Nonlinear System Analysis
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Lecture- 06
Existence /Uniqueness of Solutions to Differential Equations

Welcome everyone to the 6th lecture on Non-linear Dynamical Systems. So, we will continue with Existences and Uniqueness of Solutions to Differential Equations.

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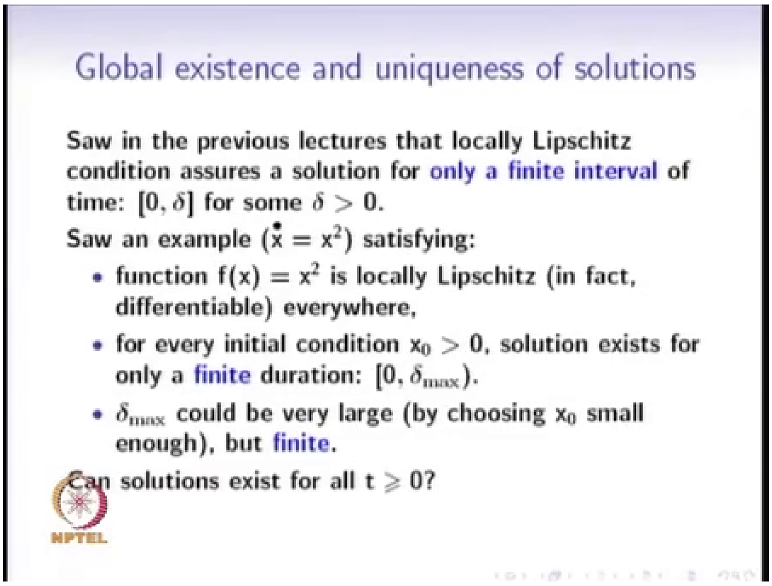

In the last lecture we already saw that under locally Lipschitz condition on f we are assured of a solution only for an interval 0 to delta, this interval 0 to delta may be very large or may be small, but it is only finite that is all that is assured by a locally Lipschitz property of the function f.

We also saw an example x dot is equal to x square, what is in point about this. The function f of x is equal to x square is locally Lipschitz in fact, is differentiable at every point yeah at any point x is equal to 100, x is equal to minus 2000 the function is differentiable hence it is locally Lipschitz at that point. Hence at every point it is locally Lipschitz, but we also saw that for every initial condition x naught greater than 0 the solution exist only for a finite interval for a finite duration from 0 to some delta max.

So, the interval is an open is semi open is what is called a semi open, it is closed on one side it is closed on this side and open on this side. So, for whatever x naught we choose as long as it is positive and nonzero it turns out that the solution exist only for a finite duration. So, this delta max could be very large this is possible when x naught is very small, but delta max cannot be assured to be equal to infinity it is possible to have only a finite duration unfortunately, because this function is locally Lipschitz. We have a finite duration of time for which the solution exists and which is unique, but we cannot have global existence.

So, the question arises can solution exists for all time greater than or equal to 0. In other words, it is can we have conditions on f. So, that the solution exists for all t from 0 to infinity. After all for linear system it is true given that it is true for linear system x dot is equal to Ax, where A is a n by n matrix. We would like to ask the question under what conditions on f little more general than linearity can we have solutions that exists globally on the interval 0 to infinity.

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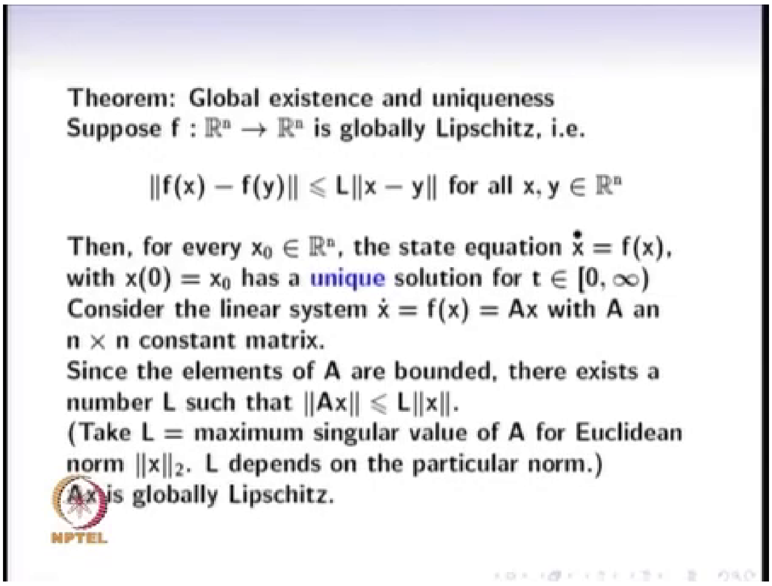

