Computational Electromagnetics
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Review of Vector Calculus
Lecture – 1.4
Corollaries of these theorems

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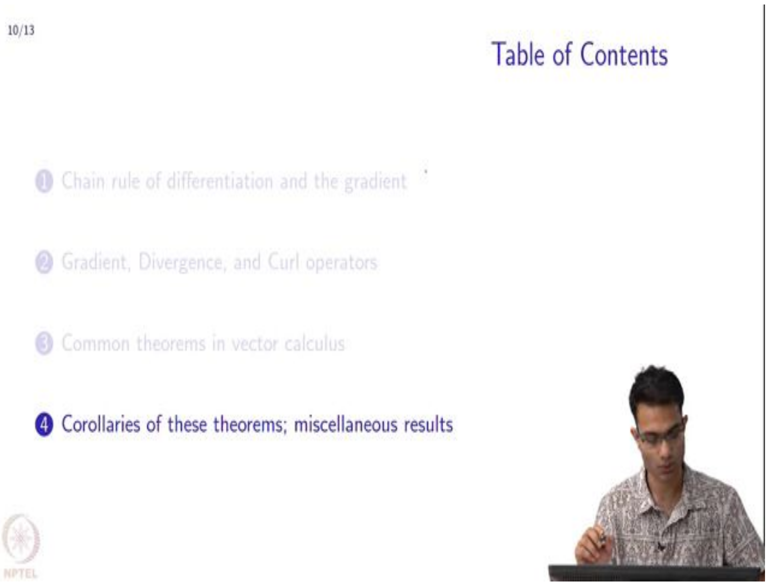

This bring us towards the last aspect of today's module some Corollaries of these theorems ok. So, the hard part is done; now let us make use of these theorems ok.

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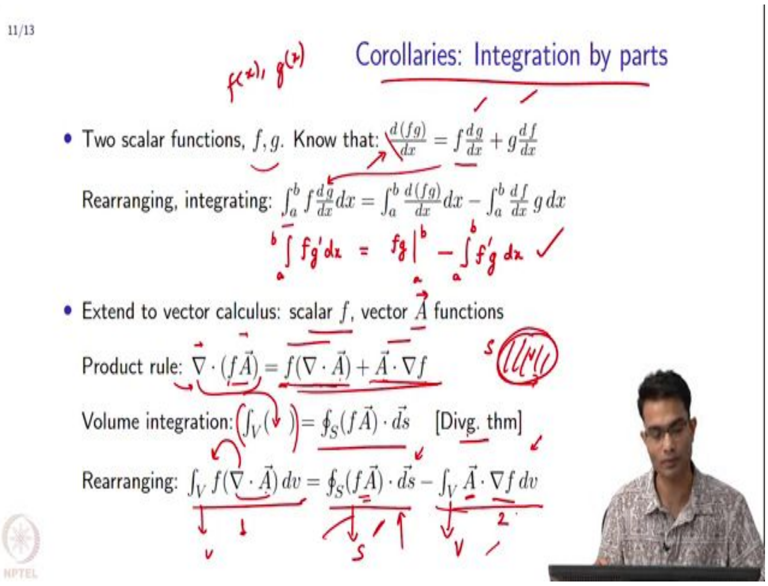

One very important part particularly in finite element methods is integration by parts ok. So, we will start with again something which you do in class 10 hopefully which was integration by parts for a scalar function. So, let us take two functions over here $f(x)$ and $g(x)$; two simple functions of one variable ok. You apply the product rule over here $\frac{d(fg)}{dx} = f\frac{dg}{dx} + g\frac{df}{dx}$ right two terms, everyone knows this. Now if I want to integrate this so what I am going to do is let us say I do an integral from a to b and just rearrange the terms ok. So, this term over here $\int_a^b f\frac{dg}{dx}dx = \int_a^b \frac{d(fg)}{dx}dx - \int_a^b \frac{df}{dx}g dx$ ok.

So, this is actually very familiar to you right. This is your integration by paths $\int_a^b fg'dx = fg|_a{}^b - \int_a^b f'gdx$ that is how you had seen it in high school ok. Minus integral ab derivative of first function, integral of second function dx right this everyone had seen in simple high school. In this course, we are going to upgrade this integration by parts to multiple dimensions.

