Computational Electromagnetics
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Introduction to Integral Equations
Lecture - 5.2
Solving the Integral Equation

So, let us what we have done is we have just formulated the Integral Equation we have not yet come upon how do we actually solve it. So, now we will look at how to solve it you will see that solving this is actually not that difficult at least in this case. So, let us see what?.

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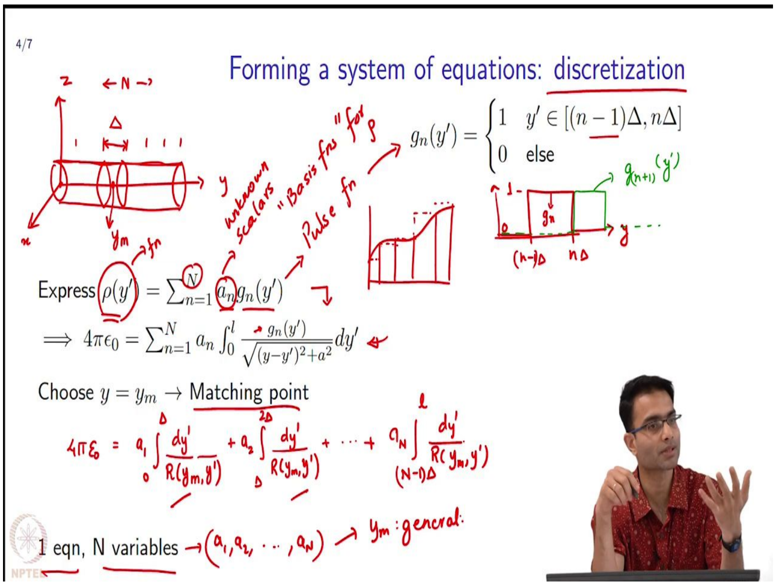

So, let us draw this once again. So, this was my z x y. As with any system of any equation that you see the first strategy to solve it is to discretize it. So, we will follow the same thing over here my variable is rho of y. So, I should discretize along the y axis right.

So, what I will do is I will discretize it like this into little segments and let each segment have some width. Let us keep it simple constant width capital delta that is how I have discretized it.

Now, this charge that I had over here the charge is sitting over here on the surface as I said as a one dimensional line charge that is the unknown. So, what I do now is this is the little bit of a trick. This unknown function over here, I write it as the sum of n functions each one with a coefficient a_n and the function g_n.

So, what is this g_n over here? I have written it over here it is what is called a pulse function. So, what is this pulse look like? So, first of all there are capital N pulses that I have used the definition of this pulse is that it is 1 inside the nth segment and 0 everywhere else right. So, if I were to plot this, so this is $(n-1)\,\Delta$ and this is $n\Delta$ right. So, this function is 1 over here and 0 elsewhere.

So for example, we can call this is your g_n. Now if I want to show you g_n plus one what will it look like? g_{n+1}; this pulse would be to the will it overlap with the previous function? No right, it will be to the right. So, if I draw this over here; let me just sorry right. So, this is going to be g_{n+1} right.

So, the idea is clear what we are trying to do. We are trying to approximate this function in terms of pulses. So, it is you know if I had a function like this; if this is the actual function, I am trying to make a piece wise constant approximation of this function right I am going to say this is how I am going to approximate my function. It is crude but what it does is it makes a math easy and later we will see how to get a little bit more sophisticated about these things.

So, these are the correct terminology for them these are called basis functions. So, we have chosen a basis function which are the simplest basis functions pulse basis functions. So, we will call this as basis functions for what for rho. Now having done this remember are unknown was this rho which was a function. Now I say that solving for a function is a difficult thing; let me make my life a little simpler let me now solve for something else.

So, when I converted from rho to this guy over here what became my unknown?

Student: a's.

The a’s right. So, these now are my unknowns. So, unknown and are they functions or just scalars.

Student: Scalars.

Scalars right. So, which looks easier to you solving for ρ or solving for a.

Student: A.

A right. I am just have to I just have to solve for a bunch of number which is much much easier than solving for a whole entire function.

So, I plug this expression for rho into the previous voltage equation and outcomes this expression. So, $4\pi\varepsilon$ I have taken up over there the left hand side was originally 1 volt. So, $1 \times 4\pi\varepsilon$ is what I got in the left hand side and the rest of it is as before.

