac{\triangle}{2}$ to $\frac{\triangle}{2}$ and the height is $\frac{1}{\triangle}$. And then, we let the pulse become skinnier and skinnier and taller and taller, all the while maintaining unit area. Therefore,

$$\lim_{\triangle \to 0} x_{\triangle}(t) \to \delta(t).$$

So, this is the picture that we have of the delta function. And when you look at the delta function from this viewpoint, it indeed seems that $\delta(0)$ must go to ∞. It is infinitely tall, infinitely thin, area is 1 and this is what the delta function is as what we are used to.

Because $\delta(t)$ is defined by these properties, here is another sequence. So, let me call this as $y_{\triangle}(t)$. The areas again unity as you can see, half times base times height. Again,

$$\lim_{\triangle \to 0} y_{\triangle}(t) \to \delta(t).$$

So, this must tend to the $\delta(t)$. Any other function that you can give an example that tends to delta of t along the lines that we have just now looked at?

Gaussian, yes Gaussian is very good. So, as the variance of the Gaussian PDF becomes smaller and smaller, the height becomes larger and larger at the origin, area is always 1; therefore, that is also a perfectly valid sequence. Anything else? Hint is signal and its transform, continuous time Fourier transform. Your rectangular pulse, right? Its transform is *sinc*, if the pulse width becomes larger and larger in the limit, the pulse becomes the, what signal?.

Student: (Refer Time: 13:46).

No, the pulse becomes the single rectangular pulse, if the width becomes larger and larger, it becomes the DC function and its transform must be the impulse. Therefore, you have a *sinc* in the transform domain that is going to become like an impulse.

So, that is another function that becomes as an impulse in the limit. Now, let us look at this triangular pulse. Let me take that and cut it into two halves and then, cut it in the middle into two halves and then, rearrange it like this. Let me call this as $z_{\triangle}(t)$.

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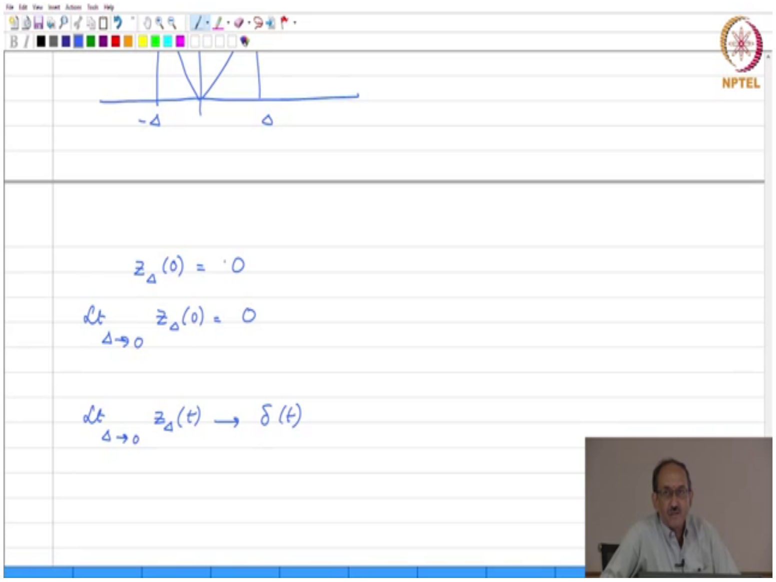

What is $z_\triangle(0)$?

Student: 0.

0. $z_\triangle(0) = 0$ for all?

Student: delta.

delta, ok; therefore, $\lim_{\triangle\to 0}\ z_\triangle(0)$ is?

Student: 0.

Very good. What is the area of $z_\triangle(t)$? 1, alright. As $\triangle$ becomes smaller and smaller, in the limit as $\triangle$ tends to 0; what does $z_\triangle(t)$ tend to? i.e., $\lim_{\triangle\to 0}\ z_\triangle(t)$? What can you say about $z_\triangle(t)$ in the limit for $t \neq 0$?

Student: (Refer Time: 16:02).

$z_\triangle(t)$ in the limit as $\triangle \to 0$; what can you say about the limit for all values of t that is not 0? No, I am just limiting myself to the question of what is the value of the limit for $t \neq 0$? 0 right; then, what is the area?

Student: (Refer Time: 16:36).

So, what does this tend to tends to?

Student: $\delta(t)$.

Tends to $\delta(t)$, but $z_\triangle(0)$ in the limit is 0. So, now you see the paradoxes that come into play, once you have this kind of reasoning and picture that you are commonly used to when you encounter delta in the usual engineering textbooks. So, all this was kind of done to tell you that the picture that we have $z_\triangle$ in terms of limits of sequences, can give rise to all kinds of paradoxes. On that intriguing note, let us end the today's lecture.