Computational Electromagnetics
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Introduction to greens Function
Lecture – 7.1
Motivations for Green's functions

So, today's module, we will talk about Greens functions. If you remember in the previous modules, we looked at the surface integral equations, we kept writing a g a function g and we said that we will worry about it later, now is when we worry about it right.

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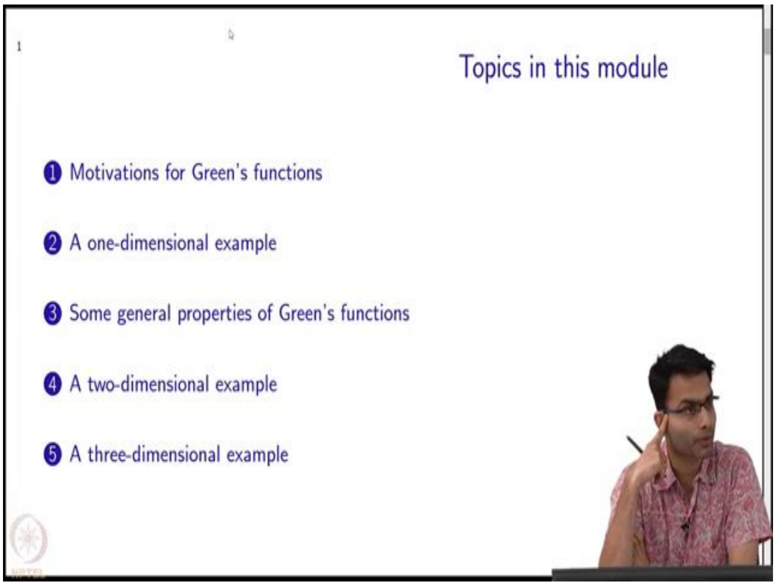

So, there is a sort of outline of this module, we will first of all give a motivation for it and after giving you the motivation for it we will run through some simple 1D, 2D and 3D examples ok. So, to many of you it will be the first time that you are encountering this object called Green's functions ok. So, to give you some familiarity and comfort with it, I will give you all of these different examples alright.

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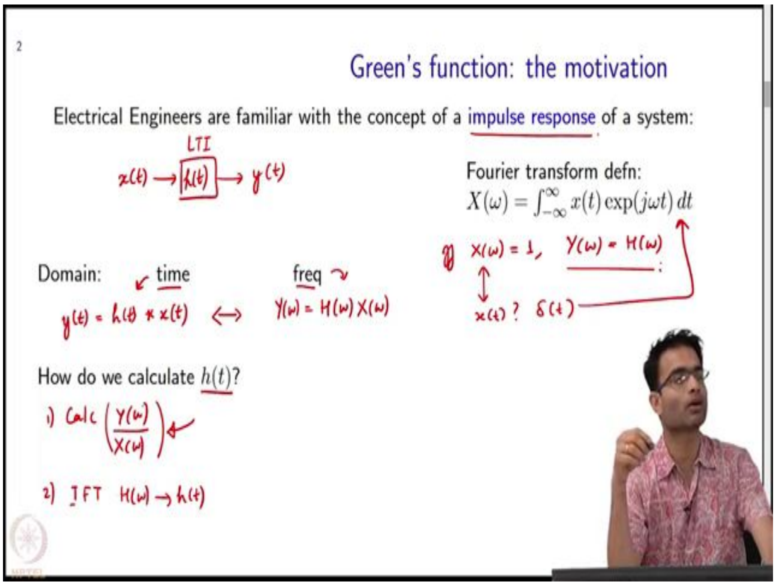

Now, some of the motivation for it is something that electrical engineers are already very very familiar with right and that is a concept of a impulse response. So, everyone who is done you know 1st year of engineering whether it is a you know double E, EP or whatever have seen singles and systems. So, what is the idea here? I send signal x(t) through some LTI system and I say that what is that LTI system characterized by? Any impulse response h(t) right. So, we will just assume that this is LTI ok, Linear Time Invariance system. How do we characterize a response in time domain if I want to express y in terms of x and h in time domain?

Student: Convolution.

If a convolution right so I just write it in time domain I write this as y(t) is equal to the convolution of x(t) and h(t) alright pretty straight forward. And I mean all of you will know by now that working in the time domain and a convolution is a little bit more painful. So, the more convenient way to handle it is frequency domain right. So, in the frequency domain the convolution becomes?

Student: Product.

Product right; so this becomes $Y(\omega) = H(\omega)X(\omega)$. So, given any system our first objective is let me characterize a system means let me find out h(t). Once I find out h(t) for a for a LTI system life becomes very easy because I either do the time domain version or the frequency domain version to get the response right. So, the question is how would I calculate h(t)? What kind of procedure would you follow for calculating the impulse response of a system?

Student: $Y(\omega)$.

