Transmission lines and electromagnetic waves
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Lecture - 15
Maxwell's Curl Equation

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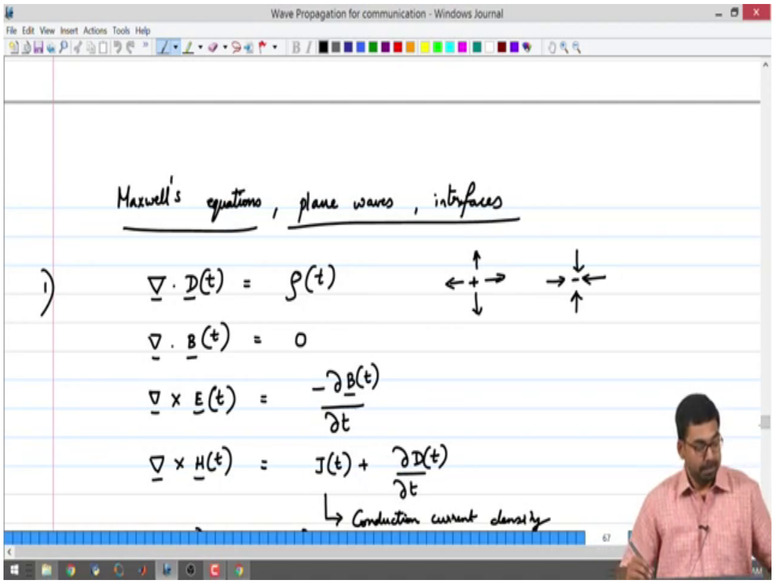

So, we will get started. So, in the last class towards the end we just wrote down the Maxwell's Equations ok in the time domain all right, consisted of two Gauss's Law divergence equations and two curl equations which talked about the coupling between the electric and the magnetic fields. We also saw that when we are writing in time domain there are a few quirks all right, you have to be clear about a the constituent relationships between B and H D and E, in the most general case ϵ is a function of frequency.

So, it is also a function of time μ is a function of frequency, it is also a function of time hence the operator is actually a convolution operator, but in the simple setting we consider μ to be not frequency dependent alright. And μ and ϵ also to be frequency independent alright, but it still gives us a thought about different kinds of materials which are possible with respect to space.

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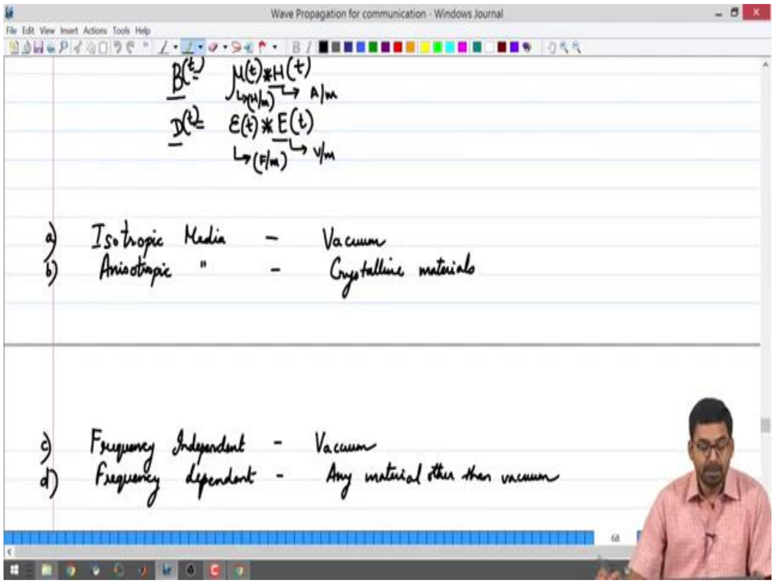

This is the temporal a you know dependence of ϵ and μ, but with respect to the space you could have Isotropic media, Anisotropic media alright and a then Frequency independent and Frequency dependent these are the broad categories of materials, and now we are assuming that we are having an isotropic medium that is frequency independent and vacuum ticks both of these boxes ok. So, it is safe to assume that vacuum is a material that we are considering for understanding most of the basics related to these equations.