So, here we have a theorem global existence and uniqueness. So, suppose f from R n to R n is globally Lipschitz that is, what is globally Lipschitz. This particular inequality f x minus f y is less than or equal to some L times x minus y norm of x minus y for all x y in R n. There is one constant L that works for all vectors x and y in R n, this inequality is satisfied for no matter which x and y in R n we put the same constant L will work. So, this is what we saw was globally Lipschitz property of the function f.

If this is satisfied then the theorem states that for every initial condition x naught in Rn, the state equation x dot is equal to f x, with x naught as the initial condition has a unique solution, define over the interval 0 to infinity. So, here we have as soon as you assume that there is one constant L that works for this Lipschitz inequality for all x and y in R n that is enough to assure us that there is a solution on the interval 0 to infinity and moreover that solution is unique. So, because we have this 0 to infinity interval we have called this theorem the global existence and uniqueness theorem.

So, consider the linear system x dot is equal f of x is equal to Ax where A is the n by n constant matrix. Since all the elements of A are bounded in fact, they are all constants. There exist a number L such that Ax is less than or equal to L times norm of Ax is at most L times norm of x. There is some constant L that will ensure that this inequality satisfied for all x.

So, what are candidates L for this particular inequality, we could take for example, the maximum singular value of the matrix A when we are dealing with the 2 norm the Euclidean norm as the norm here we have this norm and in general L depends on the particular norm whichever norm you take there will be a constant L such that this inequality satisfied for all x in R n.

So, linear systems x dot is equal to Ax in that Ax is a globally Lipschitz function and hence we have existence and uniqueness of solution over the interval 0 to infinity. More generally, even if we do not have a linear system, if we have a globally Lipschitz property that is sufficient to assure us existence and uniqueness of solutions on the interval 0 to infinity.

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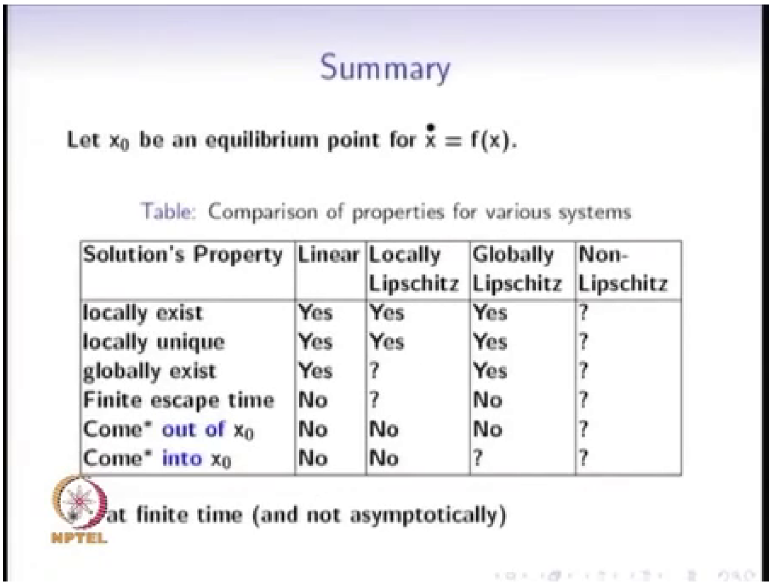

So, we will have a quick summary of the various things you have seen so far. So, x naught let x naught be in equilibrium point for the system x dot is equal to f of x. Then we have seen various properties of the solution locally existence we have yes for linear, for locally Lipschitz f also we have local existence of solutions, for globally Lipschitz f we have locally local existence of solution to the differential equation, for non Lipschitz they are not able to say anything.

What about locally unique of course, it is again yes, yes, yes. For global existence of solutions of course, yes for linear, for locally Lipschitz were not able to assure, for globally Lipschitz yes we just now saw that. What about finite escape time, is it possible that solutions exist only for a finite duration of time beyond which it goes to infinity, this is what we call escape time.

