Computational Electromagnetics
Prof. Uday Khankhoje
Department of Electrical Engineering (EE)
Indian Institute of Technology, Madras

Applications of Computational Electromagnetics
Lecture - 14.02
Inverse Problems Mathematical Formulation

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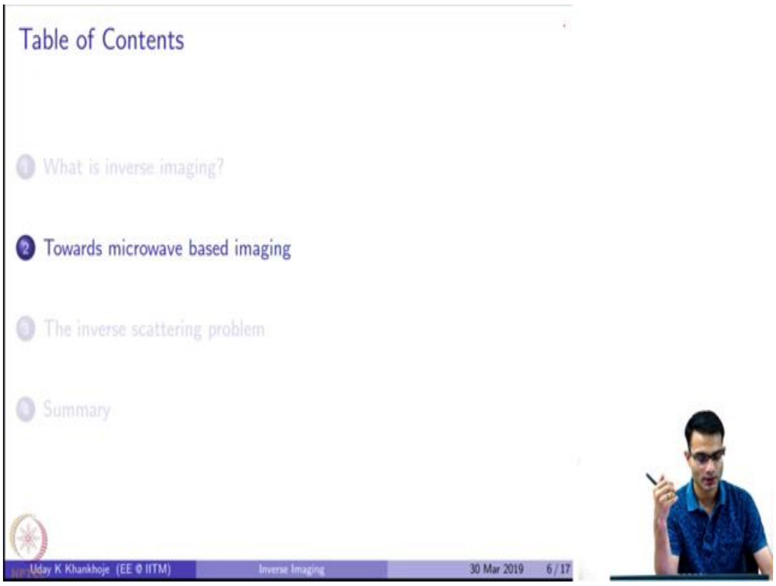

So, now let us go to towards this let us get a little bit more specific ok.

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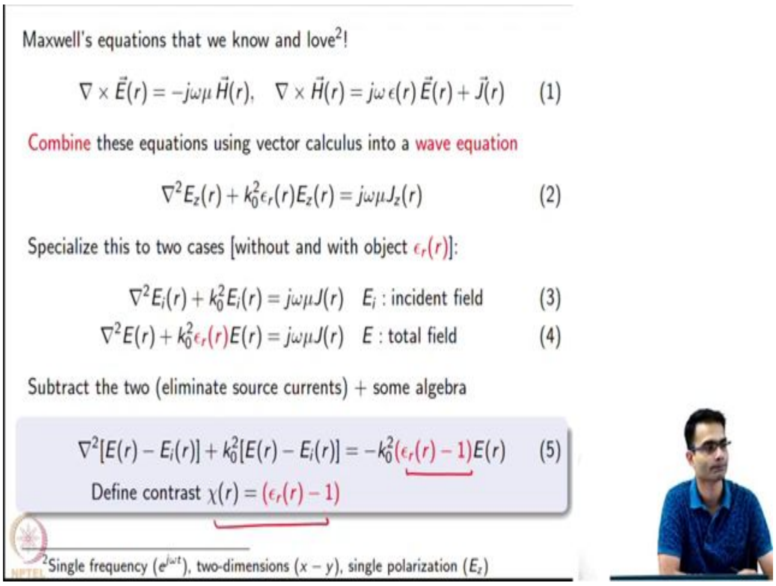

So, all the good nice pictures are done finally, you have to come back to Maxwell's equations right. So, we are going to make some simple assumptions, we are going to assume single frequency $e^{j\omega t}$, and we will make the problem 2D and we will take a single polarization E_z. E_z corresponds to which polarization?

Student: TM.

TM right. TM polarization (E_z, H_x, H_y). So, much of this is going to be familiar to you all because we are going to use a volume integral method right that is the easiest and most intuitive way to formulate this inverse (Refer Time: 01:05) problem ok. So, this is volume integral first equation well ok. So, here are Maxwell's equations

$$\nabla \times \vec{E}(r) = -j\omega\mu\vec{H}(r), \qquad \nabla \times \vec{H}(r) = j\omega\varepsilon(r)\vec{E}(r) + \vec{J}(r)$$

I have put a $\vec{J}$ term over here. What do you do next start? So, this is the revision for you all for volume integral I want to eliminate one variable. So, what do I do?

Student: Take a curl.