When I go to multiple dimensions, I can no longer use a $\frac{d}{dx}$; I will upgrade to a gradient operator right. I will no longer integrate just along the line, I may integrate a longer a volume or a surface or whatever right. So, now the first step to do is I am going to take a scalar function f of and a vector function A ok. Let us find out what the product rule says for the case of the vector calculus over here. So, this is a product rule. So, I have its like the dot product of two things over here right; this is a vector, this is a vector.

So, I can take a dot product it is a legal thing to do, it may be a slightly complicated looking expression and this does happen during computational electromagnetics which is why I have taken this example. So, product rule says first function derivative a second function; second function, derivative of first function straightforward. You can verify this by simply by taking components A_x, A_y, A_z and doing it by the definition of derivative you will get the same thing, but just to recap take the first function put it outside derivative of second function.

Second function and derivative of first function, slight complication is I have to take care of products right. So, when I took f over here it is a scalar I am multiplying it by a scalar right. So, this is a scalar A dot something this is again going to be a scalar, left hand side is dot product of two vectors going to be a scalar right. So, it is a legal thing to do; that is one quick way of checking whether you have done a correct product rule. Now what I am going to do is I am going to take this whole left hand side and integrat it over some closed volume ok.

So, this is a closed surface V and outside the closed surface is S. So, we have already seen what to do with an integral of this kind right. So, its integration over a volume of the divergence of something; divergence of something being integrated we just saw by which theorem? Gauss's theorem or divergence theorem, it will become the outward flux of that vector. What is that vector? $f\vec{A}$ right. So, that is my $f\vec{A}.d\vec{S}$ right that I use the divergence theorem ok.

But that was only applied to the left hand side; I can also substitute the right hand side and see what happens. So, if I substitute the right hand side what happens is the surface term over here is as before ok, this term over here the first so the first term over here $f(\nabla.\vec{A})$, this is as is integrated over the volume ok. Second term $\vec{A}.(\vec{\nabla}f)$ that has come on the right hand side, both of these are volume integrals and this is a surface integral ok. I have written this term number 1 and this term number 2, I have written term number 1, on the left and term number 2 on the right you could write it the other way. Term 2 here and term 1 there depends on the problem that you are trying to solve.

Why is this convenient? You will notice is for this surface and the volume that is being enclosed in it this term over here talks about the values of f and A only on the surface right, there is no volume integral over here. So, this term actually becomes a very important way of imposing any boundary conditions in electromagnetic, because boundary conditions as the word suggested applies to the boundary of the problem.

So, this is where boundary conditions can be applied and this remains a volume integral, another convenient thing here you have $\nabla . \vec{A}$. So, it is I have to take the vector field $\vec{A}$ and then take its derivative. Here what has happened is I have changed it to I have taken I do not have to take any derivative of $\vec{A}$; $\vec{A}$ is by itself. So, that can help in the case supposing $\vec{A}$ is a very complicated function, I do not want to take this derivative. So, I have transferred one derivative from $\vec{A}$ to f you can think of it that way integration by parts has given me a surface term and transferred one derivative. So, this is something that we will use when we talk about the finite element method later on in the course.

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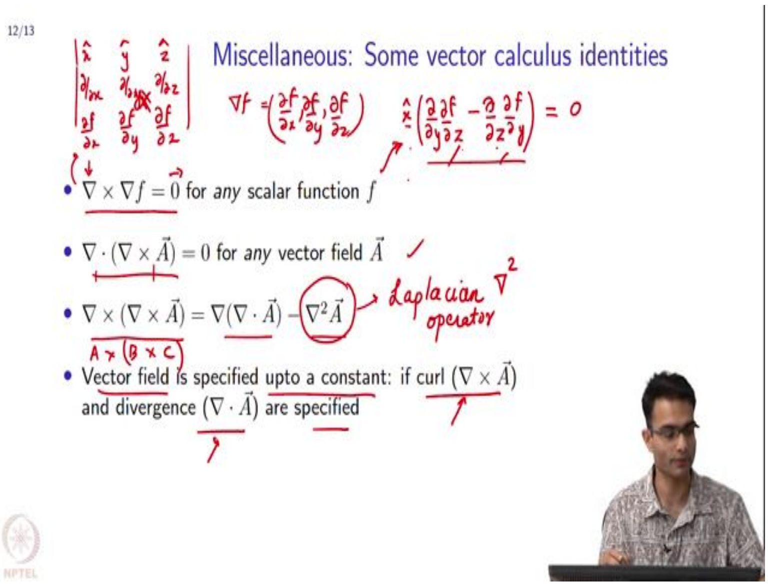