Now, we will go back to how we are decided we will solve this equation. So, to solve this equation we had said that if you look back if you remember what we did previously. We had said let us choose those points, where I know the potential and I chose those points to be along the axis of the wire right.

So, what we can do is there are n segments over here. So, for example, this is your y 1 no let us not call it y_1 this time. What I have N capital N pulses right; that means, there are capital N segments over here and my number of unknowns are N N.

So, what we will do is let us choose our the observation point let us choose them systematically. So, let us choose the centre of each of these n segments. So, for example, this let us call this mth segment and let us call this y_m. The choice of the place where I evaluate the function I also call it a matching point. Previously the y primes we call them as source points because that is where the source charges and the corresponding place where I choose along the axis, I called them matching points.

So, with this in place what will happen is $4\pi\varepsilon_0$. So, the first let us just open up this integral over here. So, the first will be a_1. This integral will go from where to where the first term 0 to Δ right. Because g_1 will be the first term and g_1 is nonzero only in the first pulse first segment and 0 everywhere else

So, in this limits of integration it will become 0 to Δ right what happens over here g_n which is 1 for that part. So, I will have a dy' and here what will I have what will be the two arguments. So, first one so y_1 I have right, so we can make it say y_m no problem correct. I chose some point over here so y_m comes over here and y' is my limit variable of integration that remains as is that was the first term.

Second term will be now you can see the pattern delta to 2 delta. Because that is why the second pulse is non zero $dy' R(y_m, y')$. How many such terms will I have? N terms right. So, this final term will have a_n this will be from $(n-1)\Delta$ to the length the total length and again dy'.

So, we just expanded the summation that is all we did. This integral is of course, going to be different right this integral and this integral they are not going to be the same because even that the function sitting inside the same. But the integral will be different because limits of integration are different.

So, starting off with the idea of discretization choosing the pulse basis functions opening up the expansion what have we arrived at we have one equation in how many variables N variables and what are these N variables a_1, a_2 all the way up to a_n.

So, you can see where this process is going right here I have kept the matching point y_m is kept general. So, but this keeping it general has given us a strategy that we can use to formulate I mean to get the solution to this. So, you there are n unknowns I need n equations right.

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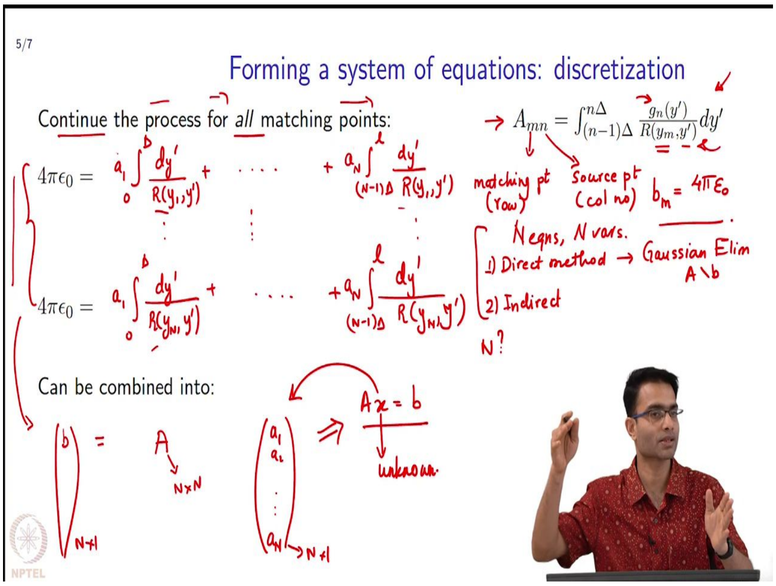

So, what we will do is we will continue this process for all the matching points. So, what matching points will we choose? We will choose y's to be the centre of each of these n segments. Since there are n segments each segment has one centre y_1, y_2 all the way up to y_n right.

So, the first equation as before what we will get is I will get a_1 integral from 0 to this thing and I have, sorry they will not be this anymore. Because I have replace it by g and this will be y' right. I have substituted the value of g to be 1 in that segment right.