Take a curl of the first equation right and play around with the little bit divergence of $\vec{E}$ in the 2 D problem over here. So, I get a wave equation right. We are familiar with all of this right.

$$\nabla^2 \vec{E}_i(r) + k_0^2 \vec{E}_i(r) = j\omega\mu\, \vec{J}(r) \quad \vec{E}_i : \text{incident field}$$

$$\nabla^2 \vec{E}(r) + k_0^2 \varepsilon_r(r)\, \vec{E}(r) = j\omega\mu\, \vec{J}(r) \quad \vec{E} : \text{total field}$$

Then we specialize this in two different cases right, with object, without object. So, with object is equation 4 and without object is equation 3. What is the difference between these two cases? The current sources is there in both cases with and without object the only difference is the objects presence came to be known due to permittivity term that is the only difference right. Any other difference you can think of? Of course, the electric fields are different. They had better be.

Student: Sir mu can also be different.

μ can also be different yeah mu can also be different, but I mean unless you are trying to image some magnetic medium will which is usually like the human body is not a magnetic medium right. So, then $\mu_r = 1$. So, that complication may be not worry about yeah.

So, the only difference is that the electric field changes from incident to total that is the one change and the second change is I have permittivity as a function of space. We have already seen this, what you do is you subtract these two equations and subtract these two equations and I want to keep the left hand side to have only k_0^2, I do not want this ε_r term on the left hand side and we will come to why. Then I have no choice, but to move the epsilon r term to the right hand side right. So, this is our familiar equation towards by now ok.

There is so, you will notice now in everything that follows this $\varepsilon_r - 1$. This is what appears in every equation, will appear in every equation because it always comes as one term together. So, it becomes a little cumbersome to every time write $\varepsilon_r - 1$. So, what do I do is I define a new variable I call it contrast and that contrast is simply this $\varepsilon_r - 1$ ok. So, there is nothing new about it, it is just a convenient way to hold that and make the right hand side look a little bit more compact. Fine, familiar to everyone right.

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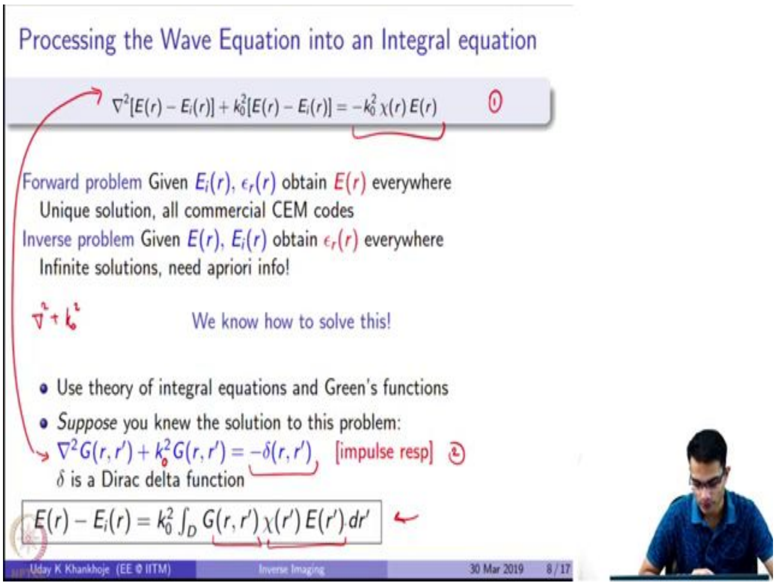

So, I have just rewritten that same equation now, I have the contrast term over here on the right hand side right. And, actually, at this point we can make a distinguish distinction between forward problems and inverse problems. We have already knew this intuitively, but forward problem looking at this equation is what? Give me $\vec{E}_i$ and ε_r right and give and as a result obtain electric field everywhere that is what the forward problem, which is what we did in the volume integral formulation right. We assume that for example, the permittivity of an aircraft that was given to you that is the forward problem.

Now, inverse problem looking at this equation is you are going to be given the electric field. So, if I give you incident field and the scattered field from that you can get total field right. So, $\vec{E}$ is given to you, $\vec{E}_{inc}$ is also given to you because you have designed the transmitting antenna right and you want to obtain permittivity everywhere ok. So, just the revision of what you already know in a little bit more formal way. And, this is a real challenge of this problem, the forward problem gives me unique solutions and the inverse problem on the other hand has infinite solutions and we will see mathematically why there are infinite solutions.

