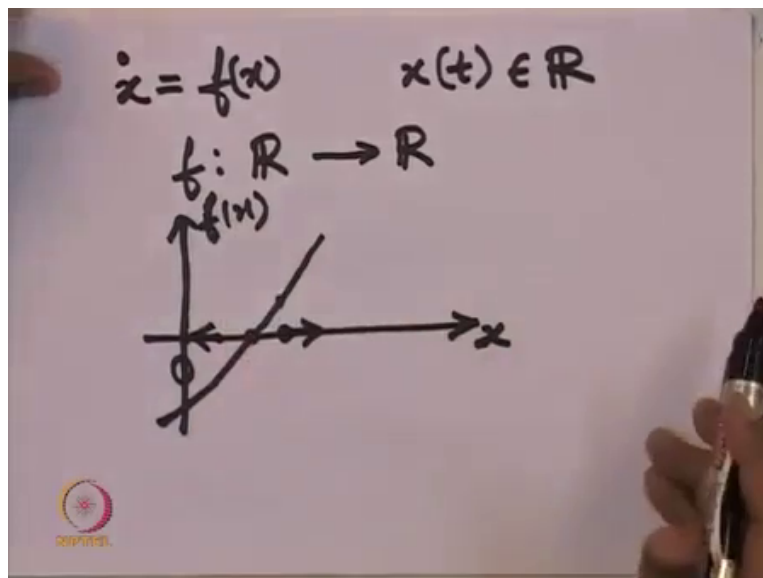


Nonlinear System Analysis
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Lecture - 14
Phase Portrait
Part 02

Hello everyone, welcome to the second lecture on Non-linear Dynamical Systems today we will continue with first order systems. First order meaning the variable x has only one component this is what we just begun last week, we will continue with this now.

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So, we will we were dealing with the case that we have the differential equation \dot{x} equal to f of x , in which x at any time instant had only one component, in other words it was a real

number. In this situation f is a map from real numbers to real numbers, more generally it was a map from \mathbb{R}^n to \mathbb{R}^n , but because x has only one component we have a map f from \mathbb{R} to \mathbb{R} .

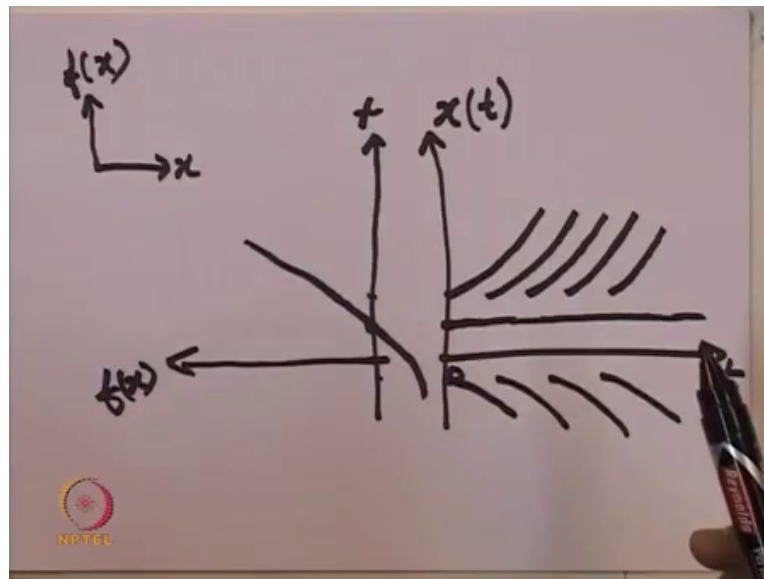
So, for this differential equation we saw that the variable x lives only on this line. So, at anytime instant suppose this is the origin at every point that is an arrow which is either towards the positive direction of x or negative direction of x depending on whether x dot at this point is positive or x dot is negative. So, whether it is positive or negative can be very conveniently be seen by plotting f here yeah this is incidentally possible only when x has only one component.

So, we can draw a graph and suppose the graph of f versus x is like this. So, at this point we see that f is positive and since f of x is exactly x dot we can see that, because the value of f is positive at this value of x we see that x is increasing at this point because f is 0 it means rate of change of x is equal to 0 when x is at this point and hence that is an equilibrium point if x starts at that point it remains there.

On the other hand if we are considering this point, then this point f is negative and hence this point x dot is negative at this point and hence it is decreasing. So, we will draw an arrow in this direction. So, for the single component case for the scalar case and x has only one component, that time this line x is filled up with arrows either to the right or to the left, the special case arrow has length 0 when there is no arrow, all this can be drawn by just seeing whether f is positive or equal to 0 or negative.

So, in other words we can plot a graph of f versus x , but since we are interested in seeing how x varies as a function of time, we are interested not just in f with respect to x .

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But we are also interested in a graph of x versus time because x the variable x is evolving as a function of time and in such a situation we would like to draw a graph of f versus x and x versus time and this is one convenient way to do it.

