Computational Electromagnetics
Numerical Integration
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Numerical Integration
Lecture – 4.3
Gauss Quadrature

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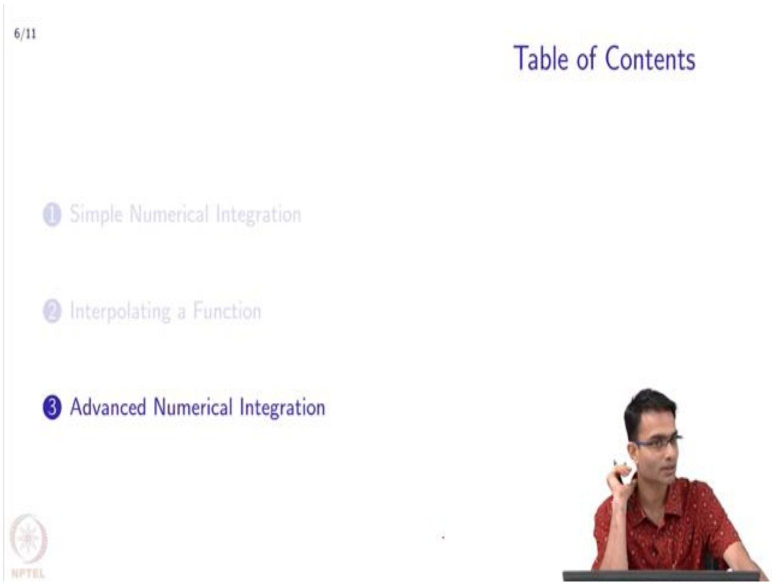

That bring us to the third section on advanced Numerical Integration.

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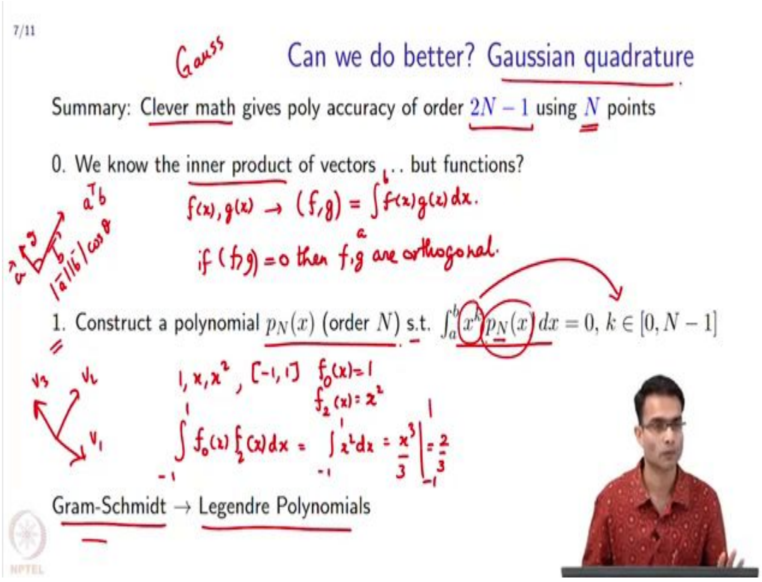

So, some there is some very very clever math, not surprisingly thanks to Gauss who is able to give you the answer to accuracy 2 N - 1; keeping the same N points right. It seems impossible, how is it possible right. So, the next few slides, we will try to understand how we can get an accuracy of order 2 N - 1 keeping the same N points ok. So, we will sort of go through it step by step. Everyone with me so far? objective is find out the integral of a function as it accurately as possible, when I do not know the analytical integration of this function.

Now, there is something called as the inner product right. So, if I give you two vectors over here say a and b with some angle theta between them ,we all know the inner product of these two vectors is; what would you write it as?

Student: $a^T b$.

$a^T b$ right; so, in the language of linear algebra I will write this as $a^T b$, in the language of vectors I will write this as right both mean the same thing. Now so, this is about inner product of vectors, how about inner product of functions, how do we take inner products of functions? That is the concept that we will need further over here. So, the dot product element wise dot product, it generalizes to a integral.

Student: (Refer Time: 01:50).

Integral right so if I say two functions say $f(x)$ and $g(x)$ I want their product right. So, the symbols is $<f,g>$ and its value is given by.

Student: (Refer Time: 02:02).

A $\int f(x)g(x)dx$ ok; so, in the case of inner product of vectors you could have the inner product of; supposing I ask you the inner product of two non-zero vectors, supposing it is 0. What does that tell you?

Student: (Refer Time: 02:28).

