Nonlinear System Analysis
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Lecture - 13
Phase Portrait - Part 01

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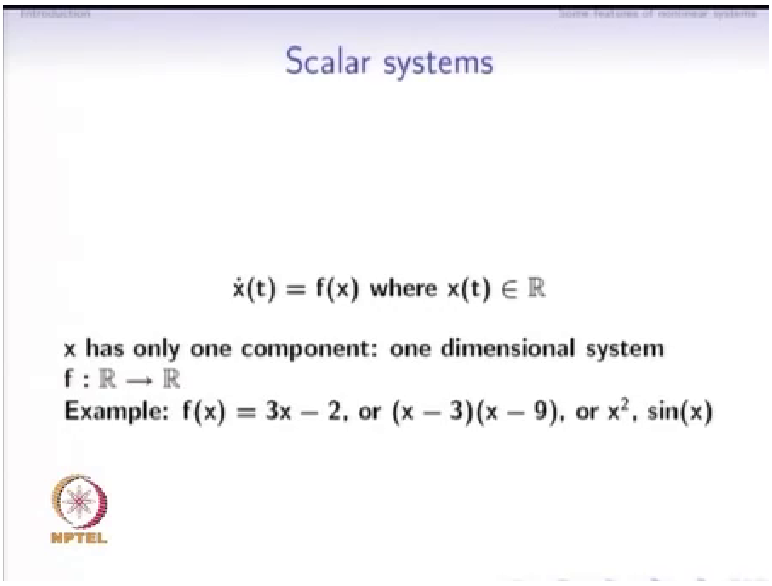

Beginning with the topic of scalar systems. So, consider the scalar differential equation what is scalar about it? x dot t is equal to f of x; where, x of t has only one component it is a real number. So, this is also called a one dimensional system. In such a situation f is a map from R to R for example, f of x is equal to 3 x minus 2 or f of x equal to x minus 3 times x minus 9 or x squared or sin x, these are examples of f that we will see today. So, this particular situation is best seen using a figure.

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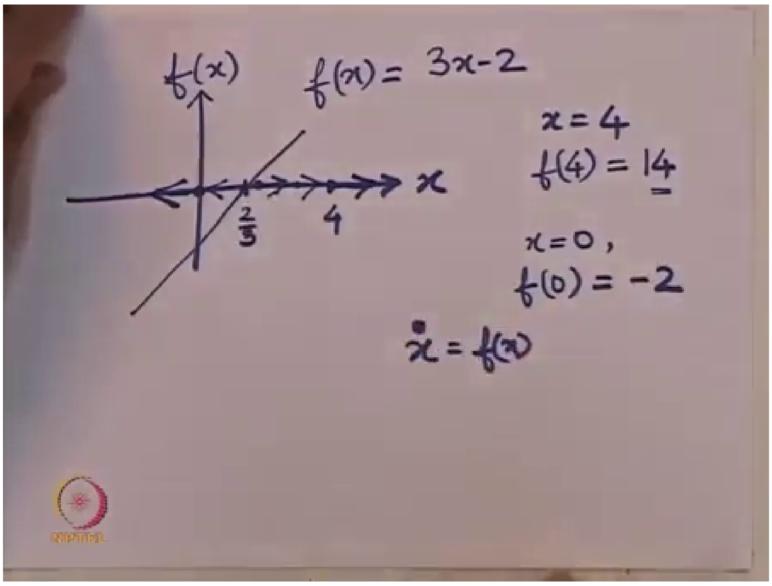

So, we are going to attach at each point a vector. So, consider f of x is equal to 3 x minus 2 and suppose we take the 4 yeah x equal to 4 for this point x equal to 4 we will evaluate f at 4 and we obtain 14. So, what it means? And at this particular point there is an arrow which is starting from the 4 and it is to the right.