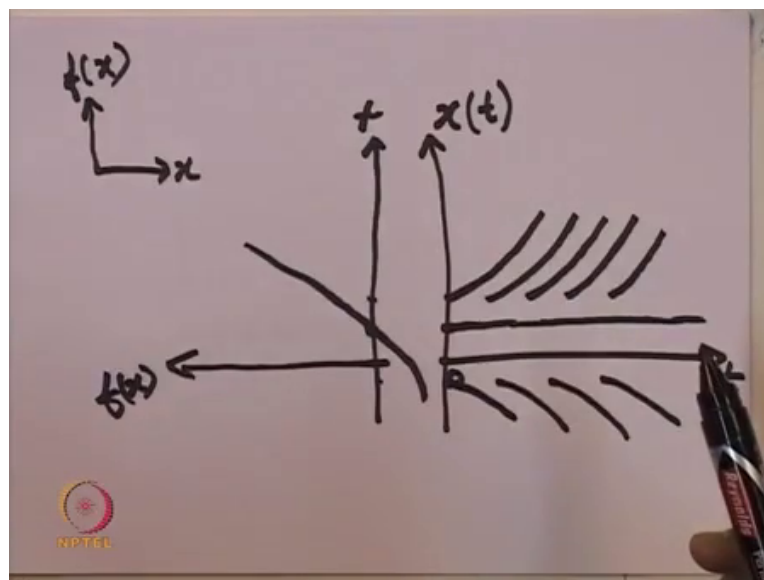
So, in our earlier situation we had x here and f of x here that has been just rotated by an amount. So, that this particular plot x is here and f of x is here, denotes precisely this plot that is being pointed by my finger. So, our previous example f was some graph like this. So, we are able to see that this point is an equilibrium point and hence suppose in this graph this is time t equal to 0 and this is increasing direction of t .

Suppose this value of x we saw f of x is equal to 0. So, this is an equilibrium point if x begins there it just continues exactly at that same value, why because at every point at every time instant t \dot{x} is equal 0. In other words x is constant, it is equal to what constant? Precisely

this value of x where it is equal where f is equal to 0. And on the other hand when x is slightly higher value suppose here then we see that f is positive because f is positive \dot{x} is positive which means x is increasing as a function of time.

So, if it is equal to this value at any time instant t it just increases, how long it will increase that depends on how f varies with x . On the other hand, if you start with the value of x that is slightly below this particular equilibrium point say here when here \dot{x} is negative. So, x is going to decrease with time. So, please note that we have dependence of f with respect to x that tells us how x depends on t , whether x is increasing like we saw here or whether x is decreasing as a function of time depends on whether f is positive or negative as a function of x . So, this we will see a little more in detail when we see an example.

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So, consider the example f of x equal to minus ax^2 plus bx plus c where a , b and c are 3 real numbers, they are all chosen to be positive numbers. So, for this situation when we draw a graph of f versus x so, this is a little unconventional way of plotting a function. So, we have plotted with the same orientation of the independent and dependent axis, x has been plotted here and the dependent variable f has been plotted in this direction.

So, this particular equation we can say has 2 roots; one positive and one negative, because a and c have opposite signs we know that the roots are real distinct and opposite signs and for very large values of x we know that f eventually becomes negative for very large unbounded values of x both positive and negative, we see that f becomes negative.

In other words the graph of f versus x is a curve like this, where these are the roots as we see one is positive one is negative. So, for this particular graph if we are to plot a function of x versus time, here the vertical axis is x and here we have the independent variable time. So, we see that this equilibrium point of course, is going to remain constant as a function of time, for slightly higher values of x f is negative yeah to the right of the x axis here means f is negative. So, x is decreasing as a function of time.

On the other hand, for value slightly below this equilibrium point we see that \dot{x} is positive, because f is positive and in other words it is increasing. This is how the curves evolve as time increases, the time axis has a very well defined direction of increasing time which we want to say time is progressing this is a situation in our world. This other equilibrium point corresponds to when f is equal to 0 again that is also being an equilibrium point x is constant, because \dot{x} is 0. So, we see that all these are cutting this t axis, why because x equal to 0 is not an equilibrium position yeah.

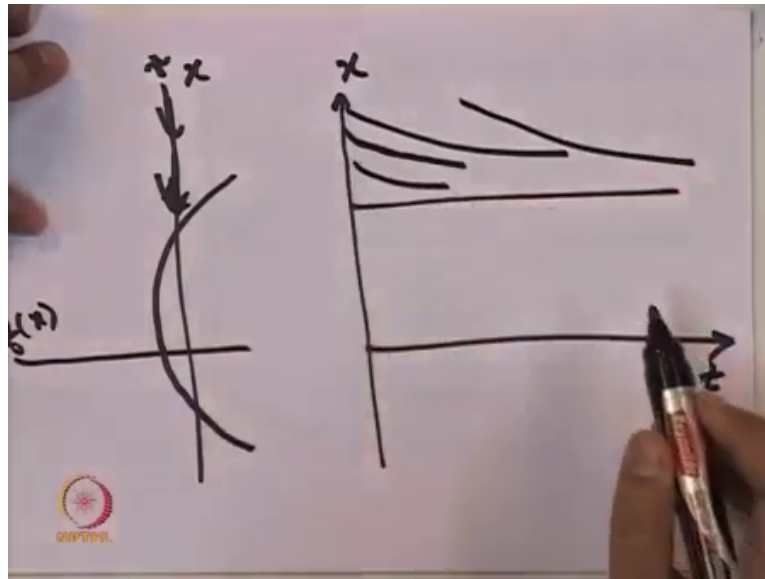
This particular line is only denoting the t axis it is not denoting an equilibrium point or an or a trajectory corresponding to an equilibrium point. On the other hand this because this value corresponds to an equilibrium point this just remains constant as a function of time similarly this. And here again because \dot{x} is negative we have these curves going away from the equilibrium point.

So, this is how trajectory is evolved as a function of time. So, we will investigate this particular example in more detail we will check whether this equilibrium points are stable or unstable and we will also ask some important questions. What are the questions we will ask? We will ask, whether this particular trajectories which are approaching this equilibrium solution this equilibrium point do they come and meet it at a particular time instant.

On other hand we will also ask here we see that these trajectories are going away from the equilibrium point we will ask the question, is it possible that a trajectory here cuts and comes to this? Or is it possible that when going along this we can move out and go here and reach that equilibrium point? These are the questions we will analyze in more detail. Another question we will ask is at every point is it true that we have a trajectory that starts at every point x and let us say t equal to 0, is it true that we have a trajectory starting from that point, going either in increasing direction decreasing direction or remaining constant.