Finally, I will have a a_N $(N-1)\Delta$ all the way to L, dy' and I will repeat this for each of the matching points final matching point will be at the very end right. So, at the very end what I will get? So again, first term will be a 1, 0 to triangle or delta first term and finally, the last term right.

So, now what we have done is we have got N equation, N variables seems like something we can solve right. So, if you do it systematically all of these guys on the right hand side sorry on the left hand side can be made into a column vector right. So, I can take all of these guys and put them into a column vector b right and you can see that all of these equations have the variables a 1, a 2, a 3 all the way up to a n in one string right. So, that looks like what the product of the product of at least one row vector with a column vector. So what should the column vector be?

Student: N cross N.

The column vector will be.

Student: a's

a's.

Student: a's.

a's will be the column vector. So, I can see so this will be a column vector over here. So, that will be a_1, a_2 all the way up to a_n. The first row of this will be what these terms right this integral this integral all of this and there is a pattern to this. So, similarly when I go to the next row this is yet another the mth row. So, all of that can be combined into one matrix and the size of this matrix is.

Student: N cross.

$N \times N$, this is $N \times N$ and this b over here is also $N \times 1$ right.

So, I have rewritten this into we can say Ax is equal to b where x is the unknown vector. So, this is going to be the objective of I mean the approach that we will follow for all integral equations somehow get it into Ax is equal to b where A and b are known solve for x. So, this is your thing x is unknown. And you can see there is a very nice general pattern that is emerging for what is the m n element of this matrix is a.

So, m is coming from what the m th matching point right. So, mth is the matching point which corresponds to the row or the column. I mean which does it correspond to a particular equation or a particular column the matching point.

Student: Equation.

Equation right so that is the row number that is why it is a_{mn} right. So, this is like coming from the matching point and n over here talks about this n over here is going from here all the way to the right right. So, that is telling me the source point right. So, your matching point is over here and your source point is over here right.

I have kept it in general g over here in case which you some other kind of function other than pulse. So, I have got a matrix element and I did not write it over here, but it is trivial to see what is bm in this case? The mth elements of the vector b. What is it?

Student: $4\pi\varepsilon_0$.

$4\pi\varepsilon_0$ right. So, this turned out to be really simple in other problems it will not be so simple.

So, this sort of completes the description of how we arrived at a system of equation. So, what did you do? You chose you took your problem discretized it chose a set of basis functions. Then use those basis functions to simplify your expand your integral equation and convert it into a discrete set of equations which finally, gave you a linear system of equations and we solved it. So, this you can ask lots of questions how do we know what n to choose that would be the first question right. Should I choose n equal to 2, 3, 4 what should we do?

Student: (Refer Time: 11:54).

Say again.

Student: As large as possible.

As large as possible. So, larger the n, the you can see intuitively the better I will be able to approximate the actual function rho the price that you will pay.

Student: Higher.

Higher computational time so that is one that is one obvious trade off for n. Also this the expression over here for a_{mn} that you can see over here this is where you can use your Gauss Legendre quadrature rules, or any other quadrature rules to evaluate this matrix element.

So, in this problem it is a very small toy problem. So, it will not matter very much even if you choose a inefficient quadrature rule, but you can imagine then real life systems let us say you are calculating the radar cross section of a aircraft. You will have close to a lakh elements right and for a lakh elements if you are trying to evaluate this integral the number of points you choose in your quadrature rule the more the number of points the more the number of function evaluations.

And if I have to do this 1 lakh times or a large number of times you know million times whatever what will happen? Your cost of just evaluating the matrix elements itself forming a x is equal to b is going to explode. So, that is why you need as efficient away as possible of giving you this a m n to desired accuracy. So, that is the cost of making the matrix a then there is the second is the cost of.

Student: (Refer Time: 17:27).

Solving a x is equal to b. So, that involves another course on numerical linear algebra. I will just roughly tell you two things there are two ways of solving ax is equal to b; one is what is called direct method which you all have studied in high school it is called Gaussian elimination. This works in you know sort of reasonably small size problems the MATLAB syntax is very simple a slash b right that is what we achieve a solution.

The second is what are called indirect methods right. So, indirect methods are iterative methods. So, something called conjugate gradient method if you heard these words there are there is a whole family of iterative methods which will solve which will not give you the answer in one shot. We will start with some guess and keep it finding the guess.