For linear system this cannot happen, for locally Lipschitz we are not able to say, for global Lipschitz it cannot happen, because you already showed that the solution exists over the interval 0 to infinity. So, it cannot escape to it cannot become unbounded in finite time, for non Lipschitz again we are not able to say anything.

The next important question is, is it possible to come out of an equilibrium point? x naught is equilibrium point. So, if a solution starts at an equilibrium point, is it possible that at some time instant, it comes out of the equilibrium point, this is not possible for linear, this is not possible for locally Lipschitz, it is not possible for globally Lipschitz, why because globally Lipschitz is also locally Lipschitz and solutions unique for some interval and hence it cannot come out, but for non Lipschitz this is possible this we have already seen.

Is it also possible to come into an equilibrium point? There is a solution that is initially out of the equilibrium point is it possible that at some time instant it comes and merges with the solution that is always sitting at the equilibrium point, this is not possible for linear, this is not possible under locally Lipschitz property. So, globally Lipschitz also it says question mark here, but it is not possible for non Lipschitz this is possible.

So, what is the significance about this? We might require to reach an equilibrium point infinite time. For example, the steady state the set point we might required to reach their infinite time this is not possible under for linear systems, for locally Lipschitz property of f, it is not possible for globally Lipschitz also, you might require a non Lipschitz dynamical system if you want to reach the equilibrium point at any finite time. So, come out of and come into an equilibrium point we mean here is at finite time.

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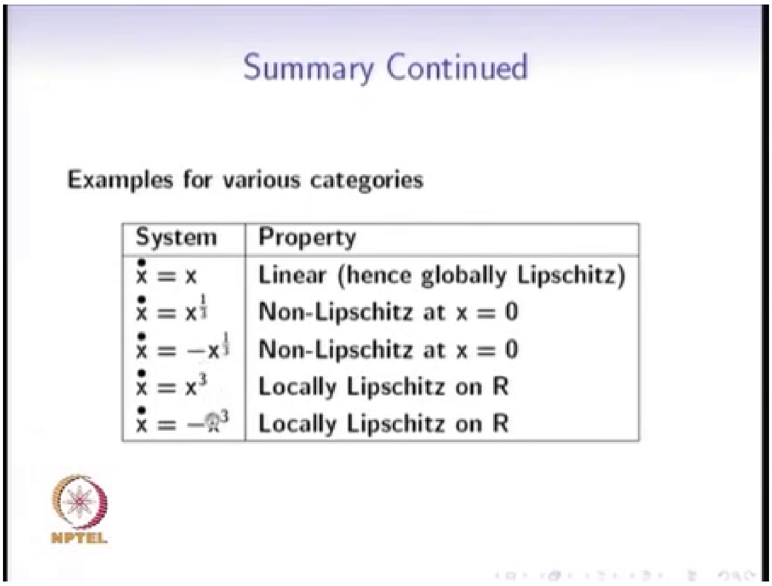

We have also seen some examples, examples of linear system, non Lipschitz unstable, non Lipschitz stable, globally locally Lipschitz, but not globally Lipschitz unstable, locally Lipschitz, but not globally Lipschitz stable ok.

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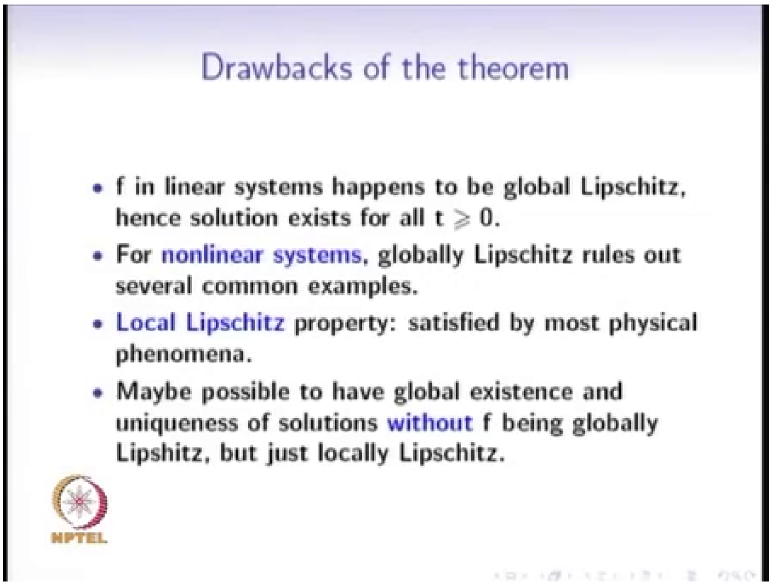

So, we will proceed with one other theorem about global existence and uniqueness that does not assume globally Lipschitz condition on the function f. So, before we see that theorem we will just analyze that theorem about global existence and uniqueness of solutions under the globally Lipschitz property of the function f.