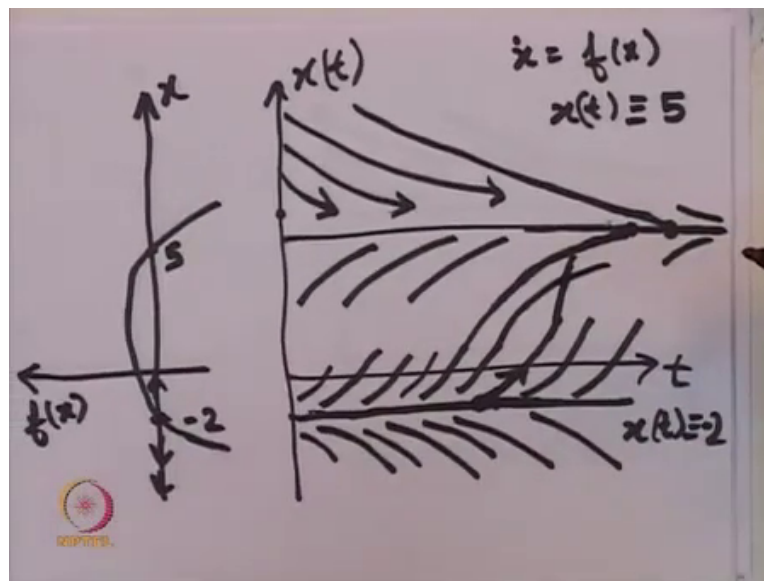
But this is an existence question, does there exist a solution from every point x , these questions we will answer in a little more detail. But we will stick to this example why? Because this particular quadratic equation with the leading term negative and the product of the constant and the leading term that coefficients having opposite signs is a situation we encounter in the Kalman filter case. In the continuous time Kalman filter in the recursive solution we see that we encounter a differential equation that has this particular signs of course, for the situation that the state variable has only one component x .

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So, this particular example we see that here all the arrows are directed inwards, this is a graph again of f versus x and on the right hand side we have already plotted x versus time. And we see that here because f was negative, \dot{x} was negative, so all the trajectories are coming downwards ok.

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Let us just see this figure again because in the previous example we have already drawn intersecting trajectories and the question is can trajectories intersect for this particular question we need to draw this figure again.

So, consider this figure in which this is the time axis and x varies as a function of time at any value of x whether it increases or decreases to decide that we have to go to this figure and check whether f is positive or negative, why do we have to do that, because we are studying this differential equation. This differential equation let us go back to our previous example in which we had a quadratic equation with one root positive one root negative.

And as I said this corresponds to a continuous time Kalman filter differential equation in which this equilibrium points correspond to constant solutions to the differential equation. Suppose this value was equal to 5 and the sum was equal to minus 2 , then x of t equivalently

equal to 5 is already a solution to the differential equation, why because at x equal to 5. If you want to find whether x is increasing or decreasing when we substitute x equal to 5 in this equation, we see f is equal to 0, why because we see that $f(x)$ graph of f versus x cuts the x axis at x equal to 5.

And hence, x of t equivalently equal to 5 is a solution to the differential equation, but for values more than 5 we see that \dot{x} is negative and hence it is all decreasing. So, the important question is, can this solution which is can this trajectory which is decreasing, can it meet the constant solution infinite time, is it possible that a trajectory here only approaches this trajectory asymptotically or can it intersect at sometime instant, this is one question we will ask. Also we will ask here we see that for value slightly more than minus 2 \dot{x} is positive and hence trajectories are increasing.

Increasing and going towards this because we see that there is no other equilibrium point in the middle this is these are how important theorems especially in second order systems are arrived at. So, we see that all trajectories are going away from this equilibrium solution, what is that equilibrium solution? The constant solution corresponding to x of t equivalently equal to minus 2 this constant solution all trajectory seem to be going away we already decided that because arrows are directed away because every small perturbation takes the solution away from the point minus 2.

We decide that this equilibrium point is in unstable equilibrium point and we can also see that from this particular phase curves from these solution curves sorry. So, we see that all solutions are going away, but the equilibrium solution itself just remains constant along this, is it possible that while it is remaining constant, it could go away from here. So, that at this particular point we have non uniqueness of solutions, there is one solution that remains constant, another solution that goes away. This is about intersection of trajectories yeah there is there is possibly one past up to here, but from here there are 2 futures.

Since, there are 2 futures for this particular autonomous differential equation in which there are no inputs we want to ask, is it is the equilibrium point an equilibrium point at all, because we see that there is one solution that also comes out. Which says that perhaps this value of 2

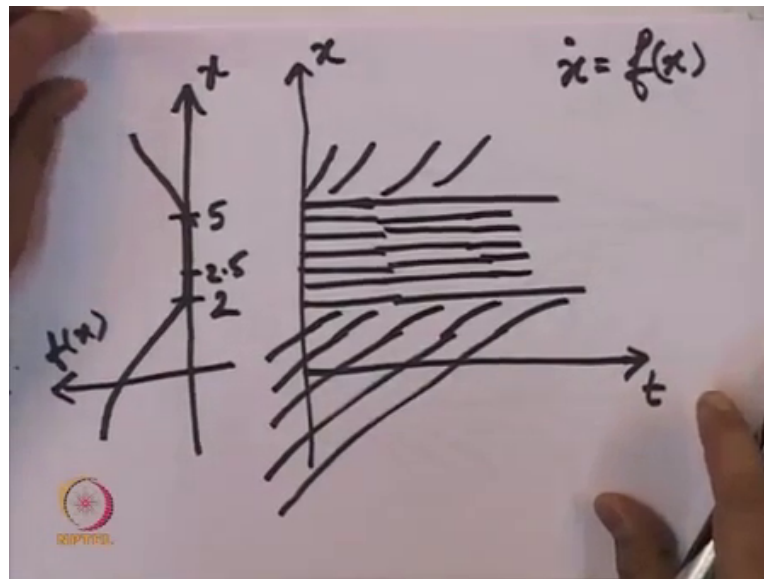
of minus 2 was not an equilibrium point, is that possible this is a important question that one has to ask in encountering non-linear differential equations.

