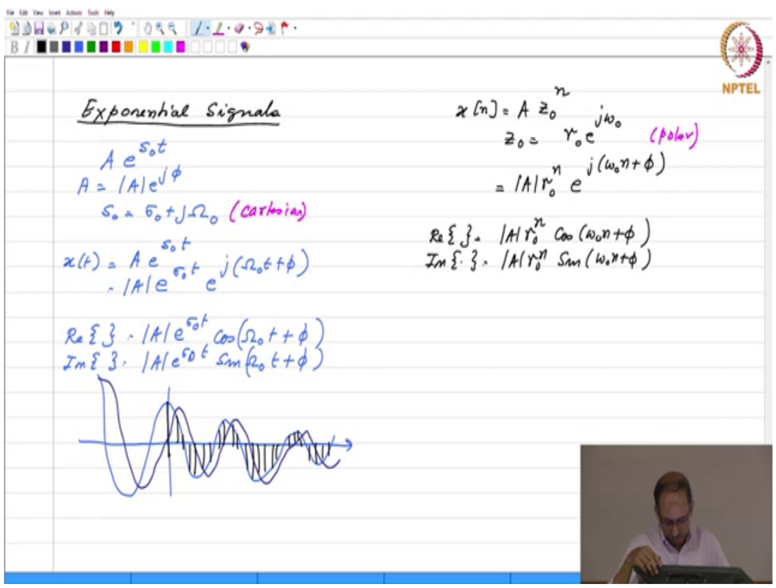
Digital Signal Processing
Prof. C.S. Ramalingam
Department Electrical Engineering
Indian Institute of Technology, Madras

Lecture 08:
Elementary Signals (2)
-Exponential signals, complex sinusoids
-Does increasing ω_0 leads to ever-increasing rapidity of oscillations?

Now, let us move on to the next class of Elementary Signals, that is really important. Again, we are, this is going to be a review for you; we are going to look at continuous-time sinusoid and discrete-time. So, let us look at, rather than sinusoidal it is more correct to call them as exponential signals.

(Refer Slide Time: 00:39)



So, what you are used to, in terms of continuous-time exponential is $Ae^{s_0 t}$. A is really a complex number, $A = |A|e^{j\phi}$ and s_0 is really another complex number which is say $s_0 = \sigma_0 + j\Omega_0$. Therefore, $x(t) = Ae^{s_0 t}$ is really, $e^{s_0 t}$ can be written as $e^{\sigma_0 t + j\Omega_0 t}$. Therefore, this becomes $|A|e^{\sigma_0 t}e^{j(\Omega_0 t+\phi)}$, right. And the real part ofcourse is $|A|e^{\sigma_0 t}\cos(\Omega_o t + \phi)$ and the imaginary part is $|A|e^{\sigma_0 t}\sin(\Omega_o t + \phi)$.

So, these are the real and imaginary parts and the picture associated with this is something like this. And remember, this is what is called as the everlasting exponential and so, rough picture. So, these are the real and imaginary parts. Now, the counter part is $x[n] = Az_0^n$ and here is the same as before $A = |A|e^{j\phi}$; z_0 is again a complex number and $z_0 = r_0 e^{j\omega_0}$.

By the way, so, s_0 which is the complex number, we wrote it as $s_0 = \sigma_0 + j\Omega_0$, this is in Cartesian form. Whereas, the complex number z_0, we are writing it as $r_0 e^{j\omega_0}$ which is in polar form. So, this is and this of course, is polar. Why is that? So, you have seen this before right, you have seen both continuous and discrete-time were introduced in signals and systems right. And signals and systems, typically it was introduced like in this form the way I am introducing it now, correct. So, did it occur to you as to why one representation is useed for continuous-time and the other representation is used for discrete-time? Very simple yes.

Student: (Refer Time: 04:22).

I am sorry, I did not hear you.

Student: (Refer Time: 04:26).

Not sure I follow your answer.

Student: (Refer Time: 04:34) Cartesian form.

But you were introduced to the discrete-time exponential; were you or were you not? If you were not introduced, then probably it is not a fair question.

Student: (Refer Time: 04:52).

So, you are not, I know the signals and systems mainly its continuous-time, But some amount of discrete-time, was it part of the syllabus or no?

Student: (Refer Time: 05:09) very brief. So, we did not pay attention to it.

Alright.

Student: (Refer Time: 05:16).

But were you introduced to this?

Student: (Refer Time: 05:22).

Even if you introduce very briefly, the very fact that you were introduced to it should raise this question, right? Hey, why is that in Cartesian whereas, this is in polar?

Student: Natural frequency.

Natural frequency. Why is that not natural there? So, have you had a course in complex variables?

Student: No.

No, that is a very good excuse. No, but even then, you might be familiar with this. If you are going to raise a complex number to the n^{th} power, what do you use?

Student: It is a discrete (Refer Time: 06:22). So, it is explained as a.

No, what is the theorem mathematical theorem that is used to?