Their vectors are perpendicular to each other right. So, inner product 0 when the two vectors are non-zero themselves means are the vectors are orthogonal to each other. If I tell you two non-zero functions have inner product 0, the same terminology applies. We say that these two functions are orthogonal to each other right. So, if $<f,g>=0$, then f g are orthogonal. You can no longer visualize it as two arrows in space ok, but that is the price you are paying for some abstraction. Now, you are abstracted the idea for dot product from vectors to functions and you can talk about orthogonality right.

So, this so called Gaussian quadrature that we are discussing now it starts with saying construct a polynomial of order N such that it is orthogonal to all polynomials of order less than N. So, this $x^k, k < N$ notice that k is maximum N - 1, you can take k equal to 0; that means, the average of p_N is 0, then I take $x, x^2, \ldots, x^{N-1}$ ok.

So, what you do is you construct a polynomial of order N such that it is orthogonal to all lower orders of polynomials ok. How would you do this? We can again go back to how we had done this process with vectors right. In vector supposing I gave you a bunch of vectors like this let say in 3D space v_1, v_2, v_3 ok. Let us say these 3 vectors wherein 3-dimensional space and I say that given these 3 vectors, can you construct a set of 3 vectors which are all orthogonal to each other? You can right.

Student: (Refer Time: 04:35).

Gram Schmidt process right; so, Gram Schmidt process is what you would follow, you take one vector start keep that the first vector, subtract of the part of the second vector along the first vector you get the next vector and you repeat this process in N steps you get N vectors that are all orthogonal to each other right. So, those who have few who have not done linear algebra you can revise this concept of Gram Schmidt. The same process I can apply to functions because, the concept of dot product orthogonality holds to a functions also right.

So, I can start with $1, x, x^2, x^3, \ldots$ all of these guys ok. So, supposing I take $1, x, x^2$ and I give you the interval of - 1 to 1. So, is the function say is 1; so, $f_0(x) = 1$ and $f_2(x) = x^2$; are these two orthogonal to each other? Let us find out; so, if I do this $\int_{-1}^{1} f_0(x) f_2(x) dx = \int_{-1}^{1} x^2 dx = \frac{x^3}{3}|_{-1}{}^{1} = 2/3$ And the answer is?

Student: 2/3.

So, these two functions are not orthogonal to each other, but there is no problem I can apply this Gram Schmidt process. And, I can arrive at a set of functions that are polynomial function which are all orthogonal to each other ok. This has of course, been done and there is a very nice name for these polynomials. These polynomials that you get after orthogonalization are called so called Legendre polynomials ok.

There is a entire family of orthogonal polynomials with different different names and different properties, but the idea is the same orthogonality between functions ok; so far so good. So, what is our sort of summary of step 1 is I have constructed a set of polynomials which are orthogonal right. So, if I fix my N I have created this $p_N(x)$ over here orthogonal to all x; x^k ok. Now, how will be use it? Let see that takes us to step 2; any questions? Clear right.

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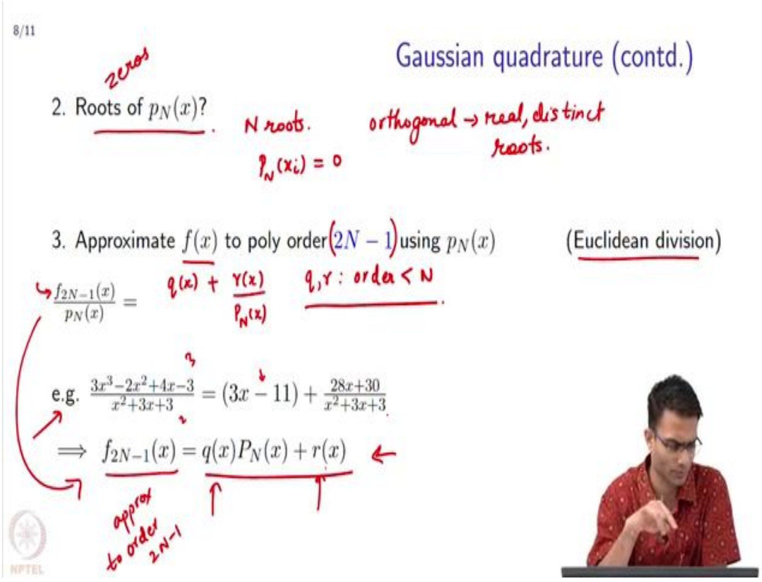

Now, step 2 what are the roots of this polynomial function? I am not asking a very specific question ok, I am just saying how many roots will there be of this Nth order polynomial? First order polynomial, how many roots or zeros you call it zeros how many roots does it have?