$Y(\omega)/X(\omega)$ right.

Student: And take the inverse.

And take the inverse Fourier transform ok. So, in other words so, we will do first is we will calculate $Y(\omega)$ for a known input $X(\omega)$ and second is do a inverse Fourier transform right. So, that will take $H(\omega)$ to h(t) or I can stay in $H(\omega)$ in the omega domain itself if all my calculations are going to be done in the frequency domain. Now this is the main thing that I want to calculate. So, what will be the, if I do not want to do any additional calculations, I just want to give some input to the systems I said my output is itself the impulse response. So, looking at this equation what is the simplest $X(\omega)$ you can think off.

Student: Delta function.

Simplest $X(\omega)$.

Student: 1.

1 right and so, if $X(\omega)$ is equal to 1 then your whatever I get from the system output is itself going to be equal to $H(\omega)$ right. Now what is the x(t) corresponding to this?

Student: $\delta(t)$.

It is a delta function right. So, if I put $\delta(t)$ into this function over here right the location of the delta function is included in the integral and I am going to get a $X(\omega)$ equal to 1 right. So, as a result whatever Y I have got I call it the impulse response ok. So, this is something very very familiar right given unit impulse to the system whatever I get as the output is my impulse response. Same idea is being called by a different name is called Green's function.

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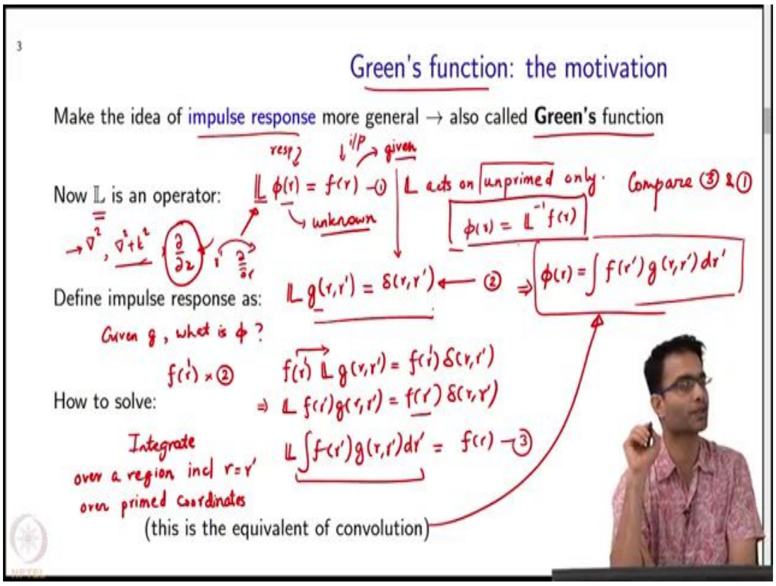

Exactly the same concept in signals I worked in time and frequency now I will work in space and spatial frequency. So, that is the only slight difference. So, this Green's function is the same as an impulse response is nothing different; what will get generalized this in space I can talk a one dimension two dimension multiple dimensions right.

So, it will be impulse response corresponding to so many dimensions that is the only idea. Now when we looked at the surface integral formulation you already saw that knowing that function g was very useful because it helped us to simplify everything else right. So, let us just do a sort of a brief review of what we already know, but in a little bit more formal language.

So, I am going to say that let us say that this L is some kind of an operator, this example of an operator can be let us say ∇^2 or it can be $\nabla^2 + k^2$ depends on the problem whatever it is let me call it L and its an operator. So, it adds on let us say some function let us call that $\phi(r)$ and I get f(r) out of it ok.

So, now the sort of notation is clear to you right there is some operator which acts on some function and output is f(r). Typically, f is known to you, f is the given forcing function and I want to find out ϕ ok. So, typically this is given. If it were the other way if I were given ϕ then this is trivial to calculate f right I just apply the operator on it and I get f. So, that is not very interesting what is interesting is given f(r) find out what is $\phi(r)$, the other thing I want yeah.

Student: What is the L operator and a sum operator?

L is any operator general we are not getting into the specifics of it can be any operator any like a like a differential operator over here plus some constants or whatever or it would be something simply something like this will keep it as general as possible. So, looking at this kind of a lets for example, take this you can notice that this operator the way I have written it only acts on unprimed coordinates. In fact, this whole equation has only unprimed coordinates in it ok. So, I am setting it up for the case of the Green's function where one more coordinate comes in the prime coordinate ok, but we will just make a note now that L acts on unprimed only.

So, now how do I define the impulse response? How did we define the Green's function earlier in terms of the language of L? Remember we had last time an equation like this and we wrote a definition for g. So, how will you write that now? So, some operator L acts on.

Student: g.

g; g remember I have defined as a function of two variables right. What should I equate this to?

Student: Delta.