What were the drawbacks of that theorem? f in linear systems happens to be globally Lipschitz and hence a solution for all time t greater than equal to 0. For non-linear systems globally Lipschitz condition rules out several common examples it is too much to ask for a function to be globally Lipschitz. Local Lipschitz of course, is satisfied by several examples and this is something we would like to retain.

So, it is perhaps possible to have existence and uniqueness of solution over the interval 0 to infinity, but without requiring f to be globally Lipschitz. The condition we saw was only sufficient condition for existence of solution from 0 to infinity was that f is globally Lipschitz, maybe there are some other weaker conditions on f under which we will have existence of solutions and uniqueness over the interval 0 to infinity.

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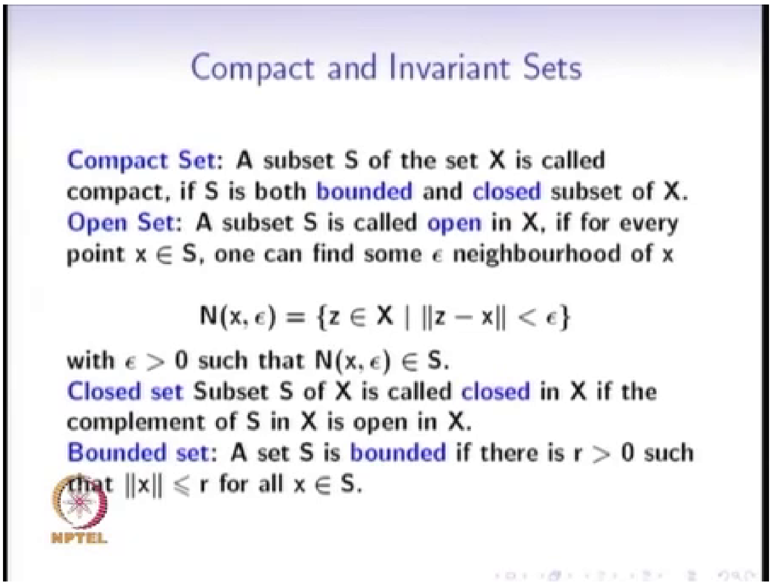

So, this require us to review compact sets open, closed and bounded sets this is something we will quickly review. A subset S of the set X is called compact, if S is both bounded and a closed subset of X. When do we call a subset S open in X, a subset S is called open in X, if for every point x in S, one can find some epsilon neighborhood of x of that point x such that that whole neighborhood is contained in the set S.

Yeah so, this neighborhood is defined like we have seen so far, set of all points which are less than epsilon distance away from the point x, this is an epsilon neighborhood of the point x the set of all points in x such that the distance is less than epsilon and if epsilon greater than 0.

So, if a set S is called open if no matter which point x you take after you have chosen the point x you are able to find some epsilon greater than 0 such that that epsilon neighborhood of the point x is contained not just in X, but in S. When do we call a subset S of X closed subset of X we call it closed in X, if the complement of S in X is open.

This has another way we can define this in another way by saying that the all the boundary points contained inside the subset S, but that we have seen before we will not review that part now. Finally, when we call a set S bounded if all the elements are bounded from some number r in the norm. So, a set S is bounded if there is some number r greater than 0 such that norm of every element x is at most r. So, all elements in S are not more than distance r away from the origin in this case.