Student: At max N.

At max N no say N they can repeated roots also right $(x-2)^2$, 2 is repeated twice right. So, there are N roots, by roots what do I mean? The places where the function goes to 0, right. So, $p_N(x_i) = 0$ fine. There is a nice theorem from numerical analysis which we will not prove over here, but it says that for orthogonal polynomials the roots are always real and distinct, we will just use it ok; so, for orthogonal real and distinct roots alright ok. So, some of you must be wondering where is this discussion going so, let us try to bring it closer.

So, remember our function is $f(x)$ which is not polynomial; it is some general function ok. Now, we want to try something interesting, we want to try to approximate it by a polynomial of order 2 N - 1. So, suppose let us just jump straight into the example ok. So, this example over here the numerator is a polynomial of order 3 cubic polynomial, denominator is a polynomial order 2. So, this is actually goes back to something which would have done in high school. It is called Euclidean division of rational functions, if I do this division this is order 2, this is order 3, the quotient will be order 1 right. So, that is this guy over here and the remainder will also be of degree.

Student: 1.

1 in this case right; so, the same thing applies over here I will get a $q(x)$ plus I will have some remainder $r(x)$ right. But what is a special property of $q(x)$ and $r(x)$? q and r what will their order be?

Student: N - 1.

N - 1 at most because I have divided by polynomial of order N and I want the function approximation to order 2 N - 1. So, the best that you could do is q can at base the order N - 1 right. So, order less than N right so that is if I just open out this formula over here I get $f_{2N-1}(x)$ that is the approximation to order 2 N - 1; as the $f_{2N-1}(x) = q(x)p_N(x)r(x)$ that is your Euclidean division.

Now, what you think would be next?

Student: (Refer Time: 10:28) removed the denominator.

Yeah, removed the denominator right I in this expression over here I just multiplied $p_N(x)$ on both sides right. So, I have a quotient and I have a remainder over here. What was our ultimate objective?

Student: Integrate.

Integrate the function right. So, we have done some we were looking manipulation next let us try to integrate and see what happens ok.

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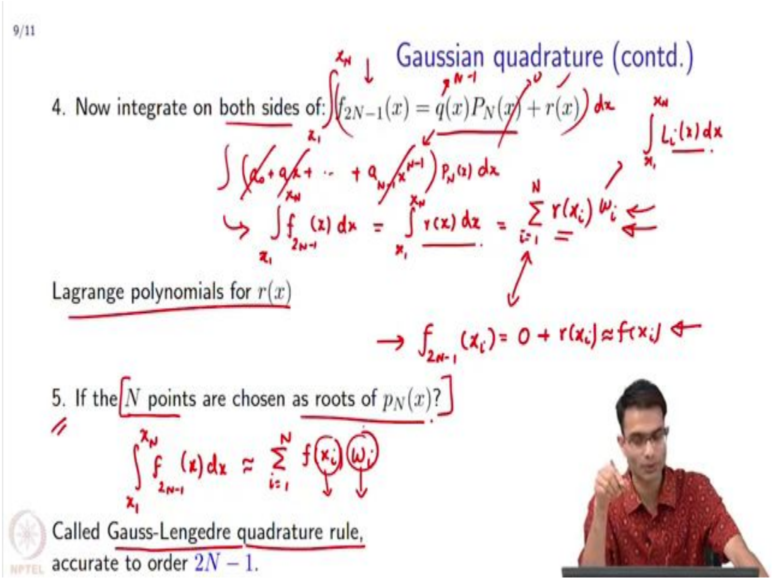

Takes as a step 4 we are going to integrate on both sides of this expression ok. So, I am going to integrate this whole thing ok, remember these are never indefinite integrals; these are always definite integrals ok. We are only talking about definite integrals alright. So, can you tell me, let us look at the first term over here ok, when I integrate this guy $q(x)p_N(x)$ what do you think the answer will be? Answer has to be 0 because so, $q(x)$ is a polynomial of order, what was the order of $q(x)$?

Student: N - 1.

N - 1 right so, this was N - 1. So, I could have written it like this $q(x) = a_0 + a_1 x + \ldots + a_{N-1}x^{N-1}$. And, what is happening is being multiplied by $p_N(x)$ P N x and integrated, but how did we construct these p_N's?

Student: Orthogonal.

Orthogonal to every degree less than it so, it is orthogonal to this guy, this guy, this guy all of these guys. So, this integration goes to the first term goes to 0. So, I am left with these integrals as $\int_{x_1}^{x_N} f_{2N-1}(x)dx = \int_{x_1}^{x_N} r(x)dx$. I cannot simplify this $r(x)$ term anymore its right, it is of order N - 1, but that is about it right I cannot do anything else.