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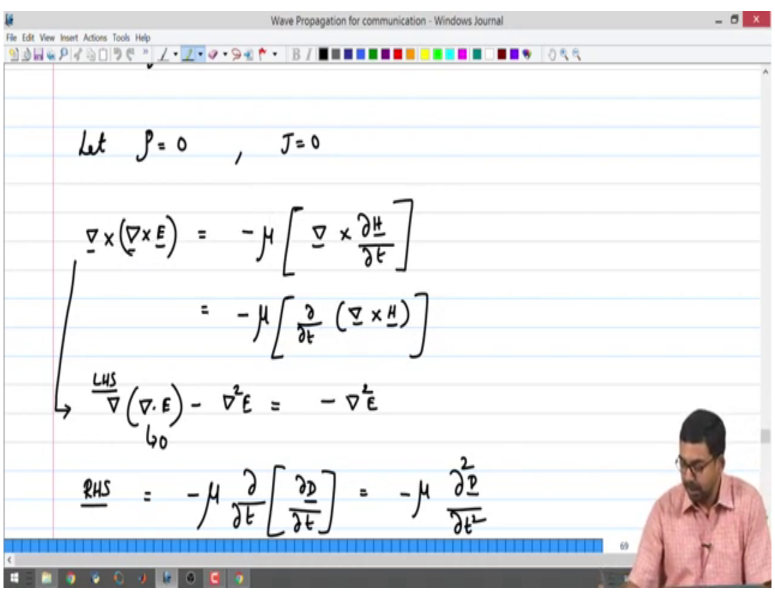

Once we did that we also set ρ equal to 0, correspondingly we set the conduction current to 0 ok.

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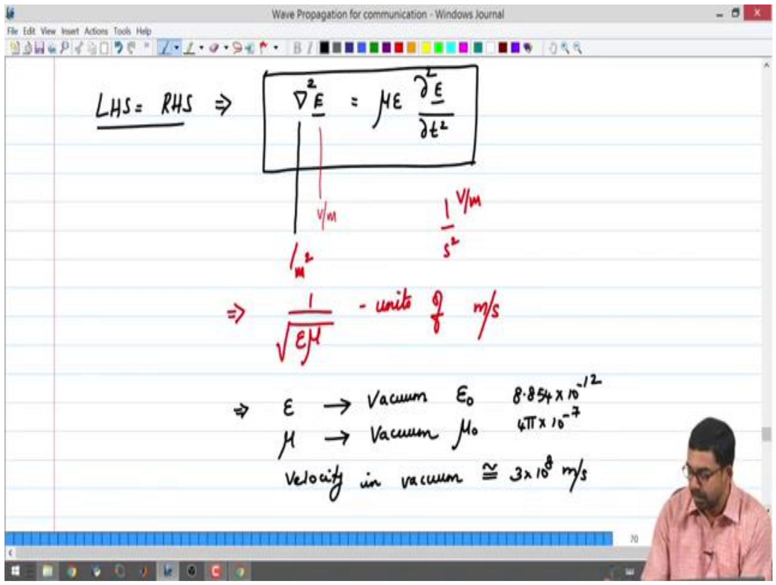

And once we did this we reduced the curl equation right, to a wave equation which is similar to what we had seen in the case of transmission lines. So, in this lecture the idea is going to be to establish the equivalence even more strongly alright, so that you are able to think about Maxwell's equations in terms of telegrapher's equations and the methods to solve them the solutions, what is the characteristic impedance, what is the forward, what is the backward wave in these cases extra ok. It has just established the equivalence very strongly.

So, I will begin once again. Ok the first thing that we can start with is, since we have arrived at the wave equation one of the things that we could do is do a dimensional analysis this is something that we had done with the transmission lines ok. So, the electric field has the unit of V/m, ok right you are taking spatial derivative twice ok, taking spatial derivative twice this means that it becomes per meter square right. On the right hand side I have temporal derivative taken twice, so it will be V/m one by second square right.

For this equation to be dimensionally stable μ, ϵ needs to have some units which will bring the units on the left and the right hand side to be common right. So, it happens that $\frac{1}{\sqrt{\epsilon\mu}}$ has the units of m/s ok, this is analogous to $\frac{1}{\sqrt{lc}}$ having the units of m/s alright. So, one thing is clear that the wave is traveling with a finite velocity ok, even in the case of vacuum ϵ can be replaced with ϵ naught ok.

This is the common notation that is used for the case of vacuum μ is replaced with μ_0 ok. In SI units the values will be $8.854 * 10^{-12}$ and $4\,\pi * 10^{-7}$ alright, and you can always calculate $\frac{1}{\sqrt{\epsilon\mu}}$ will turn out to be a finite value or a finite velocity alright. And that is approximately $3* 10^8$ m/s alright precise values, will be something like $2.997 * 10^8$ m/s extra, but here we can say that the velocity in vacuum ok ok, its $3 * 10^8$ m/s ok.