On the other hand we see that this particular solution that evolves goes towards this equilibrium solution, while it goes towards you want to ask does it go and intersect of course, we know that once it intersects if it all it intersects once we are on the equilibrium solution there is a unique future.

Because for slightly higher values of x we are coming down, for slightly lower values of x also we are going up which means we are approaching the equilibrium solution for higher and lower values of x , but while we are approaching is it possible that we intersect this equilibrium solution. This would be about non uniqueness of the past of the solution when we are at a particular point of course, we have one equilibrium solution coming and meeting this time instant x value, but there is also this trajectory possibly.

So, these are the questions that we will have to answer when studying non-linear differential equations, we will certainly answer this in more than generality; in more generality when x has n components. Another important issue is that we have these trajectories, is it possible that these 2 trajectories intersect here and then they criss cross and go.

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So, these questions one are one is going to see that this non uniqueness will be ruled out as soon as we assume a so called Lipschitz condition on f in the differential equation \dot{x} is equal to f of x we will define a particular property called Lipschitz property of a function. Once we impose the Lipschitz condition on f we will see that solutions are unique both in the past and in the future; however, only for a small interval of time, for a more longer duration of time to prove uniqueness would require more advanced properties and conditions on the function f ok.

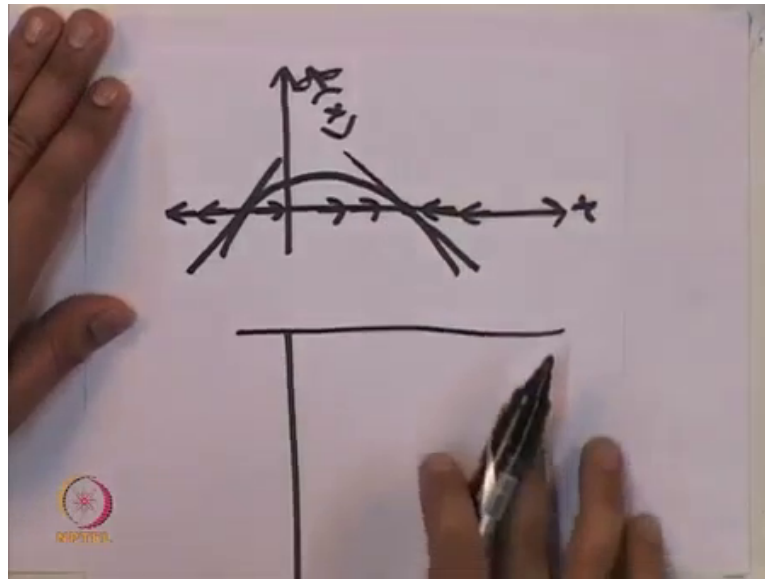
So the next important question we will ask is the question of continuum of equilibrium points. So, when would; when would there be a continuum of equilibrium points, if f were equal to 0 in this interval. What is this a graph of? This is a graph of f versus x on the left hand side of this we always plot f versus x and we will use that to plot x versus t . So, we see that because f is 0 from this value let us say 2 up to 5, because a graph of f is touching the x

axis for this entire interval we will see that from 2 up to 5 all solutions are just constant solutions.

Why because \dot{x} is equal to 0, at 5 also it is equal to 0 for slightly larger values of 5 we see that \dot{x} is positive in other words the trajectories are going away. Also for slightly lower values of x we see that \dot{x} is again positive which means that the trajectories are increasing and hence approaching this particular equilibrium point. These trajectories could have existed for negative values of time also. So, this is the situation where we have a continuum.

What is continuum? If this 2.5 is an equilibrium point, then very close to 2.5 let us say 2.50001 also is an equilibrium point. In other words all the equilibrium points are sitting very close to each other they are a connected set and they are a continuum; continuum of equilibrium points this is the language. We use to say that we have not just not isolated equilibrium points, but a continuum. So, we will also quickly see what linearization has got to do with, with solution to a differential equation and especially in the context of close to an equilibrium point.

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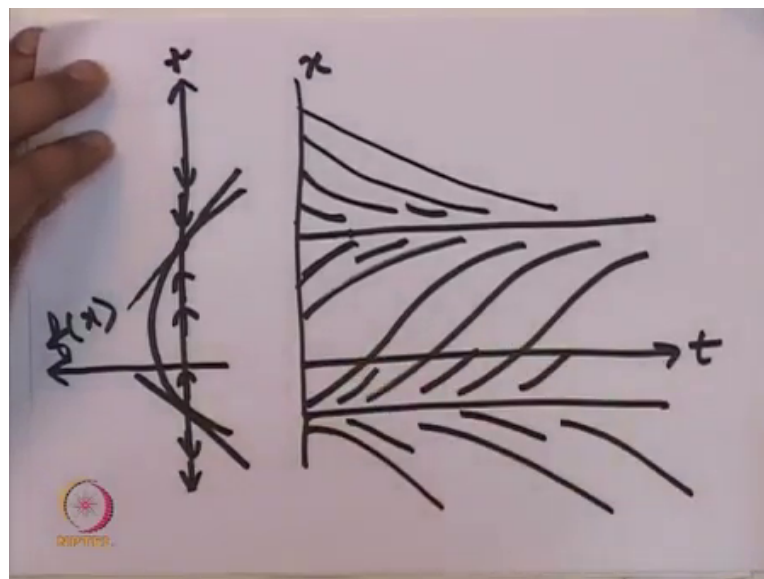
So, a good amount of information number stability can be seen by linearizing. So, suppose we consider our example so, we see that at this particular point whether f is increasing or decreasing. So, at this point we see that f is decreasing as a function of time as a function of x sorry.