So, imagine why this is a challenging problem. You have infinite possible values of ε_r which are giving you the same scattered field. That is what infinite solutions means. That means, there are infinite possible configurations of the tissue which can conspire to give you the same scattered field. So, how do you know which one is the correct one? Actually you do not know. So, you have to make a very good engineering guess about which one to choose, which is why this problem becomes the very good mix of not just electromagnetics, but also signal processing ok. Because just electromagnetics alone, you get infinite solution you cannot solve any further.

So, let us see. So, coming back to this equation we know how to solve this right and that is using the theory of Green's functions right. So, Green's functions, that is why I needed this k naught squared over here. So, this equation and this equation both of these equations, that form of the operator is the same right. So, actually write it again yeah the form of the operator is the same. What is the form of the operator? $\nabla^2 + k_0^2$ right. What is different is, here the right hand side is something and here the right hand side is delta function and how did we solve this equation? Yeah, but how did we arrive there? We took the impulse. We took this Green's function definition. We multiplied this by what?

Student: dc by d r.

The right hand side of the first equation and integrated it over r' right and the equation this equation 2 became exactly the solution to equation 1 and that is written over here over here the solution right. So, it is the convolution of this Green's function which we have spent a lot of time studying in the module on Green's function convolved with whatever was on the right hand side. So, what was on the right hand side? That $\chi(r)\, E(r)$ is appearing over here k_0^2 has to come out and what is all of these equal to? $E - E_i$ right.

So, in this equation, now you can appreciate the forward problem. ε_r is given to you; that means, $\chi(r)$ is known Green's function we have calculated. So, this is the integral equation my unknown is both inside the equation and outside the equation right. So, I solve for E given this right now. So, while that now that is clear let us go further.

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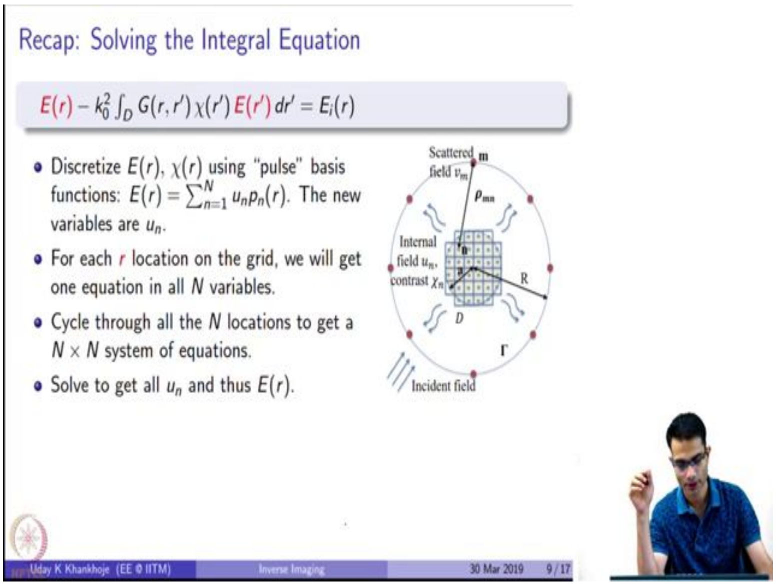

This is how we had solved this integral equation, as I mentioned the unknown is both outside and inside the integral. So, what did we do? We took a domain and we discretized it into little-little squares, in a 2D problems squares, in a 3D problem cubes right and the simplest way to solve this was using what is called a pulse basis function. So, what is a pulse basis? One inside a little square, 0 everywhere else out right; so, you all done this and solved for this your, in your assignments. And, I write this electric field now as a summation over all of these pulse basis functions PNR each of them having an unknown weight u_n right.

So, u_n 's are the unknowns earlier the field was unknown. Now my unknown is a bunch of scalars u n. Will this u_n 's be real or complex valued? In general, complex valued right your p_n is of course, real valued pulse function which is 1 or 0, but u_n can be complex and then how did we solve it? We made this r vector cycle over each one of these points one by one. So, when I take r to be one point, I get one long equation in n variables. What is the thing to be taken care of over there? When I am solving this equation, I have to choose let us there are n pixels.