Now, before going further I will also make sure that you understand the equivalence of the curl equations and the telegrapher's equations a little bit more closely, and there are some subtle differences and the subtle differences have to be clarified now itself ok. So, we will take the first curl equation right, so, I will write down

$$\nabla \times E(t) = -\frac{\partial B(t)}{\partial t}$$

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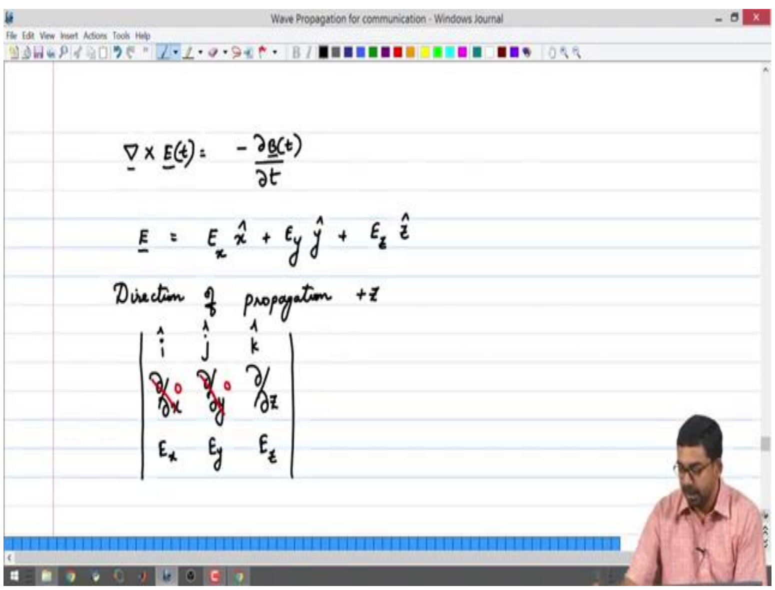

I will start with this equation, to be clear that we are writing this in the time domain one could simply use. Anyway you are taking derivatives with respect to the time on the right hand side, so it should be clear that we are writing this in the time domain, but we are assuming some property of the material.

The material that we are assuming is isotropic frequency independent material, and let us just directly say that we are going to be assuming vacuum ok. In the case of vacuum there are no free charges present, as a result of which there is no conduction current density present. So, ρ is going to be equal to 0, J is going to be equal to 0 ok. Now there are some quirks the way we write this equation let us start with the left hand side. First of all we would have noticed that there are vector notations everywhere ok del is also an operator, but it is a vector operator similarly E is a vector.

Now, one may remember that back when we were dealing with transmission lines and finite a different technique for calculating voltage in a capacitor, we started with a capacitor that was shaped like a cup with a lid on top of it ok. We calculated the voltage and at the end of the program we also calculated the electric fields, one of the things that we noticed in that program was that the electric field was going from top to bottom for sure, but it was also curving towards the sides. So, in a two dimensional plane not for you to describe the electric field you needed at least two components Ex and Ey, then only you can describe the direction of the electric field in that diagram. So, which already tells you that we will have to deal with some Ex, Ey, Ez ok. Now in the most generic case the electric field has components in all Cartesian coordinates ok, it can be written as ok.

$$E = E_x\hat{x} + E_y\hat{y} + E_z\hat{z}$$

So, where x, y and z are the unit vectors in x, y, z Cartesian access ok, Ex, Ey, Ez are the components along x, y and z. This is the way one would describe an electric field and that means that the left hand side becomes much larger than what we would have expected. The second thing that we have to see is this $\nabla \times E$ operator and its consequence ok. In the case of transmission line we had assumed the direction of the wire to be a z direction, we assume that the propagation of the voltage and the current wave was in the z direction voltage is a scalar does not have any direction associated with it and current is also a scalar quantity.

So, we did not consider too much about voltage and current extra, but the direction of propagation was along the wire and we fixed a coordinate system along the wire to be z direction. The problem now is we do not have any wire ok, but we can begin with the simplest case let us assume that the wires have vanished, but the direction of propagation is still z ok, this is a first step that we have to make in order to understand this curl equation a little further. Let us say that the direction of propagation is z, ok I have put z I have not explicitly marked plus z or minus z yet ok and there is a reason right.

Because the wave equation solution will have a superposition of a forward and a backward wave the forward wave will go along plus z direction and the backward wave will go along negative z direction alright, but for now we are dealing only with one curl equation. Just to keep it simple I am just marking this direction of propagation to be z alright, but particularly you can say that I am starting with let us say plus z ok. If this is the case all right, the left hand side is $\nabla \times E$ so we can write down

$$|\hat{\imath}\, \hat{\jmath}\, \hat{k}\ \frac{\partial}{\partial x}\, \frac{\partial}{\partial y}\, \frac{\partial}{\partial z}\ E_x\, E_y\, E_z\,|$$

I have to find the determinant of this matrix ok. Now one of the things that we have to be clear about is if the direction of propagation is z. We assume everything to be uniform in the x and in the y directions ok, which means that, the uniformity in x and y directions mean that there is no derivative present if you take a derivative it will become equal to 0 the x and y directions.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

So, the first and foremost thing that we can do is assume that the direction of propagation is z, we have to cross out these things, and make them equal to 0 ok this is the first step. The next step that we can do is we have to pick between Ex, Ey, Ez or a combination of everything ok, we can say that lets for you know sake of just picking something if you did pick direction of propagation to be ok.