There are a few identities of a vector calculus for example, there is $\nabla \times \nabla f = 0$ for any scalar function f ok; again there is a very simple way of proving this. How would you do it? Well so we have ∇f is going to be in terms of components I can write it like this right and I am going to take. So, I am going to take the vector triple product vector cross product.

So, it is going to be $(\hat{x}, \hat{y}, \hat{z})$ that is what we write down first and first term over here. I will write down its corresponding to this guy. So, partial derivatives y and here is going to be ok. So, let us just look at the x component first ok. So, what is this over here? So, $(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y})\hat{x}$. So, these are the two terms I am going to do right. now there is a.

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It will be done I have only.

Student: (Refer Time: 07:47).

So, when we write this expression out over here, you can see that there are double derivatives second order derivatives. But it is of the same type, there is a theorem from calculus which says that for a well behaved function; the order of differentiation does not matter. So, the in these in mixed derivatives as they are called this order does not matter. So, these two terms are actually one and the same when you get a 0 ok.

So, now, this same this was for the x component, you can apply to the y to the z you will get 0 0 0; you will get actually I should write this as the 0 vector. So, this is it says that the curl of the gradient is 0 right; the next theorem is the divergence of the curl is 0 ($\nabla.(\nabla \times \vec{A}) = 0$). Same kind of a argument, you can apply calculate each term one by one take the divergence, you will get it to be 0. Again this proof depends on the mixed derivatives being independent of the order in which they are applied ok.

So, this will also go to 0 and then there is the analog of a vector triple product; vector triple product earlier I mean in high school, you have seen $\vec{A} \times \vec{B} \times \vec{C}$ something like that right that same thing is generalized over here. And some amount of algebra will give you this expression over here. This term we will encounter a few times; this is called the Laplacian operator right. So, del squared is the Laplacian operator. It occurs throughout engineering wherever there are partial differential equations ok.

The final theorem that we will use is that a vector field $\vec{A}$ is completely specified up to a constant; if the curl and the divergence are specified ok. So, if you tell me the curl and divergence of a vector field, the vector field is fully specified ok up to a constant because these are all derivative operations. If I add any constant, it will not survive right. And so, this is something that is used quite a bit in electromagnetics and we will use it further ok.

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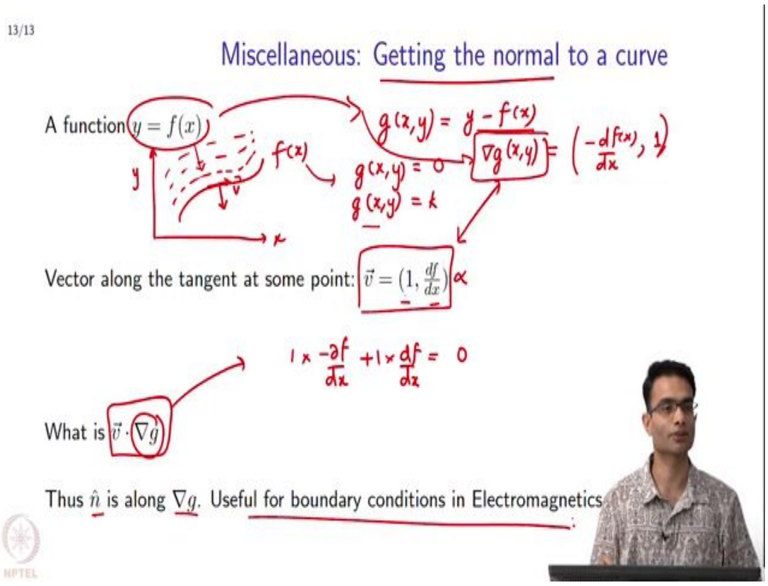

And that brings us to the final miscellaneous use of what we have learned so far which is to get the normal to a curve ok. So, this is a useful trick to learn when we want to apply boundary conditions, we want to get an equation for the normal right. So, I am going to take a simple function y, x. So, this is $f(x)$; it can be any kind of a function and what I want is I want to get the normal ok. So, normal to this curve over here is what I want to get ok. So, it does not seem very straightforward how gradient is going to help me in this.