So, when we try to put a tangent to this curve at the point at the equilibrium point you see that this tangent has negative slope. So, for those who are having difficulty to answer this question with these axis tilted let me just turn this page this is how we should be imagining. So, this is a graph of f versus x and we see that at this equilibrium point, if we draw a tangent to this particular curve f and this tangent has a slope which is negative yeah which just means that f is decreasing as a function of x .

On other hand here this tangent is a line with positive slope. So, we are going to differentiate the function f not at any point, but at the equilibrium point and the equilibrium point we see that this tangent is a line with positive slope while this tangent is a line with negative slope. This positive or negative slope itself decides whether the equilibrium point is stable or unstable so, that we can see also here. So, here we see that the arrows are all like this.

So, to see that the slope already gives the information whether this equilibrium point it is stable or unstable and similarly this is stable or unstable, we can see we can as well analyze this equation this differential equation which corresponds to this line. So, consider the, before we see the differential equation corresponding to a line we will just complete the trajectories for this example.

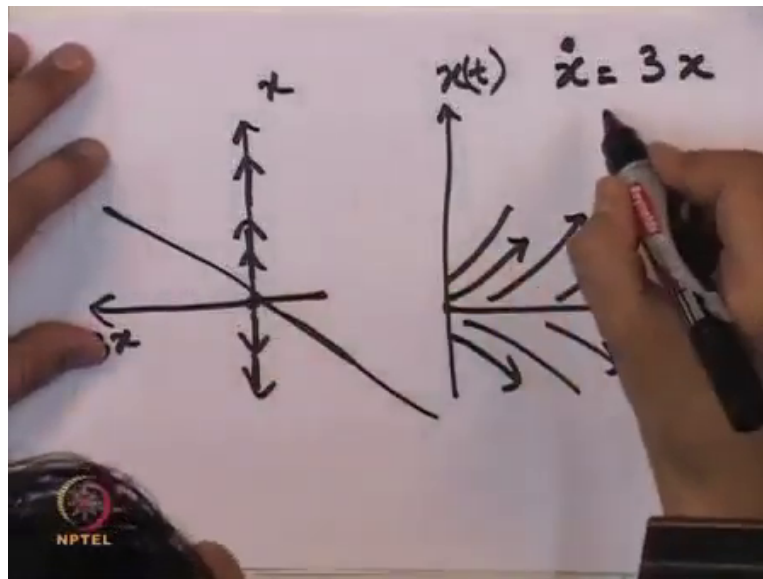
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So, we see that the equilibrium point remains constant similarly this one also remains constant. And here the trajectories are all decreasing and approaching this equilibrium point here they are all increasing. So, I can be a little fast here because we have already seen this in more detail. This is an equilibrium point, this is also an equilibrium point, this is a graph of x versus time t . So, the fact that this equilibrium point is stable, why because trajectories close by are coming towards this, we would like to define this asymptotically stable in the sense of Lyapunov which we will do in a few after a few lectures.

On the other hand this one is unstable because there are small perturbations near it which are going to take the trajectory away from this equilibrium point and this much information we claim we can see already from the slope of the tangent of the function f with respect to x at the point. Tangent drawn at which point on the curve, tangent drawn at the equilibrium point, because this tangent has positive slope this particular equilibrium point trajectories are going away from it. Because this tangent has negative slope we will see that this trajectories are coming close by the tangent is going to decide whether the equilibrium points are stable or unstable, this we will see from studying the corresponding linear system.

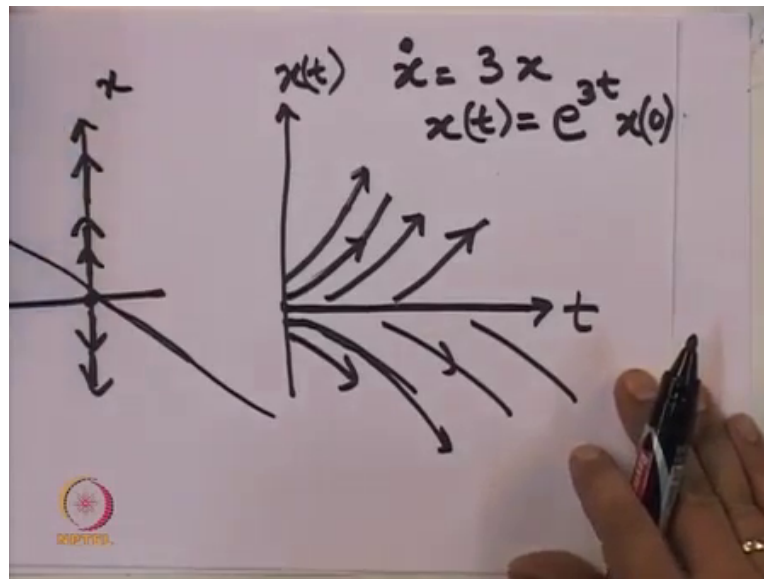
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So, consider the example \dot{x} is equal to $3x$. So, here the line $3x$ versus x , on the other hand here is a graph of x versus time. So, we see that all trajectories are going to go away this we are able to see, because the equilibrium point we are able to see because we can draw the arrows here of this function $3x$ versus x . So, we see that for positive values of x $3x$ is positive this you could also consider drawing a graph of $3x$ versus x before this rotated axis.

So, you can draw it in this particular situation and after you have drawn this because we are actually interested in graph of x versus time when we turn this we see that this line with positive slope 3 corresponds to this and we see that it is indeed unstable.