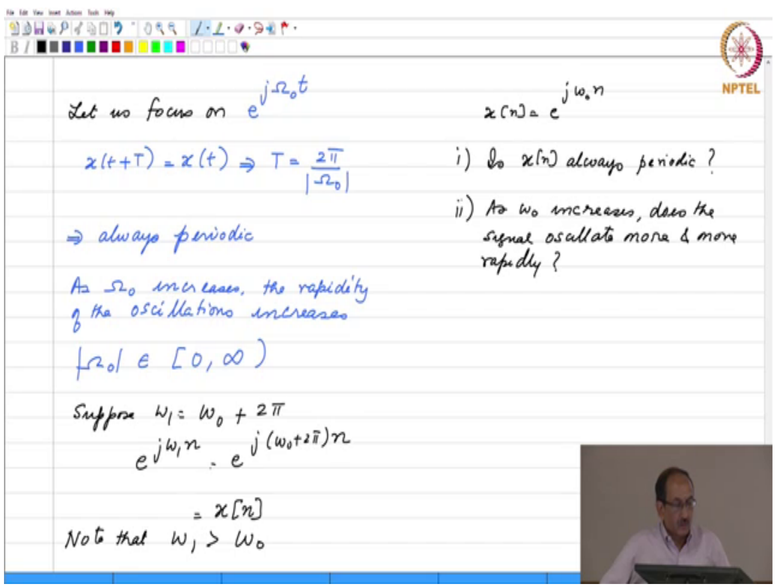
Student: (Refer Time: 06:28).

Let not difference here as pronounced that, but that is the theorem that we use. Here, you are going to raise this complex number to the n^{th} power. And, if you are going to raise a complex number to the n^{th} power, the easiest thing would be to express that in polar form and then proceed, alright. So, this becomes $|A|r_0^n e^{j(\omega_0 n+\phi)}$.

And ofcourse, similar. And the imaginary part is $|A|r_0^n \sin(\omega_0 n + \phi)$ and the reason why you are using polar form is de Moivre's theorem and raising a complex number to the n^{th} power, where n is an integer; that theorem makes this very simple. That is the reason why you are using polar form here and Cartesian form for the continuous-time case. And then, in terms of pictures, again you can plot two different plots here. In my cleaned up notes, I will put up MATLAB generated plots which is very accurate, here I am just going to give you some feel.

So, this is the so called stem plot, right. The independent axis will be n and then you will plot these samples of $x[n]$, real part versus n, imaginary part versus n. So, looks like these two are very very similar; one after all seems to be the taking sample values of the other. Now, let us probe the similarities and differences a little more to get a better understanding of this.

(Refer Slide Time: 09:11)



So, let us focus on $e^{j\Omega_0 t}$. Here, we are trying to compare and contrast continuous-time sinusoids versus discrete-time sinusoids. So, here just to make sure everything is in place, this is an everlasting exponential. So, this goes from $-\infty$ to $+\infty$ and if σ_0 were negative, for positive values of time, the envelope will decay whereas, it will blow up for negative values of t. On the other hand if σ_0 were positive, for positive values of t, the exponent the envelope will be exponentially growing.

Whereas, for negative values of t, it will be decaying and if $\sigma_0 = 0$, the envelope will be constant, that is when you will have your sinusoid. And if $\omega_0 = 0$, then this will be purely a real valued exponential, right. Of course, assuming we are having the amplitude also to be real value. And the corresponding counterpart here, the role of σ_0 is now played by r_0. So, if r_0 were less than 1, then for all positive values of n, the envelope will decay whereas, for negative values of n, the envelope will blow up.

If r_0 were greater than 1, positive values of n, the envelope will blow up, for negative values of n, the envelope will decay. And similarly, if $\omega_0 = 0$, you will get the corresponding counterpart for the case where $\Omega_0 = 0$. Again, if you assume the $|A|$ to be real valued, you will get a pure exponential. We will again delve into a little more of this as we go along. So, looks like these two are very very similar, now are there differences to see, what differences are we focus on $e^{j\Omega_0 t}$.

So, this is a pure complex exponential and $x(t+T) = x(t)$. That is, this signal $e^{j\Omega_0 t}$ is periodic with period T implies that T is; what is the value of T for this signal?

Student: (Refer Time: 12:10).

2π?

Student: (Refer Time: 12:13) Ω_0.

By $2\pi/\Omega_0$, is this correct?

Student: (Refer Time: 12:19).

No, remember T is the smallest number for which this is true, right. So, this is not fully correct, there is something not quite right with this formula for T.

Student: (Refer Time: 12:46) Ω_0 equal to.

That is Ω_0 not being 0 is fine.

Student: (Refer Time: 12:55).

Suppose Ω_0 is not being 0, is this still correct?

Student: (Refer Time: 12:59).

Yeah.

Student: (Refer Time: 13:03).

No, remember T is the smallest number for which this has to be true, right.

Student: (Refer Time: 13:11) non-zero number.