Why it is to the right? Because this number 14 that we have obtain is positive. And moreover in addition to being to the right towards the direction of increasing x it is a vector of length 14. At another point for example, x equal to 0 we can check what is f evaluated at x equal to 0 and for that we get minus 2.

So, it means that at 0 we draw a vector which is towards the negative direction of x and it has length equal to 2. So, for a scalar differential equation x dot is equal to f of x; it means that at each point, we can draw a vector to the right or left depending on whether f at that point is positive or negative. So, in this particular situation we see that f of x is equal to 0 precisely at x equal to 2 by 3.

So, suppose 2 by 3 is a point here and f of x is a line. So, now, we are going to plot a graph of f verses x even though x itself was a function of time, we are plotting f as a function of x and we see that we get this line which passes through the point x equal to plus 2 by 3 at this point f becomes equal to 0.

So, at this point the vector has length 0, everywhere to the right we see that this vector is pointed to the right of this point why is it to the right? Because we see that f to the right of this particular point is all positive.

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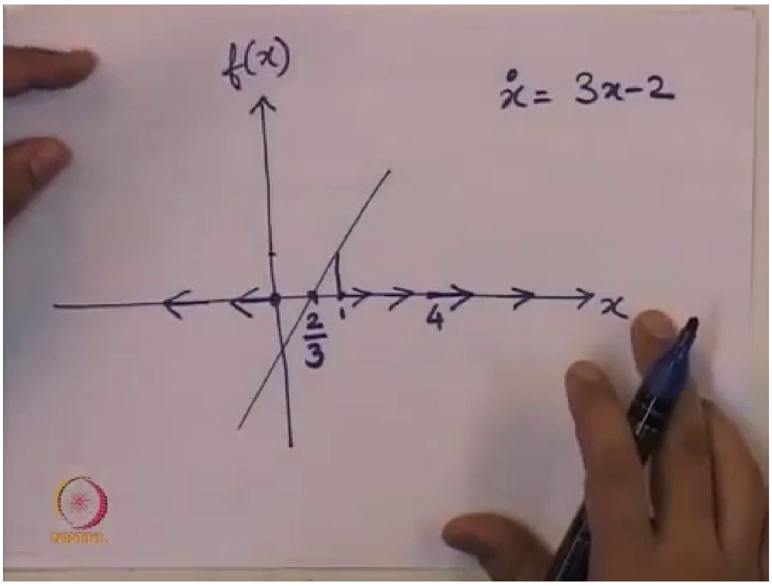

So, this particular point let me draw a slightly bigger figure, were considering the differential equation x dot is equal to 3 x minus 2 and we are going to plot f verses x even though x itself was a function of time and we will also later plot x as a function of time, but for now we are interested in drawing the vector field.

We took a point 4 there is a point 2 by 3 here and there is a point 0 at the point 4 we already saw that the vector is directed to the right at the point 2 by 3 the vector has length 0 and at the point 0 the vector is directed to the left towards decreasing direction of x.

So, what does this mean? That when we draw when we plot f of x versus x we see that to the right of the point 2 by 3 the arrow is mark to the right why is it to the right? Because at this particular point say 1 we see that f is positive at x equal to 1; because f is positive it means x dot is positive in other words x is increasing.

So, more generally we can see that, if we are given with a function f and if f is scalar, we can draw a graph and decide at which points x is increasing which points x is decreasing by just seeing whether f fist positive or negative at that particular value of x, this is a way we will analyze the other examples that we saw.