Now, we want to estimate which of the electric field components can exist and which of them can produce a magnetic field on the right hand side, there are not all the options you cannot have Ex, Ey and Ez and I will show the reason why. Suppose you did pick the direction of propagation to be z, and suppose you think that the electric field has a z component ok and let us say the electric field has only the z component ok, it does not have any x or y component which the simpler cases right.

$$\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

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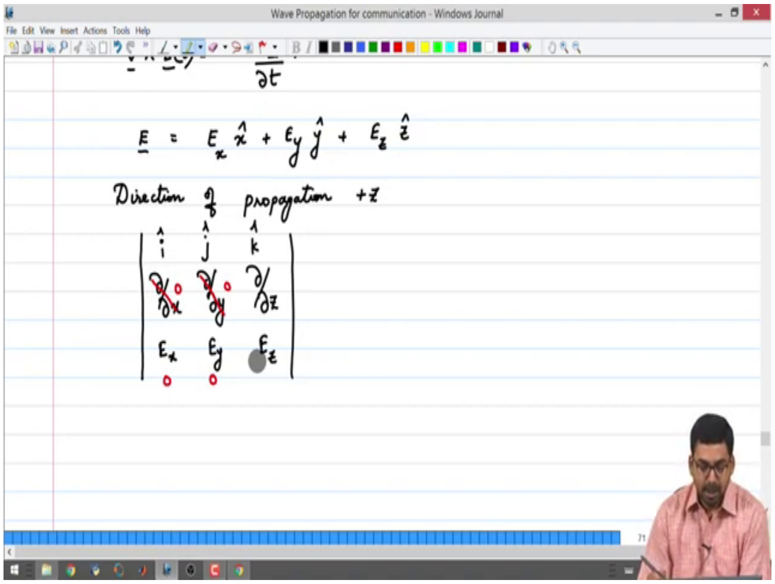

What one can do is you can mark this to be 0 0 and Ez right. If you did that, what will happen is when you are trying to find out the left hand side you will have

$$i(0-0)-j(0-0)+k(0-0)$$

The left hand side becomes 0.

And consequently you can say the right hand side also should become 0, the right hand side is your derivative of magnetic field, which means, that a this spatial change in electric field having only a z component is not creating any time change in your magnetic field ok, so, this is not a possibility the simplest case ok. So, when your direction of propagation is z it is incorrect to choose z in the simplest case ok, and we are dealing with something analogous to transmission lines later on we will see a full vector expression for the electric field and try to see how we can solve them.

But for now we are considering that the electric field is going to be along one Cartesian direction ok so, that means, the option is not to choose Ez all right. So, if the direction of propagation is going to be z you can cross this out ok.

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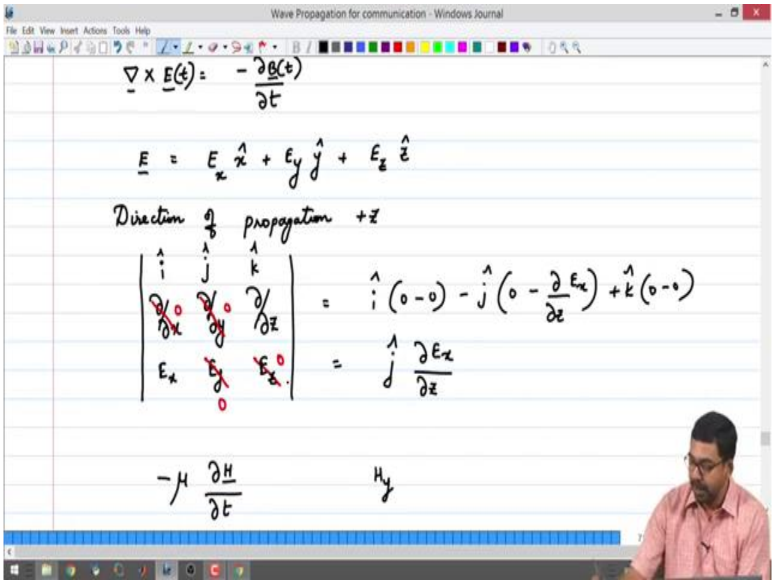

Then you have a choice between Ex or Ey, both of them will give you a non zero magnetic field on the right hand side ok. So, both of them are valid, but just to make you know our job easier, we will say that the electric field has only an x component ok. Let us say that the electric field has only an x component, so, what we are doing now is we are forcing this to be 0 ok we are forcing this to be 0.

So, we are creating some electric fields which have only x components, in other words ok. Now if you take the left hand side right this will become,

$$= \hat{\imath}(0-0) - \hat{\jmath}\left(0 - \frac{\partial}{\partial z}\right) + \hat{k}(0-0)$$

$$= \hat{\jmath}\frac{\partial E_x}{\partial z}$$

So, I started with an electric field that has only an x component ok.

$$\nabla \times E = \hat{\jmath}\frac{\partial E_x}{\partial z}$$

which means, that the right hand side should have a unit vector in the y or the J direction here alright. So, now that gives me an idea about what happens with $-\frac{\partial B}{\partial t}$ ok, so, we can write down $-\frac{\partial B}{\partial t}$ right. So, we can write this as $-\frac{\mu \partial H}{\partial t}$ alright, this means that a μ is a constant for all our cases we have considered isotropic frequency independent materials.