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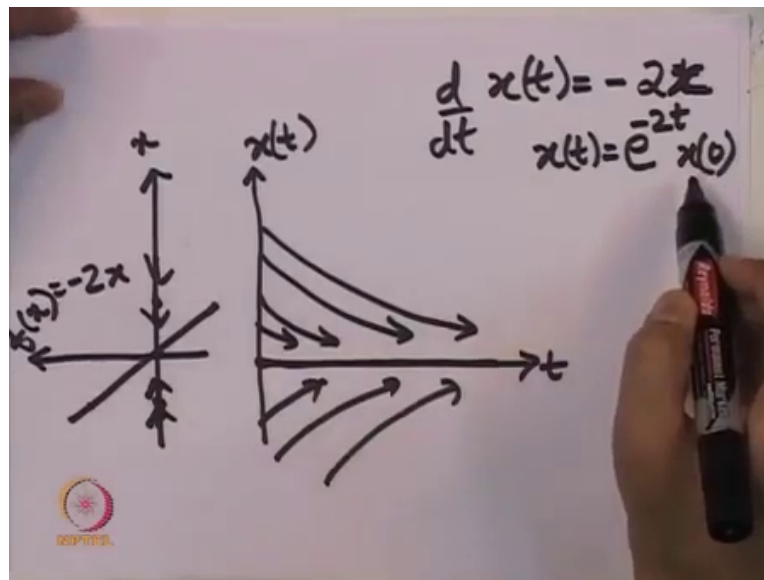


And of course, we know the solution to this is exactly x to the power x of t is equal to e to the power $3t$ times the initial condition. So, we see that with this particular line has positive slope, that positive slope 3 is going to come in here. So, that if the initial condition is non 0 then we say that e to the power $3t$ becomes some number that is very large as t increases and hence x to the power x of t is going to become unbounded. It will become positive and unbounded if x of 0 is positive, it will become negative and unbounded if initial condition was negative.

And that is indeed what we see, for a for a negative initial condition the trajectories are going away and for a positive initial condition the trajectories are going to become positive and unbounded. In any case it is going away from this equilibrium solution, which equilibrium solution. The initial condition corresponding to 0 in which situation we see that e to the

power 3 t times 0 will always be equal to 0, no matter what time instant t we are interested in. So, we will quickly take an example where the slope is negative.

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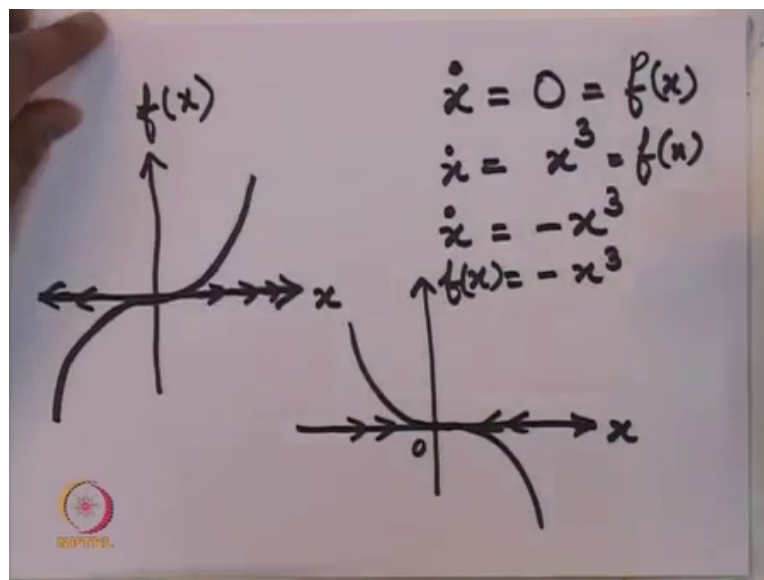
So, we take \dot{x} is equal to minus 2 times x in which case we know the solution x of t is equal to e to the power minus 2 t times $x(0)$. So, for this example we see \dot{x} is equal to minus 2 times x . So, let us draw that same graph here.

This is a line with negative slope with slope equal to minus 2 graph of f of x versus x , in this case f of x is equal to minus 2 x . So, we see that this is a line with negative slope and hence for positive value of x because f of x is negative all arrows are directed towards negative direction of x , in other words \dot{x} is decreasing. On the other hand for when x is negative if

we want to decide where to draw the arrows because f of x is positive we draw it in positive direction of x .

Now, we will study how the solutions look for linear systems 0 has to be an equilibrium point we see that already here. This is how the trajectories are evolving. So, this 0 is a stable equilibrium point, why because close by trajectories are approaching towards it is asymptotically stable. That we can see because this line has slope negative which line the graph of f versus x at the point 0 . In fact, at every point it is a line with slope negative and that is deciding, it is deciding the exponent and because that corresponding to a line with slope negative. We see that this is e to the power negative number times t and it is going to decrease in magnitude no matter which initial condition x of 0 we with start with.

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But then to see that when the slope is equal to 0 we will find our test is inconclusive to see that we will quickly see 3 differential equations, \dot{x} is equal to let us say 0, \dot{x} is equal to x^3 and \dot{x} equal to $-x^3$. So, for these 3 differential equations we will see that in each of the cases the tangent has slope 0 at the equilibrium point. So, we are drawing graph of f versus x , in this particular example f is equal 0 it is a 0 function which is always equal to 0. So, all the points are equilibrium points and at any point the graph of f versus x has tangent equal to the x axis.