So, if the problem size is very very large you need to resort to iterative methods and as the course goes I will give you little bit more information on these things, MATLAB also has these things in built alright. So, that is about the ways of solving ax is equal to b. But let us get back to this the first question which I asked you how you chose n right. So, you might start out conservatively choose some small n and then start making n larger and larger right.

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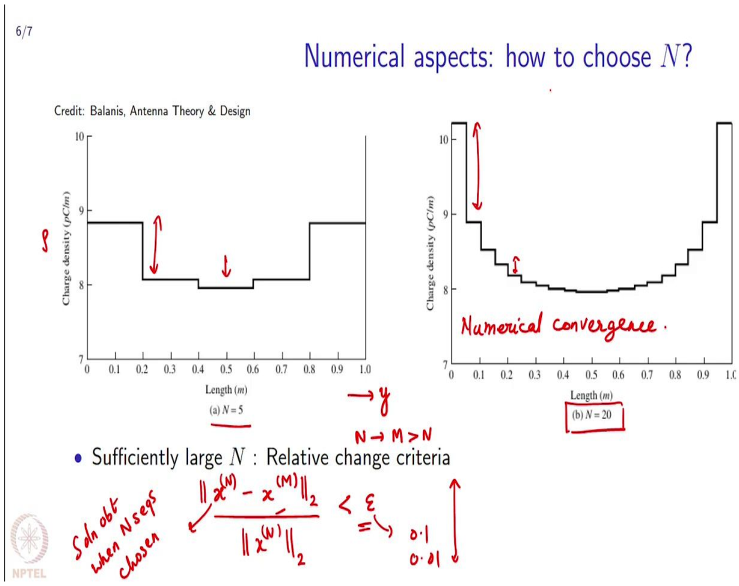

So, I have some graphs over here. So, the first one is N is equal to 5 and what is being plotted is your rho as a function of this is my y axis.

So, I just chose 5 segments what is system of what will be the size of my a 5×5 right. I solve it I get the solution and this is what the sorry the so charge density this is what it looks like. Intuitively it makes sense the charge is tending to bunch of towards the end there is of course, at the centre it is not 0, it is not gone to 0. Because a charge it will still be there right as you said rightly in the start of the lecture there will be some sort of a equilibrium that will be establish where the forces are in balance.

So, this is why N is equal to 5 as we went further to n equal to 20 what happened? You see that the solution is beginning to look a little bit smoother now right. Here there were big jumps over here now the jumps are there is still one big jump over here, but these jumps are now smaller. So, I am able to approximate the function better and this was so N is equal to twenty. Again we will take hardly a fraction of a second on to solve this.

So, now this sort of brings a question how do I actually choose n? So, one popular way of doing it is that you take your so let us say your solution that you got at resolution N. Now let say I increase from N, I go to a larger number M, where M is greater than N. What I can do is I can find out the relative change. So, x this is just the difference between the two I can take the norm of this vector to find out how much energy is in the difference. And I can find out for example, compared to the referring solution what is the norm? So, these when I say norm, I am taking the two norm of other way.

Student: So x n.

So, x n is the solution obtained when N segments chosen and x m therefore, is the solution with m larger number larger than N chosen. Now I can say that this relative change over here should be less than let us say some epsilon; this epsilon can be something like you know you know 0.1 or 0.1. So, you can say I want my solution to stabilize or converge within say 1 percent, 5 percent whatever depending on your requirement.

What you will find is as you begin increasing N more and more and more at some point your cost of computation becomes very large , but your improvement in solution becomes very less. So, your pass the point of I mean you are getting diminishing returns. So, at that point you should stop. So, the systematic way is to implement inside your solver something like a check on the relative change in solution and stop when this threshold has been met then you can be sure that the solution has converged.

So, the word for this process is called numerical convergence. Again this is the key strategy we will use throughout this course in solving systems of equations right. Is it fine?. So, that is one way of increasing the accuracy increase N. Any other ways that you can think of improving accuracy looking at the solution that I have obtained can you think of some other cleverer way where I can keep n small yet get a more accurate solution.

Student: Basis function.

Basis function should be can be chosen little bit more intelligently right. Here my basis functions were pulses. So, I was forced to approximate my smooth function by set of pulses that is why I need large number of pulses right.