Delta right, this is the definition of Green's function right, $Lg(r,r') = \delta(r-r')$. This is the response of the system when the given input is a delta function right, as we discussed that is the very definition of this system. So, in other words the system response when the input is a delta function is called the impulse response and in this language that we are using now its called the Green's function. So far we have not done anything fancy we have just used definitions.

Why are we assuming its a LTI we are.

Student: We are assuming is delta.

We are assuming it is a LTI system. So, the impulse response will be the response to a delta input, L is like a think of L just as for example, like this.

Student: Here by.

A differential equation system.

Student: Some system (Refer Time: 09:08).

Yeah. So, a system can also be characterized by means of a differential equation right. So, just think of it like that yeah direct delta is a input and g is the response of the system to it ok, it is that function which satisfies this equation right. The only function that will satisfy this equation is the impulse response itself and hence the name in impulse response ok. So, if I look at this system, so, this is the input and this is the response, I know that the notation looks a little different ok, but its.

Student: Opposite.

Yeah, I mean its opposite means when I am trying to solve this problem what I am actually formally trying to do is I am trying to find out this is my objective right, but I can I am writing it in this notation for now ok. So, all your intuition of signals and systems carries forward over here and this equation you have already been once before when we wrote down for the wave equation right.

Remember we had $(\nabla^2 + k^2)g = -\delta(r - r')$. So, I have not put a minus sign over here previously we had a minus sign. So, that is another matter of convention with these Green's function some people will define with a minus some people define with a plus does not matter ok.

Now, this impulse response is. So, if I if you are given the impulse response like in the previous slide how do you find the response of the system convolution right given h(t) convolved with the input get the ok. So, now, if I am given g right, so, if I am given now I am given g what is ϕ that is my question? In other word I want to solve this equation formally, how do I do it? So, if I tell if I call this equation 1 and call this equation 2 I want to use 2 in order to solve 1. So, if you recall the tricks that we had used for that surface integral what would you do? Multiply.

Student: (Refer Time: 11:23).

In fact, this answer is simpler.

Student: (Refer Time: 11:25) d of another.

So, I multiply the second equation by what?

Student: f(r).

If I multiply the second equation by f(r) why?

Student: Because then you can a phi of if you integrate then you can have.

Exactly he has a right idea right. So, if I take 2 multiplied by f of what should I multiply it by? Let us start by saying let us multiply by f(r) what will happen is its an operator remember. So, I am multiplying on the left hand side ok. So, maybe I should let me just write it like this f(r) into equation 2. So, $f(r)Lg(r,r') = f(r)\delta(r-r')$ ok. Can this f(r) move across this L? No, right what do I want if I ok. So, as he suggested the intuition is that if I multiply f(r) and now integrate this delta function on both sides what will the right hand side become?

Student: f(r').

If it will become f(r') right so, I will get the RHS of equation 1 almost, but the left hand side does not look like L applied to something, its f multiplied by L acting on something. So, I am not exactly getting what I want. So, is there a small change I could do?

Student: f(r).

Multiply by f(r') instead. So, if I do this f(r') . Now L operator it acts only on unprimed. So, it is a constant as far as L is concerned right. So, L for example, is you know d/dr, if I multiply r'; r' has no relation to r right. So, r can move all the way across this L operator in that case this becomes $Lf(r')g(r,r') = f(r')\delta(r-r')$. And now as already been suggested I integrate right. So, if I integrate over a region, what is important over here I should include what point?

Student: r'.

r=r' otherwise I am integrating 0 on the right hand side. So, this will become, and I am integrating over which coordinates primed or unprimed?

Student: (Refer Time: 14:12) primed.

Primed coordinates.

Student: Yeah.

Right because.

Student: Because there might be a.

Then another reason is that then I can move that integral past L because L acts only on unprimed right. So, now, integrate over primed coordinates. So, this becomes that is the left hand side and right hand side becomes right. Now, if I look at this equation. So, let us call this 3. So, look at compare 3 and 1 what is the right hand side of these two equations? Is the same f(r) f(r) what is the left hand side L acting on something; L acting on something right. So, if I just compare these guys what can I say.

Student: $\phi(r)$.

So, you notice I took the impulse response and I did this integration right this integration is actually what you have seen in signals and systems to be convolution ok, but this is the more general way of writing it. So, given the impulse response given the input to the system f I give you the output. So, all the hard work goes into finding out g, a lot of hard work will go into finding out g that is solving equation 2. Once I solve equation 2 give me any number of different inputs all I have to do is evaluate this integral right.

So, that is why this Green's function holds the same importance as the impulse response does in LTI systems, it completely characterizes the system ok. What I have not mentioned in this very general derivation is that this Greens function should also obey the boundary conditions of the problem that is implicit in this; when we take the explicit examples it will become clear, how these boundary conditions are being satisfied.