So, the tangent also has slope 0 at every point and all the points are also equilibrium points, this is the conclusion we get for this linear system, when the tangent itself is the graph f and both have slope 0. But this particular example \dot{x} is equal to x^3 when we draw f of x which is equal to x^3 we see that we get this curve. So, one can check that this particular curve is tangential to the x -axis at this equilibrium point which is, what is equilibrium point? We equate x^3 to 0 and we see x equal to 0 is the only solution only real solution. So, the tangent also at this point happens to have slope 0.

On other hand we can see that the arrows for this example would always be directed away from the equilibrium point 0. However, the tangent having slope 0 is also the situation for the third example where f of x equal to $-x^3$, for this example here all the arrows are directed towards the equilibrium point and hence this equilibrium point is a stable equilibrium point it is asymptotically stable all trajectories are eventually going to approach 0 as t tends to infinity but here also the tangent has slope 0.

So, these 3 examples, one where \dot{x} is equivalently equal to 0, second where \dot{x} is equal to $+x^3$ and third example where \dot{x} is equal to $-x^3$, each of the 3 examples the tangent to the curve f of x versus x at the equilibrium point has slope 0, but in one case all the points are equilibrium points in one case the equilibrium point is unstable, while in another case the equilibrium point is stable.

So, these 3 different situations can easily emerge when the tangent is having a slope 0. So, this we will see is a very important situation where the general case also when the when the

linearization has Eigen values on the imaginary axis, the test of stability or instability using the linearization will end up being inconclusive.

So, having seen this we will proceed to the situation of second order systems after having seen the first order case. What is first order? When there is when x has only one component the scalar situation after having seen this we will proceed to the second order case where x has 2 components.

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Classification of equilibrium points Existence and Uniqueness of solution

Second order systems

Differential equations in first order with the state having two components:

$$\dot{x}(t) = f(x) \quad x(t) \in \mathbb{R}^2$$

x can be thought of as evolving in the plane \mathbb{R}^2 .
Easy to visualize. Many more features possible as compared to scalar case.
The general situation $x(t) \in \mathbb{R}^n$ is just a small extension of this.
(A polynomial of any degree with real coefficients can be written as a product of first and second degree factors (with real coefficients).)

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The situation when x has 2 components has many different possibilities and these different possibilities will also throw some light about what can happen when x has more than 2 components. So, consider the differential equation \dot{x} is equal to f of x , where x at anytime instant has 2 components. So, x of t is in \mathbb{R}^2 . So, we can now think of x evolving in the plane \mathbb{R}^2 .

So, this is relatively easy to visualize, but there are many more features that is possible. Let me emphasize it is easier to visualize than the general case where x has n components even though it is little more complicated than the one dimensional case where x had only one component. But this going from one component to two component introduces so many possibilities, but all these possibilities can also happen in \mathbb{R}^n and compared to \mathbb{R}^n , \mathbb{R}^2 is easy to visualize.

Moreover the general situation is only a small extension of this, why is it only a small extension? Because a polynomial of any degree with real coefficients can be written as a product of only first and second degree factors, if we want the coefficients to be real that is.

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Classification of equilibrium points

Existence and Uniqueness of solution

Equilibrium points : linear systems

Consider the linear system

$$\dot{x}(t) = Ax \quad A \in \mathbb{R}^{2 \times 2}$$

Eigenvalues of A , i.e. roots of $\det(sI - A)$, decide key features.

Suppose no eigenvalue of A is zero.
The origin in the plane is **only** equilibrium point.

The different types of equilibrium points are

1. Center
2. Node - Stable, Unstable.
3. Focus - Stable, Unstable.
4. Saddle point.
5. Some more (non-regular) cases.

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So, consider the linear system \dot{x} is equal to A of x where A is a 2 by 2 matrix. So, we will start with linear systems, because we are interested in a qualitative study of the differential

equation. Qualitative study meaning we want to only extract out certain features whether it is stable, asymptotically stable, whether there are periodic orbits, for studying this situation it turns out that linear systems have a rich a rich set of possibilities and that is the reason that we are starting with the case a is A matrix 2 by 2 matrix and \dot{x} is equal to A of x.

So, the eigenvalues of A that is the roots of the determinant of $sI - A$ decide the key features for this case. So, let us begin assuming that A has no eigenvalue at 0. So, for this situation the origin in the plane is a only equilibrium point, because A has no eigenvalue at 0, it is nonsingular A is nonsingular and hence if f of x is equal to 0 it means x has to be equal to 0. That is what is being said by the origin is the only equilibrium point, but this equilibrium point can be of different types depending on different phase.

So, the different types we will see are, the equilibrium point could be a center, we will define what is center is. It could be a node in which case it could be a stable or unstable node. It could be a focus a stable or unstable focus. It could be a saddle point and for certain situations for more general for more for some non regular cases it could be none of these which we will also see.


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Classification of equilibrium points Existence and Uniqueness of solution

Eigenvalues, eigenvectors

When A is a 2 by 2 matrix, eigenvalues can be real/complex.
Convention (for the next few lectures):
complex means not-real,
purely imaginary means nonzero
(not a standard convention, just for the next 3 lectures).
 $\lambda \in \mathbb{C}$ is an eigenvalue of A if there exists a nonzero vector $v \in \mathbb{C}^2$ such that

$$Av = \lambda v$$

 When A has real eigenvalues, then v can be taken to have both components real.

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So, let us quickly review what eigenvalues and eigenvectors are. So, when A is a 2 by 2 matrix the eigenvalues can be real or complex. So, for the next few lectures we will use this convention that we will use complex if the eigenvalues are not real, we will also say if something is purely imaginary, then we will assume that it is non 0 why because if it is 0 then we will say it is 0, even though the origin is also on the imaginary axis, even though purely imaginary includes the situation that it is equal to 0. We will follow the convention in the next few lectures at purely imaginary means non 0. So, this is not a standard convention just for the next few lectures.