So, what I will do is I will introduce a new function $g(x, y)$ and I will write it as $y - f(x)$ ok. So, far I have not done anything new. So, this curve over here is given by $g(x, y) = ?$ is equal to what should I set it equal to get this curve? Supposing I set it to 0, then $y - f(x) = 0$ or $y = f(x)$. So, I have captured this curve in a now a function of two variables which is g. Now you can supposing I talk about $g(x, y) = k$ where k is some constant, what will happen to this curve? It will just get shifted right. So, it will get like this right.

So, these are contours or level functions level sets of this function over here. Now if I asked you at this point over here, I want to find out the vector corresponding to the tangent at this point ok. So, I have a function over here we all know that the slope along a curve is given by the derivative right. So, this vector V over here is going to be given by this quantity over here right, you can see that if I take the ratio of the y component to the x component what do I get? df/dx right that is the slope of this function over here.

So, this is there can be some alpha which multiplies over here. I just want a vector along the tangent over here ok. So, this is something that we will keep. Now that I have got this ∇g this sorry, now that I have got this $g(x, y)$. Supposing I want to calculate in what direction does this function change the most perpendicular to this level curves right. So, it is going to change the maximum over here so let us try to just find out. What is ∇g? Look a little mysterious, why we are doing it, but we will soon see and I calculate this. Now ∇g; g is a scalar so ∇g will give me a.

Student: Vector.

Vector right that vectors the components will be. So, let us take the derivative of this with respect to x what will I get? $-df(x)/dx$ because y does not depend on x if I take derivative with respect to y, what do I get?

Student: 1.

1; with respect to z, 0 ok. So, we can as well remove this 0, because we are just working in two dimensions ok. So, this is my ∇g and this is my v. Now find out what is this? $\vec{v}.\nabla g$ what will I get? $1.(-df/dx) + 1(df/dx)$ equal to.

Student: 0.

0. So, what we have found is that this ∇g quantity is actually normal to the tangent over here ok. So, given the curve over here, I constructed a two dimensional function g; I found out its gradient ∇g and this ∇g gives me the direction of n right. So, this is a physical interpretation of gradient, it points in the direction of the maximum change of the function ok. So, this is again very useful for imposing boundary conditions in electromagnetics.

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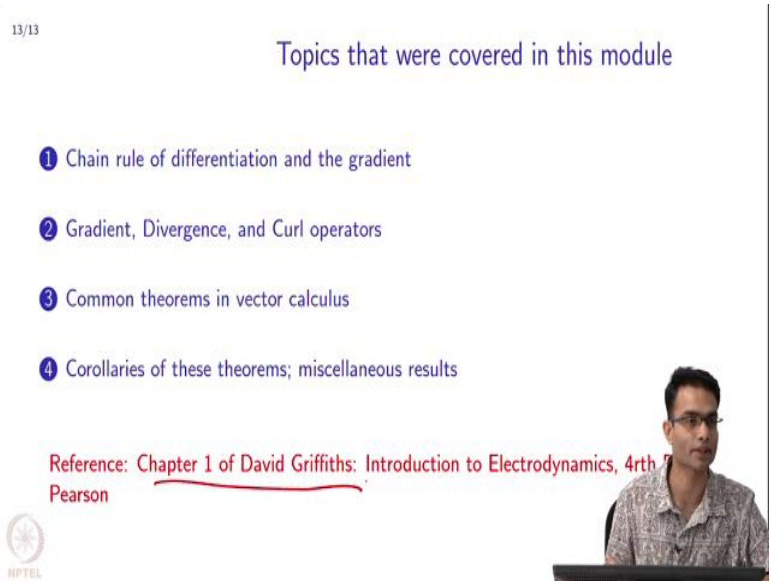

That brings us to an end of this module as reference; I will suggest that you read Chapter 1 of a Griffiths excellent book Introduction to Electrodynamics alright.

Thank you.