Student: A singularity.

I have to worry about the singularity when I take my r which is in the red font, when I take r to be one point in the pixel, I iterate over all the other pixels all other pixel except the self pixel. There I have to do the singularity little bit carefully, but it is not a problem because is Green's function we know has an integrable singularity. So, I get it in close form right that is how I get my $N \times N$ system of equations right. So, first solving the forward problem no problem, that is how I got my $E(r)$ right. So, we this is the strategy that we use for the forward problem, now the reason I am spending so much time on recapping the forward problem is because the inverse problem is very similar in structure ok.

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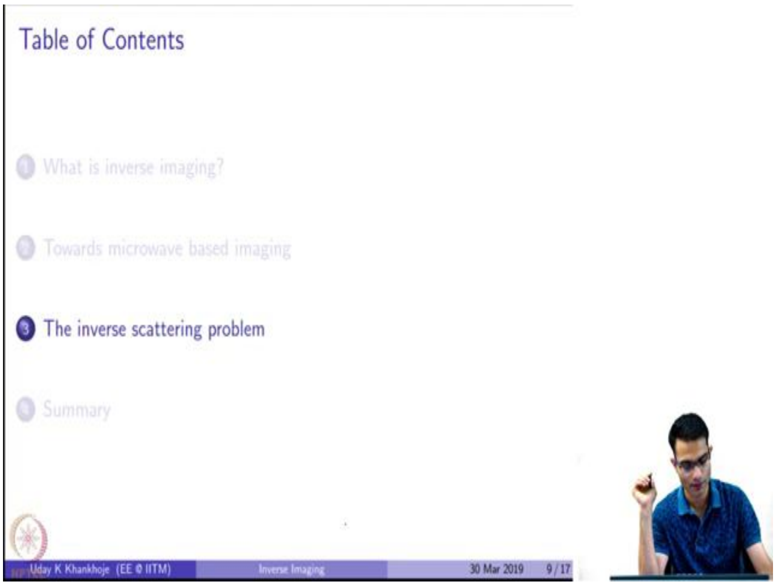

So, let us look at what happens. So, the inverse scattering problem now back to our favorite equation right.

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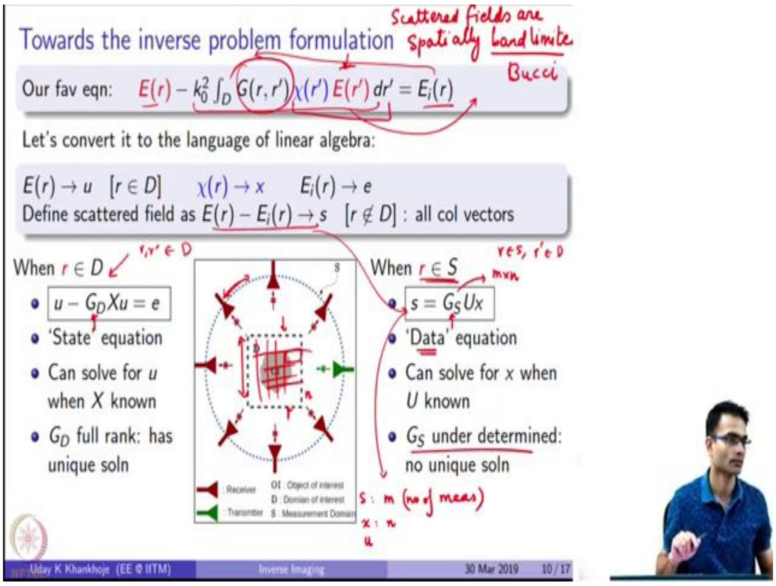

So now, when I want to solve it, I am going to make a few steps to get it into the language of linear algebra right because we do not solve it in continuous variables we discretize it. So, this electric field inside the medium, I am going to call it u it this $\chi(r)$ I am going to call it x, and this incident electric field I am going to call it e ok. So, these are just column vectors; what is given to me is the scattered field right. Now this domain was D. Do I know the scattered field inside the domain or outside the domain?

Student: Outside the domain.

Outside the domain; obviously, because it is a non-invasive method I cannot go inside the domain and measure field. So, I have this electric field that is measured only outside the domain. So, what is scattered field total minus?

Student: Incident.