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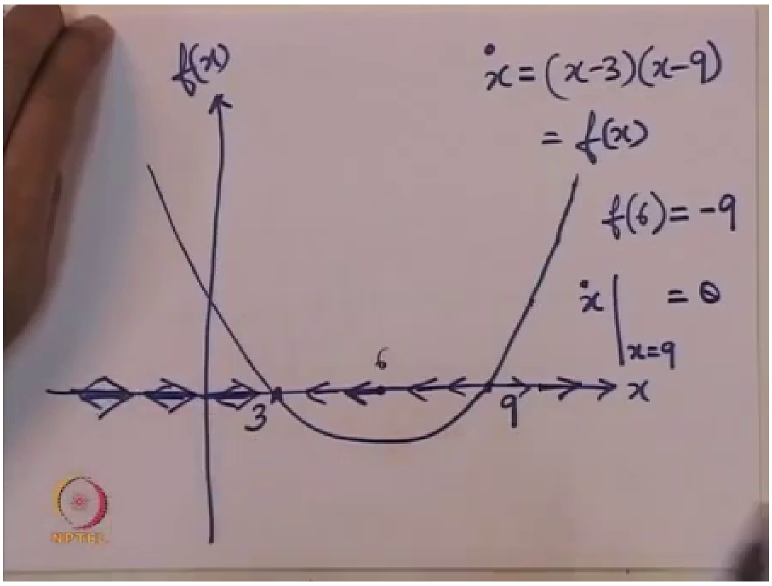

So, consider the differential equation x dot is equal to x minus 3 times x minus 9; which we want to call as f of x. So, this is equal to 3, this is equal to 9 and the graph of this function looks roughly like this. This function has roots at x equal to 3 and 9 and hence it is passing through 0 precisely at x equal to 3 and 9 and if we take a point 3, then we see that the vector at the point 3 has length 0 and hence we plotted neither to the right nor to the left.

On the other hand consider the point 6 at x equal to 6 we expect that f will be negative we can check that f at 6 is equal to 3 times minus 3 which is minus 9. So, since it is negative we can also see that from the graph here, this is a vector pointing to the left towards decreasing direction of x and that we can also see because x dot is equal to f of x. So, at x equal to 6 x dot being negative x would eventually start decreasing it would decrease and that is precisely what this arrow shows the arrow shows the direction in which x will evolve.

On the other hand suppose we take x equal to 11 at x equal to 11, we can easily draw the graph the arrow to the right why because at x equal to 11 the function f takes a positive value. So, we are able to draw all the arrows for this particular example, all we have to do is we have to see where the equilibrium points are and to the left. And the right of the equilibrium points we can draw the arrows towards increasing direction of x or decreasing direction of x depending on whether f takes positive values there or negative values..

Another important point we can notice that if we start at the equilibrium point x equal to 9 then we will of course, remain at 9 because x dot is equal to 0, x dot evaluated at x equal to 9 is equal to 0 and hence x does not change at all, it will remain in the point 9 similarly x equal to 3 is also an equilibrium point.

So, as we can see we have made a mistake here, all these arrows for x less than 3; because f is positive, it cannot be towards decreasing direction of x on the other hand we should be seeing that all the direction although these directions have to be reversed they are all towards increasing direction of x. So, this particular figure I will quickly draw again.

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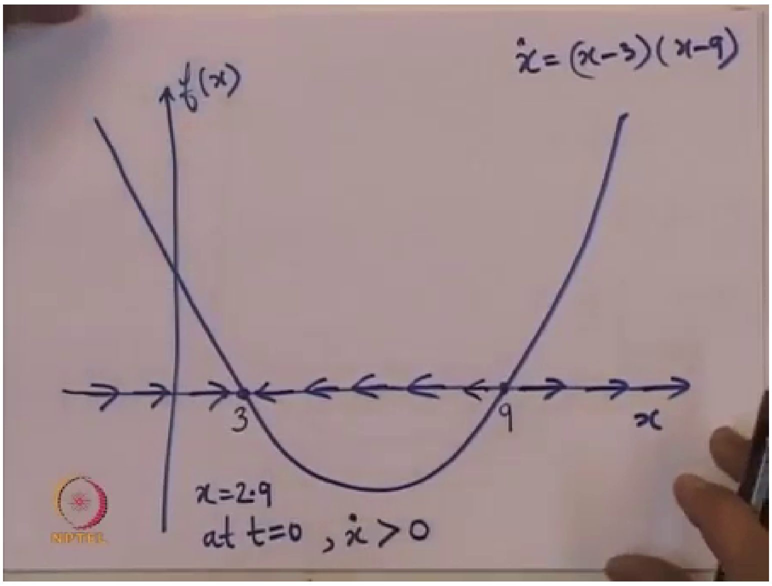