So it is a constant all right, the only thing that can have any vector property is the magnetic field itself, that means, $-\frac{\partial H}{\partial t}$ should be having some J alright, it has to for the equation to be stable and have equal a you know equality you should be having the same directions pointing alright. Which means that the magnetic field can only have a y component ok, so, this means that your magnetic field will end up having a y component.

So, if your direction of propagation is going to be fixed to the electric field, you will suppose only an x component, and the magnetic field will automatically start having only a y component ok ok. This is something that we need to get used to in the case of voltages and currents we did not deal with any directions, but here we need to deal with directions extra.

This is something that you will have to keep in mind you should never use $-\frac{\partial E_z}{\partial z}$ for a z direction propagation, because it does not create any time varying magnetic field right.

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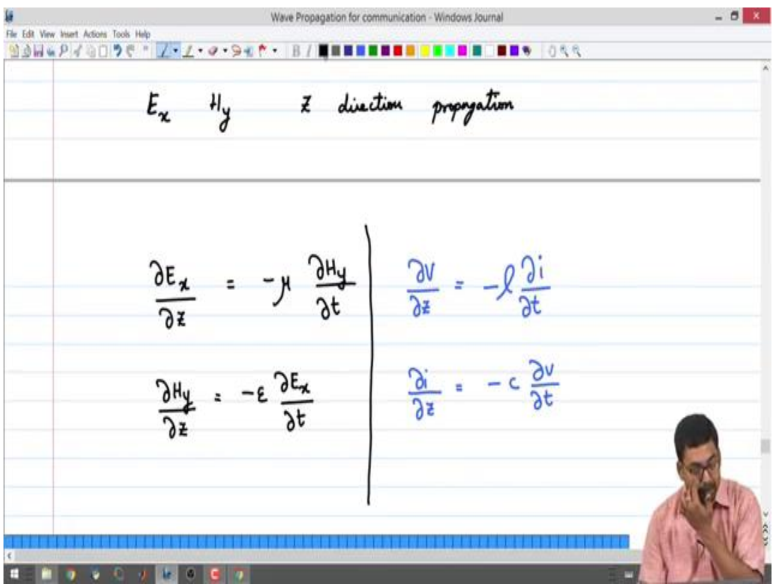

The system is now clear I am going to be having Ex, Hy propagation in the z direction all right ok. Now if this is the case, my curl equation actually does not have any derivatives in x and y, $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ it's just 0, so I will just end up having $\frac{\partial E_x}{\partial z}$ ok.

So,

$$\nabla \times E = \frac{\partial E_x}{\partial z}$$

And I have here minus $\mu \frac{\partial H}{\partial x}$ ok alright. Now let us just go back to our notes and write down what equation we originally had for our transmission line ok. So, I will just write down the transmission line equation on the right hand side to just to establish the equivalence ok. I had $\frac{\partial V}{\partial z} = -I$ ok. So, it is exactly the same if you just change the variables. It's exactly the same, what it means is the computer programs that you have written already are going to be the same to solve Maxwell's equations ok. You do not even need to change the variable because it is just the name of the variable, does not say anything more than the name of the variable.

So, in essence the program that you have written in the past actually solves Maxwell's equations, actually solves wave equations for electric and magnetic fields. It is just that you just call it by different names. That is alright. So, its equality should be very clear now if I want to write down the other telegrapher's equation, I will begin with the other curl equation that I have. I will use the same rules z direction propagation which will mean that I can now only Ex, Hy or Ey, Hx extra, but I am fixing Ex, Hy over here right. Ex, Hy then I will end up with another equation, I will write that equation down right ok.

I will end up with

$$\frac{\partial H_y}{\partial z} = -\epsilon \frac{\partial E_x}{\partial t}$$

the equivalent equation that you saw in transmission line will be

$$\frac{\partial i}{\partial z} = -c \frac{\partial V}{\partial t}$$

And at this point should be abundantly clear that all we are doing compared to transmission line is changing variable names, wherever a current occurs you just changed it to magnetic field and the unit for the magnetic field is ampere per meter alright. You are changing the units so, if you are making the left hand side from ampere to ampere per meter the right hand side should change from say volt to V/m so, the right hand side is becoming E instead of V that is alright.

So, it is almost exactly the same alright, now which means that the conclusions that we drew from transmission line for telegrapher's equation and wave equation are going to be exactly the same as what we can arrive with Maxwell's equations ok, and what could those conclusions be. Let's go back to our, you know, transmission line thing and take each uh quantity that we have calculated and then calculate them back and then try to denote them with different variables.