Incident right. So, $E - E_i$ is being recorded into a column vector s of course, r should not belong to d. So, all of these are complex valued column vectors, straightforward. This is my domain again. So, S is the ring of receivers and D is the domain of interest ok. Now, this equation over here has can take two avatars or two forms as you can see. First is when this r vector belongs to the domain ok. So, this r over here, it is my choice where to set it. So, when

I want to solve the forward problem what do I do is, I choose r to belong to this domain and I cycle over all the internal points get $N \times N$ system.

But let us see what it what happens in terms of linear algebra. So, let us just discretize this equation one by one, one term by one term first term electric field, what happens? It becomes u just take this equation and symbol by symbol let us write it on the language of linear algebra. So, the first term is $E(r)$ which becomes u, next is all of this $k_0{}^2 \int_D G(r,r')\chi(r')E(r')dr'$ is known in the forward problem ok. So, we are going to call it X and then I have a u which is an unknown and the right hand side is known which is e.

So, we call this what this is given a name is call the state equation because it tells something about the state of the domain and if I am if I if you give me X I can solve for u right I can take u common from the left hand side I will get $I - G_D X$ which I have to invert and get the electric field right. And, this G_D matrix it has full rank it is a square matrix $N \times N$ not only its square it also has full rank therefore, it is invertible. So, $I - G_D X$ is also invertible right and it has a unique solution right. So, this all we know now I can choose to move r out of D right.

So, that gives me the second equation. So, when r is outside D ok. So, now what I will do is this E_i term over here I can move it to the left hand side and this whole integral term over here I will move it to the right hand side. So, $E - E_i$ becomes s right for all $r \in S$ this becomes the scattered field small s ok. Then I have this whole integral over here. Now, in this equation when $r \in S$, the kind the form of this is going to be different than what it was over here $r, r' \in D$.

Here what happens is $r \in S$, but $r' \in D$. So, the values of the matrix coefficients will; obviously, be different. So, the integrals will be different. So, I am going to call this a different matrix G_S just to distinguish it from the G_D I there is a G_S matrix and all of these my χ has been replaced by the x. So, it appears over there and electric field is unknown u.

Yeah. So, this is called the data equation because it is giving me a prediction of what my scattered field data should be; because that is what I measure I only measure scattered field.

I measure s, s is given to me ok. So, $G_S Ux$ is a kind of a prediction of what the scattered field should be if you give me U and x. Actually I don't even know U and x that is the problem, but if you want to give it to me I can tell you I can predict that this should be the scattered field.

Student: So, if you are going to figure it out to marked up to d matrix (Refer Time: 15:58).

If you could. We will see what is the problem in that right. So, now we can try to understand the whole problem on infinite solutions why are there infinite solutions. This G_S matrix over here is actually underdetermined ok. So, how many measurements will I make right. So, what will be the length of this s vector? How many of measurements I have that will be the length of s right, if I have 10 measurements on the outside ring then s has only 10 entries.

Only 10 points where $E - E_i$ was measured right. So, if I call that say m measurements, that is your s. Now this domain over here I am going to I want to determine to a very high resolution right. So, I am going to chop it up into various such pixels over here. So, some number n right. So, x is going to be of size n and at each pixel over here, x is the unknown, but what is also there? U : the internal field over here right this E which is sitting inside the integral that is also unknown. And, that will be and the number of unknown electric fields inside the medium is the same as a number of pixels, that I have slash this domain into right. There will not be any difference wherever if you look at this integral wherever there is a χ term there is a E term.

So, if you do not know χ you also do not know E at that point. So, the number of the size of x and the size of u is identical. So, x and u both are of size n. So, in this equation therefore, G_S is of what size? It is a matrix of size $m \times n$. So, this is the reason why this is the underdetermined system of equations and what is the problem with underdetermined systems of equations, how many solutions from linear algebra we know, how many solutions that is under underdetermined system have? Infinite solutions right; so, you can have a solution in the row space, but there are infinite null space components, which can all give you exactly the same s. So, this is the challenge of this problem and.

Student: Can you can you explain (Refer Time: 18:08) ways is equal to m.

m is the number of measurements.

Student: Ok. So, what is the (Refer Time: 18:20).

Ok very good I was waiting for you to ask this question why cannot I; if I want to let us say discretize and find out my permittivity in a 5×5 grid is it 25 unknowns, what prevents me from taking 25 measurements, then I will get a square system of equations no problem.