So, this is the differential equation x dot is equal to x minus 3 times x minus 9. Another important features we can see here is that if we start slightly to the right of the equilibrium point of the equilibrium point 9, if we start to the right then x is going to increase and it will go further away from 9.

So, for a very small perturbation of 9 to the positive side of 9, takes that initial condition further away from 9. Even though we noted that at the point 9, the trajectory remains at 9 for all future time, but slightly to the right for a very small perturbation, the trajectory goes away from 9. Also slightly to the left of the point 9 we see that the trajectories are again directed away from 9 why? Slightly to the left of 9 the function f is negative so, the x will become further away from 9 it will further decrease.

So, we will like to say that this particular point is an equilibrium point, but it is also an unstable equilibrium point. For very small perturbations both to the right and left, we see that trajectories are going to move away from this equilibrium point. On the other hand please note that 3 is also an equilibrium point, but for very small perturbations to the right all the arrows are pointed towards the point 3 and we expect that for small perturbations towards the positive direction of 3, the trajectories are moving back towards 3.

On the other hand, if you move slightly to the left of 3 meaning if we start from an initial condition for example, x equal to 2.9 at t equal to 0, then we see that x dot is greater than 0 that is why the arrow is marked to the right and hence it will increase and approach 3 again.

So, this equilibrium point we will like to call is a stable equilibrium point in the context of Lyapunov stability, we will see more precise definitions of stable unstable asymptomatically stable equilibrium point, for now for a scalar system looking at the graph of f versus x we are able to decide which are the equilibrium points we are also able to decide whether this equilibrium points are stable or unstable.

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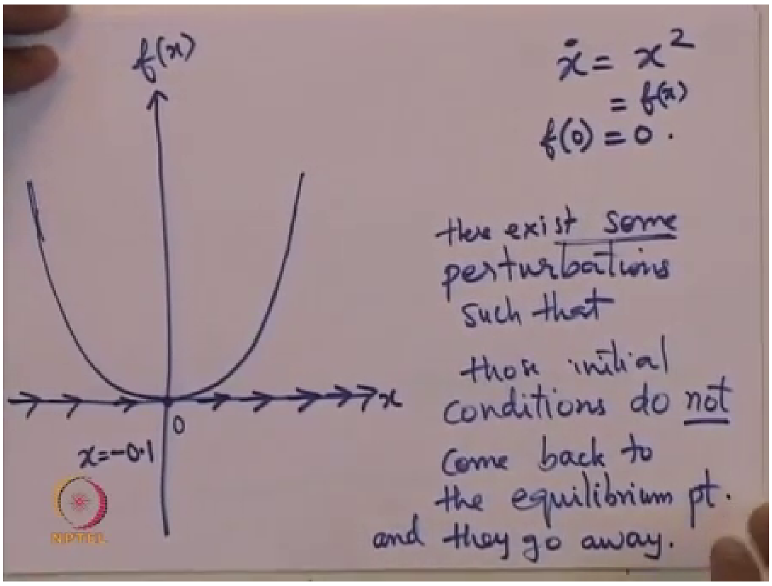

So, now we will see another example from the list of examples we saw. Considered this graph we see that f. So, this is equal to f of x, you see that f at 0 is equal to 0 why because 0 is a root of this. So, we see that the point 0 itself is an equilibrium point if you start there we are going to remain there slightly to the right, we see that f is positive and hence the arrows are directed towards the right.