So, that it becomes clear that we are talking about either a transmission line or we are talking about wave propagation in vacuum ok, that is the only thing that we can do now ok. So, we will have some left hand side quantities we will have quantities on the right hand side, but we have to be clear about whether we are talking about a transmission line or a wave traveling in vacuum ok, just to make it clear we will just mark with different variables ok.

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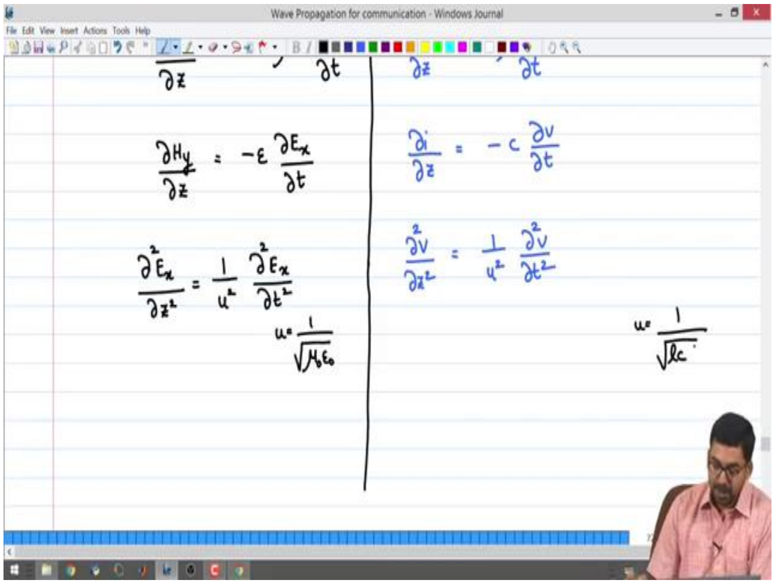

One of the things that we can do is take the wave equation ok, so, I will write down the wave equation for the transmission line on the right hand side. So, I will have

$$\frac{\partial^2 V}{\partial z^2} = \frac{1}{u^2}\frac{\partial^2 V}{\partial t^2}$$

This is the equation that we had in the case of transmission lines ok. I want to write down this equation for the electric field, all I need to do is replace V with Ex right, $1/u^2$ 1 by u square remains $1/u^2$, the difference has to be pointed out, u is 1 by square root of we are dealing with vacuum. So, I am putting μ_0, ϵ_0 ok and in this case we are dealing with some capacitances and ok.

$$u = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Now when we talk about a transmission line it is very common to use the letter u for the velocity, when we are talking about electromagnetic waves u is not very commonly used right, and especially when we are dealing with vacuum extra the latter used is c. Just we will replace some variables to make it clearer that when we are talking about an equation, whether we are talking about an electromagnetic wave or we are talking about a transmission line, ok ok.

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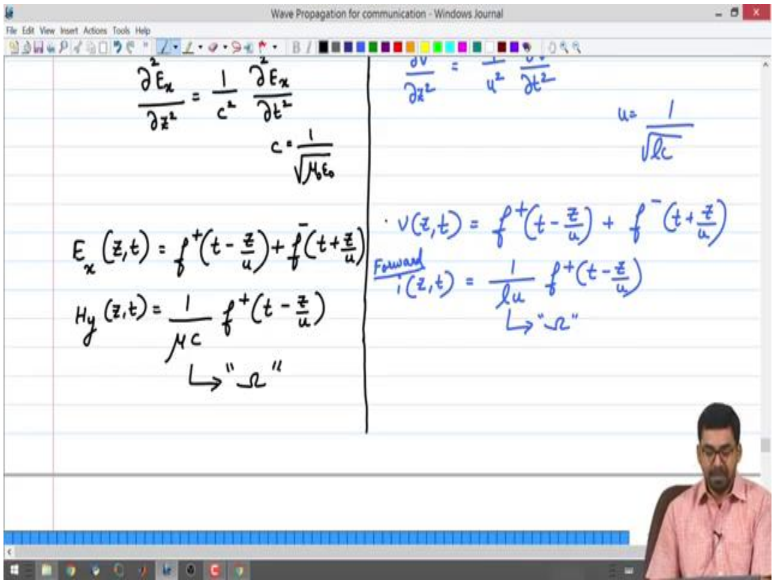

So, it is

$$\frac{\partial^2 E_x}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2}$$

The next thing that we did in the case of transmission line is, we wrote down the analytical solution to the partial differential equation for the voltage ok.

So, we had voltage is a function of position and time is equal to

$$E_x(z,t) = f^+\left(t - \frac{z}{u}\right) + f^-(t + \frac{z}{u})$$

I should have written this in blue, here also I should have written this in blue sorry (Refer Time: 24:59) ok. So,

$$V(z,t) = f^+\left(t - \frac{z}{u}\right) + f^-(t + \frac{z}{u})$$

on the left hand side solution is Ex is going to be varying in z and t. This is something that students have difficulty when we say, Ex it is varying in z and t sometimes they have confusion because they put subscript x you know and then they put it as x comma t extra ok.