But what do we have with us? We have N points at which I have a possibility of evaluating my function ok. I have so, supposing I want to use those N points, can I write this can I simplify this further in terms of what we already learnt, can I can I construct a Lagrange polynomial for $r(x)$? I can right because I have a expression for $f_{2N-1}(x)$ so, I know q, I know p, I know r. So, I know $r(x)$, $r(x)$ is some polynomial of degree N - 1 right.

So, I can use what we have discussed earlier the Lagrange polynomials so integrate this guy out right. So, what will this integral be? Again very simple i is equal to 1 to N, what should the first term be? I am integrating a function r so, first term should be I am using that the integration formula which I had derived. The quadrature rule which I had derived for Lagrange polynomials. First term should be the r of x_i exactly the value of the function at some node multiplied by.

Student: Weights.

The weights so, w_i the weights do not depend on the function right. So, remember these weights were given by the $\int_{x_1}^{x_N} L_i(x)dx$ pre-computed so, time saved. So, I wanted the integral of this guy and I am somehow left with the value of some other function only ok. So, far have we specified the locations x_i? No, right we just left it generally ok. So, now let us play the final card of tricks which is step 5, is that if the N points are chosen to be the roots of $p_N(x)$ ok.

So, what can I say about $f_{2N-1}(x_i)$ what will its value be equal to? Right so, it will be $r(x_i)$ because, it is the root of $p_N(x)$ right. So, putting this and this together $\int_{x_1}^{x_N} f_{2N-1}(x)dx$ is equal to; now I will I will use this guy over here, but I can replace $r(x_i)$ by right I can just replace. So, anyway this was whole thing was anyway and approximation of x itself right. So, this will become so, let us get this is approximately $\sum_{i=1}^{N} f(x_i)w_i$.

So, does this look like a quadrature rule? It looks like a quadrature rule, why because the node values are to be given; the weights are given which are independently calculated. So, I have got a quadrature rule because, of the extreme cleverness of both Gauss and Lengedre the name of Gauss Lengedre goes after them. So, you have a Gauss Lengedre quadrature rule which is accurate to order 2 N - 1 right. What was, did you have to pay a price for getting this very good a result?

Student: (Refer Time: 16:08) $p_N(x)$.

We have to figure out $p_N(x)$, but that is the 1 time affair; any other any other price that I have to pay?

Student: Sir value of N will be.

Value of N will be.

Student: High.

Value of N will be high, no value of N is up to you; if you tell me that I whole I have access to the function only at 4 points then you will create $p_4(x)$. It is in your hand how much accuracy you want, but when you choose N equal to 4 your answer is accurate to what order? Order 2 N - 1 which is.

Student: 7.

7 in the earlier high school approach our answer was accurate only to order 3 right; for N equal to 4 because earlier what we said with N points I can fit a polynomial of degree N - 1. So, my answer if I give you N equal to 4, your answer is at best you can approximate a cubic function are integrated. Now, you are saying with 4 points you can do a 7th order polynomial ok. The price that you have to pay is that you do not have choice on where to evaluate the function.

You have forced to choose the x_i's to be the roots of $p_N(x)$, you cannot just say I will uniformly chop up my interval at spacing h; you cannot do that. You have to use the roots of $p_N(x)$ otherwise what happens this property does not hold true ok, but that is a small price to pay. So, in the case of these Gauss-Lengedre quadrature rules both the x_i's and the w_i's these are pre-computed. There are there are mathematical libraries, MATLAB has its C++ has it just invoke them they will give you the set of x's and w's; plug it into your thing and you have done ok.

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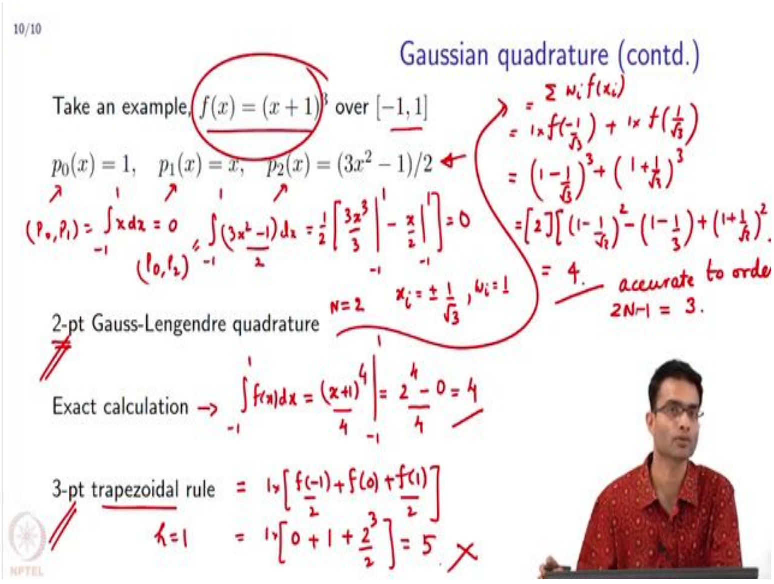