Student: 20 to 25 receivers.

I have to put, I have a lot of money I can buy 100 receivers if I want; what is the problem? So, actually you do not know enough to answer that question. So, I will give you the answer. There is another theorem in electromagnetics which we have not studied which says that the fields I mean the degrees of freedom in this field on the boundary is fixed. In some in other words the number of variables the maximum number of variables required to characterize the field on this boundary is fixed and it is fixed by something that is determined by the size of the object.

So, if the object size is fixed, the number of degrees of freedom available on the scattered field is also fixed. So, it is if I put more sensors over there, I am not going to get more information; it is like over sampling a band limited field. Once I have satisfied Nyquist criteria will I get more information by sampling finer and finer? No, right so, in fact the theorem is result which says at the scattered fields let me I should write this. So, scattered fields are spatially bandlimited, this is the very famous paper by Italian person called Bucci.

Student: Is there any relation between n and m?

There is no relation between n and m, I mean I mean m for example, the number of measurements you can choose it to be at best how much? What is given to you by Nyquist rate ok? So, this theorem is telling you that the fields are spatially band limited.

Student: So, (Refer Time: 20:30) will not will not be get anything more.

You will not get anything more.

Student: Yeah, but then you still have a number same

You have the number same, but your G_S matrix now is a rank deficient.

Student: Oh.

Right you will get a square matrix, but it is not full rank. And why it is not full rank? Because those yeah sampling signal beyond its Nyquist rate is what happens is that, one row will be linearly dependent one column will be linearly dependent on the other column because there is no new information. So, that is that this is the real reason why you cannot go any further over here.

So, what should you do? You should typically choose a number of receivers to be at the sampling limit. So, that you do not lose any information, but beyond that cannot help you right. So, this is actually one of the reasons why this underground imaging is even more challenging because you are losing important information below the ground, there is no way for you to recover it right.

So, even though I mean yeah that information is lost forever and this band limit depends on basically what is the dimension over here. So, if I want to image you know head size, the head size is fixed for this head size the band limit is fixed. If I make the object bigger then there is more information, but if for a given head, fixed.

Student: If the object size is very small.

Yeah.

Student: Then we can add more receivers.

If the object size is small then the band limit is even smaller so; that means, as very little information. So, you need very few receivers.

Student: We should.

No, see the amount of information is fixed by the size of the object. So, the size of the object gives you the maximum frequency the spatially band limit and that tells you how frequently I should sample this is sampling in the space domain. Now if my object is very small considering the smallest possibility one pixel white right. So, how many unknowns are there? There are only two unknowns over there. So, in principle just two receivers you will get everything.

Student: (Refer Time: 22:36).

Just 1 second. Is that clear?

Student: But where compare to.

We are not dealing with rays this is wave optics.

Student: (Refer Time: 22:39).

So, some more the other field will come to the receiver right there is no probability here. Once I send the wave with probability one is going to hit the object.

Student: Yeah.

With probability one is going to get scattered.

Student: Yeah.

And with probability one it will be received. If it were a ray then there is a chance it hit or not, but this is in the wave optics reigning for sure.

Student: Sir.

Student: We know that m is the basis.

m is the number of measurements

Student: Number of measurements.

Yeah, but yeah good question; so, if I know the band limit to be m why should I not take m measurements only? I mean m pixels m unknowns, but resolution will suffer I can solve the problem, but resolution that I will get will probably be course.

Student: Thank you for. So, how?

You will well, you will not get a clear idea you will get a fuzzy idea what is there.

Student: What.

You will get a fuzzy idea of what is in the domain no that is correct. So, that can be a that is a strategy meant to get a good initial guess, that is correct and let us say a good idea. So, you choose $m = n$ you can at least get a system which you can solve and that is your good initial guess ok. Now, I can give you some typical numbers in the microwave range if I take of cross section of best issue or something the number of receivers is usually no more than 10 to 20. So, m is no more than 10 to 20.

So, n is the square of the number of pixels right m n is the number of pixels. So, if I even say 25 measurements what is the best you could do 5×5. So, if I have let us say an object which is $10\ cm \times 10\ cm$, I am getting information only $2\ cm$, but maybe I need more resolution in that. So, it is workable, but not good enough. So, this is a challenge.