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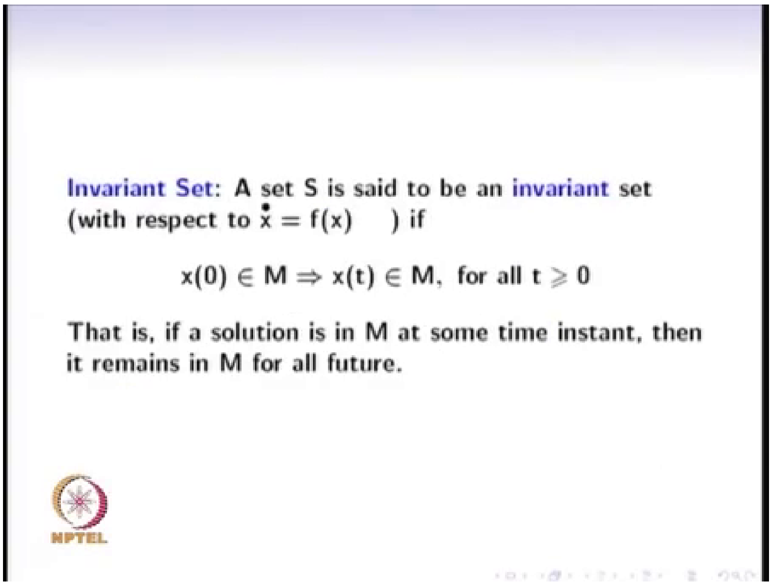

We also need the notation of an invariant set. So, a set S said to be invariant, this invariant here is with respect to some operation in this case it is with respect to the dynamics of x dot is equal to f of x. So, set S is said to be an invariant set with respect to x dot is equal to f of x if, whenever the initial condition is inside sorry there is a small mistake here this S should be replaced by M.

So, a set M is said to be an invariant set, if whenever the initial condition starts inside M, the trajectory is inside M for all t greater than or equal to 0. In other words if the solution is in M at some time instant then it remains in M for all future, that is the definition of a set M to be invariant. So, please replace this S with M.

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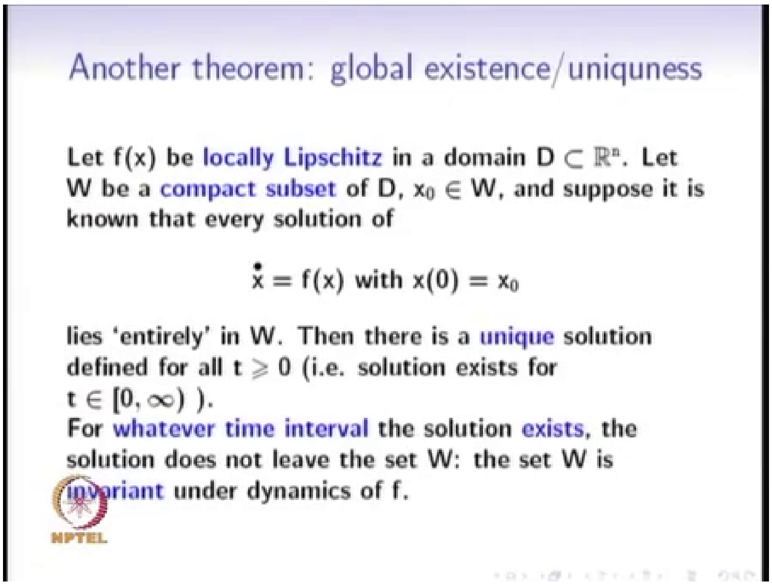

So, finally, we have another condition for existence and uniqueness of the solution over the interval 0 to infinity. So, let f of x be locally Lipschitz on a domain D. So, we are assuming only locally Lipschitz please note, let W be a compact subset of D and this initial condition some initial condition x naught is in W. Suppose it is known that for every initial condition inside this compact subset W we have that the whole solution lies inside the compact subset W.

Suppose it is known that every solution of x dot is equal to f of x, with the initial condition in x naught lies entirely in W and this is required to be true for every initial condition x naught in W. If it is known then there is a unique solution defined over the interval 0 to infinity, the solution exists and is also unique over the interval 0 to infinity.

Notice that we have only local locally Lipschitz condition on f, but we have this additional property that there is some compact subset W such that whenever it starts inside W the whole trajectory for whatever interval it is defined that trajectory remains inside W inside this compact subset.

In other words this W is an invariant set. For whatever time interval the solution exist the solution does not leave the set W, in other words the set W is invariant under the dynamics of f. If somebody gives us this compact subset W which is invariant under the dynamics of f and f is just locally Lipschitz then we have a solution defined not just over a interval 0 to delta, but in fact, 0 to infinity. So, for whatever duration the solution exists that is an important thing here.

So this completes existence and uniqueness of the solution our study about that, we saw locally Lipschitz property, globally Lipschitz property and finally, we have seen that if there is a compact set that is invariant, then also the solutions can be assured to be assured to exists and it is unique over the interval 0 to infinity.