On the other hand slightly to the left of the point you see that f is again positive and hence again the arrows are directed towards the right. In other words the value of x is going to go on increasing whether it is to the right or to the left of the points 0 and only at the point 0 the value of f being 0 x does not change x dot is equal to 0. So, now, we will like to ask is this equilibrium point stable or unstable?

The property of stable or unstable we like to give only to equilibrium points, this x equal to 0 is an equilibrium point now we see that slightly to the left we see that when x is slightly negative. For example, x equal to minus 0.1, the value of x dot is positive and hence x is going to increase an approach 0. So, can we call the 0 and a stable equilibrium point? We can answer after we analyze to the right of the point 0 to the right x dot is again positive and hence x is going to further increase and become away from 0.

So, we see that for certain perturbations it comes back to 0 and for certain other perturbations in goes away from 0. So, we can say that there exists some perturbations such that such that those initial conditions do not come back to equilibrium point. So, in such a situation they are going to say that this equilibrium point is unstable, when will we call it unstable? There are just some bad perturbations there exists some perturbations such that those initial conditions they do not come back to the equilibrium point and they go away.

In such a situation that equilibrium point is unstable, we are not going to be satisfied with some perturbations which come back to the equilibrium point. We are unhappy that there are some perturbations that are going to go away from that equilibrium point and hence that equilibrium point has been classified as unstable.

So, we will see more about stability unstability asymptotic stability in the following lectures, but we will end this lecture by seeing a similar graph which we will like that is done as homework.

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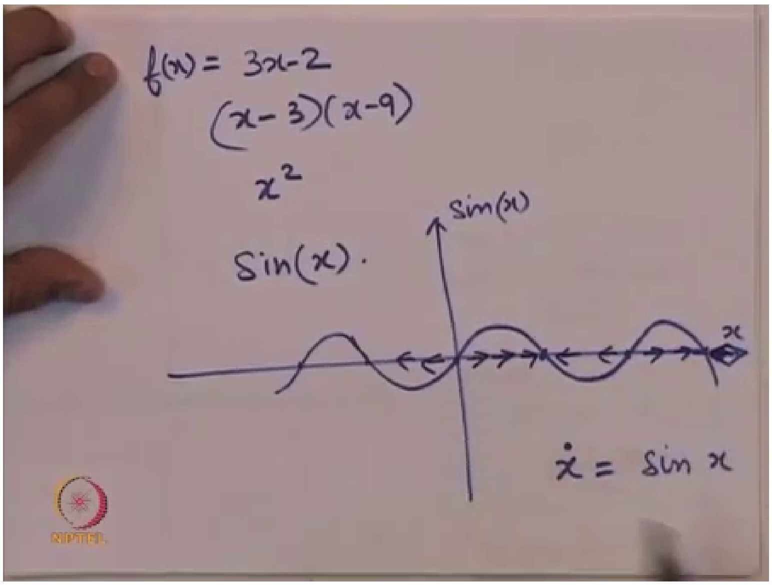

For which examples, we have already seen 3 x minus 2, we have also seen x square. Now, we will quickly decide what are the equilibrium points for this particular function? Sin of x and whether they are stable or unstable.

So, there are several equilibrium points equilibrium, there are several equilibrium points for the differential equation x dot is equal to sin of x. Please note an x itself is not a sinusoidal trajectory it is a differential equation in which sin comes in suppose this is an example. So, here we see that all the 0 crossings are equilibrium points and depending on whether before and after that equilibrium point whether this sin of x is positive or negative based on that we are able to classify this equilibrium points as stable or unstable.

So, here we can draw these arrows to the right and here to the left, similarly here again we can draw. So, we see that this equilibrium point is unstable, this one is stable, this is unstable this is stable. For this particular differential equation, we are able to see that there are several equilibrium points and alternatively they are alternately they are stable or unstable. So, this is something that we expect the viewers to carefully verify. With this we end today's lecture, we will continue with these aspects from the next lecture in more detail.

Thank you.