Non-zero number. So, what change do you have to make?

Student: (Refer Time: 13:15) $2n$ by.

Now, just following up on what was said. Smallest non-zero, can you follow up on that and then correct the error in this formula?

Student: (Refer Time: 13:23).

No, where will n come? n will not come here.

Student: (Refer Time: 13:33).

What is missing is this, right. Because, we need the smallest non-negative number. So, this is always period with period T. No matter what Ω_0 is, this is always periodic with. So, the first important implication is, this is always periodic. And the other important thing is, as Ω_0 increases, what can you say about the rapidity of the oscillations? It will also increase, the rapidity of the oscillations increases.

And there is no upper bound on Ω_0, right. $|\Omega_0|$, there is no upper bound.

So, $|\Omega_0|$ belongs to the interval 0 to ∞. So, very natural to ask this question about the discrete-time counterpart, $e^{j\omega_0 n}$.

Student: (Refer Time: 15:04).

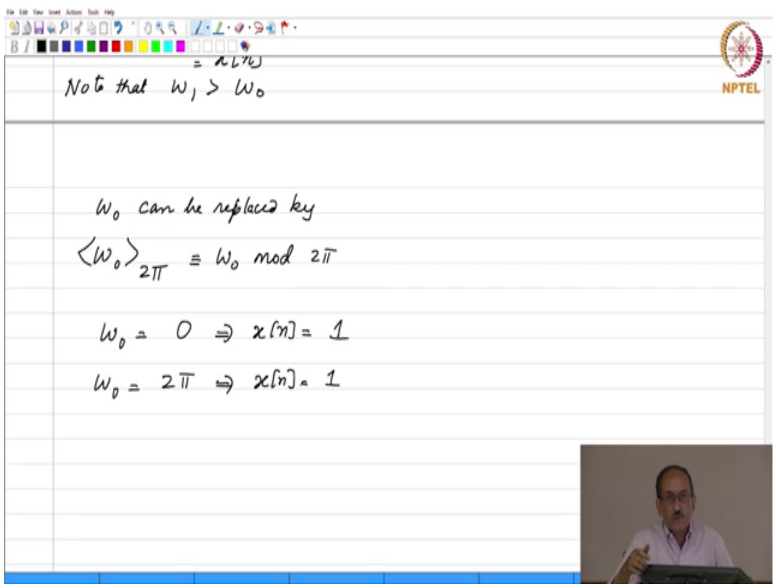
Say that again.

Student: (Refer Time: 15:10) open bracket 0.

Because, Ω_0 can also be the DC signal, correct? There is no problem with Ω_0 being 0, that will give you the DC signal, right. For the DC signal, what is the period, that is a minor issue. But, Ω_0 can indeed take on the value 0 which will give you your familiar DC signal. Now, let us ask the same question for its discrete-time counterpart. We will ask these two questions: Is $x[n]$ always periodic? As ω_0 increases, the signal oscillate more and more rapidly?

So, these are the counterparts of the question that are clear for the continuous-time case. Suppose, we consider two different frequencies; ω_0 and $\omega_1 = \omega_0 + 2\pi$. Then, $e^{j\omega_0 n}$, which is $e^{j(\omega_0+2\pi)n}$. So, this is exactly $x[n]$ because this is nothing, but $e^{j\omega_0 n}e^{j2\pi n}$, $e^{j2\pi n}$ ofcourse is 1. So, note that $\omega_1 > \omega_0$. So, you get exact same signal, if you replace, let us look at an actual signal, let us vary ω_0.

(Refer Slide Time: 17:59)



So, this suggests that as far as the signal is concerned, ω_0 can be replaced by $\omega_0 + 2\pi$. So, this is simple, we are going to use in this context. So, this is nothing, but $\langle\omega_0\rangle_{2\pi}$. Now what we are going to do is, we are going to keep on increasing ω_0 and we see that ω_0 and $\omega_0 + 2\pi$ will yield the identical signal. Therefore, if $\omega_0 = 0$, this implies that $x[n] = 1$. In our language, this is called as the DC sequence.

You will also get exactly the DC sequence when $\omega_0 = 2\pi$ because, ω_0 and $\omega_0 + 2\pi$ are the same. Therefore, you get an identical signal when you increase the frequency from 0 and make it 2π, again the DC sequence comes. So, this leaves us with the question what happens to the rapidity of the oscillations as ω_0 increases from 0 to 2π. So, what we will do is we will look at a simple real valued sinusoid, we will vary the frequency of oscillation from 0 to 2π and, then we will see how the oscillations vary.

Do they increase? If they do increase, then it appears that you will have to have a maximum rapidity of oscillations. And then, the reason why this has to be true is, because again when you keep on further increasing ω_0 when you hit 2π, you have to get back the same DC sequence. So, intuitively, it appears the frequency of oscillation will increase and then probably start to decrease and then slow down and reach the DC sequence once more. So, we look at waveforms next class.