This is something that you have to remember, because I substituted Ez for a z direction propagation it will not create a time varying magnetic fields. So, this is something that you have to get used to. This is a common mistake and it happens for many people. Ex varies along z and t ok is equal to, I am going to use the same form right. F of t minus z by u ok. So, we are establishing one to one equivalence in many cases all right, other things that we saw here was that the current ok.

In the forward case it was let us just be clear about this the forward case ok, in the case of the forward wave, right the current was simply voltage divided by the characteristic resistance ok. And in our previous notes I am seeing that we have written

$$I(z,t) = \frac{1}{lu} f^{+}\left(t - \frac{z}{u}\right)$$

From here we had also concluded that the unit of Lu has to be that ofΩ is to satisfyΩ is law this is what we have written, I am seeing the notes ok. Therefore, 1/lu has the unit of (Refer Time: 27:33) so, Lu has the unit of (Refer Time: 27:35) so, Lu has the unit of Ωs ok.

Therefore, I will just mark this has as the units ofΩ's. Now we go to the left hand side, the equivalent of current in the case of the curl questions that we have written is going to be magnetic field right. So, we can just write this down as

$$H_y(z,t) = \frac{1}{\mu c} f^{+}\left(t - \frac{z}{u}\right)$$

Which means that μc in this case has to have the unit of Ω's because all you did was, the right hand side was current alright, I mean, your current was one by lu times voltage alright.

So, you said that lu has the unit of ΩIf you divide current by space ok, to get magnetic field ampere per meter the same way you have divided voltage by distance, so we got voltage per meter, which means that the unit of the in-between thing remains the same ok. This means this also has the unit of Ω so, that means that we are bringing in similar concepts ok. Now we can also write this down in a slightly different way ok. μc, c is already given to be $1/\sqrt{\mu\epsilon}$ and we are talking about vacuum so, I will just mark it as $\mu_0 c$ ok.

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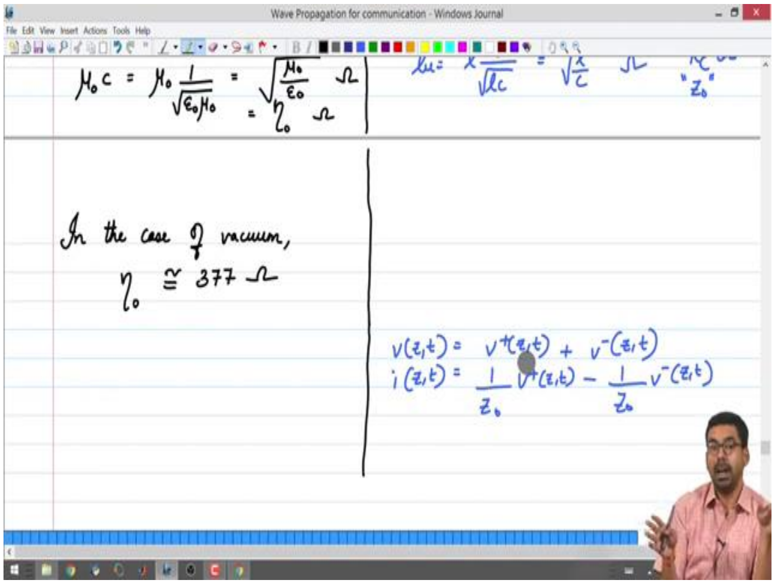

So,

$$\mu_0 c = \mu_0 \frac{1}{\sqrt{\epsilon_0 \mu_0}} = \sqrt{\frac{\mu_0}{\epsilon_0}}\ \Omega = \eta_0\ \Omega$$

On the right hand side you had this to be lu this is going to be

$$lu = l\frac{1}{\sqrt{lc}} = \sqrt{\frac{l}{c}}\ \Omega$$

We called it as characteristic resistance ok, so, we denoted it with R c in the beginning lectures, later on we also used Z_0 ok.

So, I will just put Rc or Z_0 ok. So, when we talk about Rc or Z_0 we are talking about characteristic impedance of a transmission line, in this case to just denote that we are not talking about a wired transmission line, but we are going to talk about fields we just use a different variable alright. So, we say that this is η_0, Greek letter η_0 to denote that it is vacuum and it is inΩ's ok.

In the case of vacuum can also extend this a little bit more, in the case of vacuum the value of ϵ_0, μ_0 are known ok, and I have just written down the value of μ_0 and ϵ_0 a few minutes back. η_0 is approximately 377 Ω's. So, that characteristic impedance of vacuum is 377 Ω's ok, what this means is. If the magnitude of the electric field is going to be 1 V/m, you can expect magnetic field of 1 V/m divided by 377 ΩThat means that your magnetic field is going to be orders of magnitude lower than the electric field is usually ok.