So, we will use these quadrature rules extensively in integral equation approaches. To drive home the idea of Gaussian quadrature that we have studied so, far what we will do is we will take a simple example. And, the example that I have taken is a function $f(x) = (x+1)^3$ and the interval is - 1 to 1 ok. So, the choice of how many points of quadrature rule that I want it is up to me. So, for example, this is the first Gauss-Lengedre polynomial or Lengedre polynomial, this is the second and this is $p_2(x) = (3x^2 - 1)/2$ right.

So, $p_0(x), p_1(x), p_2(x)$ to just test our understanding let us see if they are orthogonal to each other right. So, if I take the inner product of $p_0(x)$ and $p_1(x)$ right. So, what do I get $\int_{-1}^{1} xdx$ and you can see that this is going to be 0 ok. Let us take $p_0(x)$ and $p_2(x)$ so, $\int_{-1}^{1} (3x^2 - 1)/2dx$; what am I going to get? So, $1/2(x^3|_{-1}{}^{1} - x|_{-1}{}^{1})$ right and you can see that this will also come out to be 0.

So, this shows us that these our polynomials are indeed orthogonal and what remains for us to do is to evaluate this integral using let us say three different methods. So, the first example would be for example, I mean you can take the exact calculation just to know what we are supposed to get. So, integral $\int_{-1}^{1} f(x)dx$ this is a polynomial functions so it is trivial to integrate. So, I am going to get $(x + 1)^4/4|_{-1}^{1} = 4$ ok.

Let us try a little bit more expensive evaluation, but using a inferior rule which is the trapezoidal rule, the extended trapezoidal rule. So, here the spacing between the points so, it is 3 points between - 1 to 1. So, the spacing your h is actually just equal to 1. So, the 3 point trapezoidal rule will give me $f(-1)/2 + f(0) + f(1)/2$.

So, this evaluates to 5. So, you can see that the trapezoidal rule is already over estimating the exact answer and not by some small amount 5 when the actual answer should be 4; so, that is the big change. Let us see what a 2 point Gauss quadrature rule gives us. So, as I have mentioned before a quadrature rule is defined by the nodes and the weights.

So, if I chose a 2 point rule right so, N is equal to 2 this is my, the function that I am going to take. So, I already know the nodes I mean look at this function you can see that the roots of this polynomial are simply $x_i = \pm 1/\sqrt{3}$. And, some extra information is needed those are the weights. In this case I will just give you the extra information that the weights are both equal to 1, in general they need not be and you have to calculate it carefully. So, let us evaluate this over here; so, I am going to get its going to be sum of $w_i f(x_i)$ that is my general formula.

So, it is going to be $f(-1/\sqrt{3}) + f(1/\sqrt{3})$ ok. And, this is going to give me $(1 - 1/\sqrt{3})^3 + (1 + 1/\sqrt{3})^3$ and we can evaluate this quite simply. So, $a^3 + b^3 = (a + b)(a^2 - ab + b^2) \Rightarrow (1 - 1/\sqrt{3})^3 + (1 + 1/\sqrt{3})^3 = 2((1 - 1/\sqrt{3})^2 - (1 - 1/3) + (1 + 1/\sqrt{3})^2)$ And, you can work through the math you will get an answer 4 in this case also.

So, let us observe that what we did is by using only at 2 point Gauss quadrature rule, I was able to get the exact answer 4 right. This should not surprise us because, we know that 2 point Gauss quadrature rule is accurate to order 2 N - 1 right 2 N - 1. And in this case N is 2 so, this is accurate to order 3 my given function over here is a cubic polynomial. So, this is no surprise I got the correct answer because, all the conditions are satisfied.

So, this just shows you that you can get the same answer by having only 2 function evaluations in this case whereas, if you use a slightly inaccurate quadrature rule even a 3 point evaluation gives you the wrong answer ok. So, hopefully this drives home the importance of having of using a proper quadrature rule alright. So, this brings us to an end of this module on numerical integration or quadrature as it is more popularly known.