So, for 1 V/m electric field which varies with position, you will be having a time varying magnetic field ok whose value is going to be 1 V/m divided by 377 so, approximately 400 times smaller magnetic field, in terms of value effects is a different question ok. So, electric field values are usually high magnetic field values in amperes per meter are usually going to be lower (Refer Time: 32:44) right, but does not mean that the effects are, I mean the same or lower or anything, but this is just to keep in mind. If you get a solution where your electric field is something and your magnetic field is higher than that maybe you need to relook at your solution and just validate ok what is happening.

The other thing that we can start looking into is this, forward backward extra. In the case of the transmission line the forward wave had a characteristic impedance of Z_0, backward wave had a characteristic impedance of $-Z_0$ because, the current flipped its directions in order to account for the direction of propagation, same things will happen here if you run the simulation. Whatever we had before and assume that whatever you are doing is with electric and magnetic fields you will get exactly the same.

So, there is no harm in thinking that whatever we are seeing here is actually transmission line, so you could model vacuum as transmission line air as transmission line anything which does not have wire, but you have a source and a receiver. Assuming some condition for example, only Ex B, Hy direction is z extra, you can safely consider them to be transmission lines and you know what to expect from a wireless transmission system ok.

And to be more complete I will just write down both the forward and the backward case ok. So, here I had

$$V(z,t) = V^{+}(z,t) + V^{-}(z,t)$$

In the case of transmission lines I should have used blue. I keep forgetting this ok.

And the current is

$$I(z,t) = \frac{V^{+}}{Z_0}(z,t) - \frac{V^{-}}{Z_0}(z,t)$$

And you could write down the equivalent by merely changing V to Ex ok, I to Hy ok and you will have a forward wave of electric and magnetic field and a backward wave of electric and magnetic field ok. Which also means that we could start looking at impedance reflection coefficients ok. So, even in the case of your a homogeneous isotropic medium if it is terminated somewhere ok with a load, a load is usually going to have its own value of ϵ and μ alright, it will have its own value of impedance ok.

If that is the case you can start looking at reflection coefficients for these cases right, then you can start looking at impedance matching for electric and magnetic fields, then you can start to think about quarter wavelength transformers for this alright. Whatever you could do before you could do here all right, but in order to be complete I will go over a few of these in detail ok and leave the rest for you for example, I do not think, I will be doing bounce diagram for this ok, because you already know how to construct a bounce diagram alright. But you can be sure that in the case of transmission lines, one of the things that we saw was the use of a bounce diagram was, suppose you had an abnormal termination of your transmission line somewhere you can expect a reflection or a bounce all right.

And from there you can figure out from the time of flight of the pulse, how far away and depending on the sign of the voltage that you are having whether it is a short circuit open circuit extra. The same thing can be done here also all right. If you are trying to send an electromagnetic wave through vacuum and there is an abnormal thing happening there. There is something that is blocking that and you are getting a reflection back you can tell where the object is ok and you can tell the kind of the object also, maybe the equivalent of short circuit and open circuit should be something over here ok.

So, whatever was possible that is possible here, so this process where we estimated the place where the abnormality is in the case of transmission line was known as time domain reflectometry, just means that in the time domain you measure the reflection two things you will measure, one is the time it took for you to get this pulse back and the other thing is the sign of the pulse itself ok. The same way you could do it here and this would be just known as optical time domain reflectometry or electromagnetic time domain reflectometry that is it alright. So, the equivalence should be clear: the fear of Maxwell's equations should go away, the most important equations are the two curl equations and for us the two curl equations are just two telegrapher's equations ok.

But then what about the Gauss's law, why is that plugged in, why are there two more equations and how do I draw an analogy between that and the transmission lines could be a question to ask? The answer that I always give is do not worry about it for now ok, those are known as implicit boundary conditions ok, we will get to that a little later drawing straight analogy for everything is going to be a little bit tedious not impossible at all ok, however, at this stage you have to worry about the two curl equations the curl equations are the only ones that give you a relationship between electric and magnetic field, remember that the two Gauss's law do not have electric and magnetic field on left and right side so, they do not give you any coupling ok.

I will stop here, in the next class we will write the program also ok for the three cases now that we are proficient in writing these programs and we also have a base for writing these programs, we will try to make some programs for the wave equation. We will also try to solve the telegrapher's equation which is Maxwell's equation using two curls and then try to understand the boundary conditions and what they could mean. What is the equivalent of a short circuit when you do not have a wire?

What is the equivalent of an open circuit when you do not have wires? Ok and what is a reflection coefficient in this case, is the reflection coefficient going to be real complex? I think we know the answer is going to be having a real part and an imaginary part extra. So, we will look at these things, but with a little bit more detail ok. So, I will stop here.