So, a complex number λ is called an eigenvalue of A if there exists a non 0 vector v such that Av is equal to λv . So, this equation if you are able to solve for a non 0 vector v then that λ is called an eigenvalue and that v is called an eigenvector. So, in general

we might require the 2 components in v to both v complex, but when A has real eigenvalues, then we can ensure that the eigenvector has both components real.

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Classification of equilibrium points Existence and Uniqueness of solution

A: diagonal

For the sake of simplicity, when A has real eigenvalues, we assume A is diagonal.
This is like assuming that A has two independent eigenvectors.
In the new basis (basis of eigenvectors), A looks diagonal.

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

For this case, different features depending on signs of λ_1 and λ_2 .



We will start with the situation that when A has a real eigenvalues A is already diagonal, this is like assuming that A has 2 independent eigenvectors. Because it has 2 independent eigenvectors in a new basis, in which basis in the basis comprising of the eigenvectors A looks diagonal which we also say A is diagonalizable. So, in this new basis, A already looks diagonal that is the reason that we are assuming this situation.

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Classification of equilibrium points Existence and Uniqueness of solution

A: nondiagonalizable

When A has repeated (real) eigenvalues, possibly A is nondiagonalizable. We can assume in some basis A is like

$$A = \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$$

For this case, different features depending on sign of λ_1 .



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So, in this case there are different features depending on the signs of λ_1 and λ_2 and when A is not diagonalizable then it is forced to have repeated eigenvalues, because A is a 2 by 2 matrix then they are forced to be real eigenvalues. In that situation we will assume that in some basis A is of this form $\begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix}$ this is so called the Jordan canonical form. Here the different features depend on the sign of λ_1 .

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Classification of equilibrium points Existence and Uniqueness of solution

A has nonreal eigenvalues

When A has complex eigenvalues (and 2 by 2 case), repetition is ruled out.
Because A has real entries, eigenvalues come in complex conjugate pairs.
Sign of the **real** part of the two complex eigenvalues plays the key role.



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And when A has non real eigenvalues; that means, it has complex eigenvalues then for the 2 by 2 case repetition of eigenvalues is ruled out, why because they have to occur in conjugate pairs. Since A has real entries, eigenvalues have to occur complex conjugate pairs, that time the sign of the real part plays a key role real part of the 2 complex eigenvalues.

(Refer Slide Time: 40:00)


Classification of equilibrium points Existence and Uniqueness of solution

Real Jordan form

A matrix A with complex eigenvalues can be brought to the following form (using new basis in \mathbb{R}^2).

$$A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

α and β are real numbers.
Eigenvalues of A are exactly $\alpha \pm j\beta$
In this form, non-diagonal terms correspond to rotation, and diagonal terms correspond to shrinking/expansion.
 $\beta = 0$ means real, and $\alpha = 0$ means purely imaginary eigenvalues.



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We will also need to know, what is real Jordan form. So, matrix A with complex eigenvalues can at least be brought to the following form using a new basis in \mathbb{R}^2 what form. The diagonal elements have α and the off diagonal elements β and $-\beta$ and α and β are assumed to be real numbers.

So, eigenvalues of such a matrix A are exactly $\alpha + j\beta$ and $\alpha - j\beta$. So, in this form the non diagonal terms β correspond to rotation and the diagonal terms α correspond to whether the trajectories are shrinking or expanding this is what we will like to emphasize for the next few lectures. Also $\beta = 0$ means that the 2 eigenvalues are real and $\alpha = 0$ means the 2 eigenvalues are purely imaginary they are $\pm j\beta$.


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Classification of equilibrium points Existence and Uniqueness of solution

Nodes

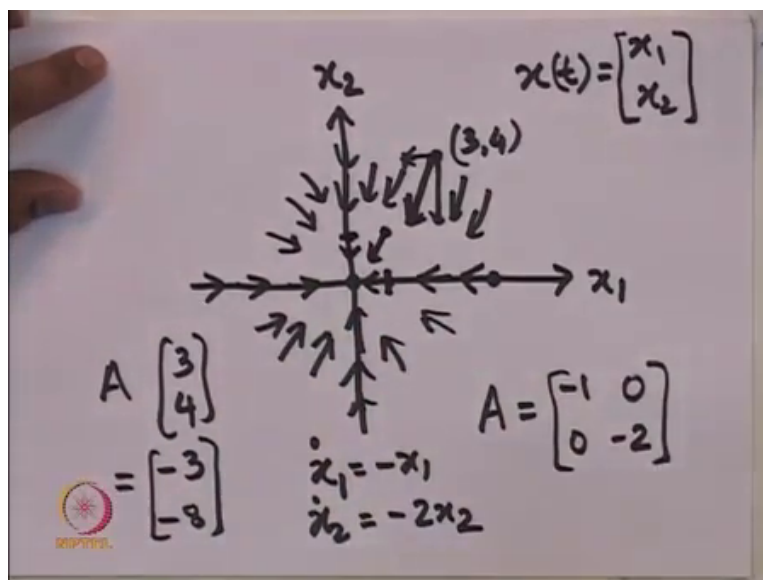
Suppose A has distinct real eigenvalues.

- Eg.
 $A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$ has a **stable node**.
- Eg.
 $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ has an **unstable node**.



So, suppose A has distinct eigenvalues distinct real eigenvalues. So, let us see these cases one by one for example, take A equal to minus 1 0 0 minus 2. So, this is what we call a stable node, why is this? We will see in more detail when we see a graph of how the trajectories evolve.

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So, now x has 2 components hence the components are called x_1 and x_2 both are a function of time. So, when we are on the x_1 axis we are dealing with a case when A is equal to minus 1 0 0 minus 2 this is the example we last saw on the slide.

So, differential equation corresponding to this A because A is diagonal is a set of 2 decoupled differential equations \dot{x}_1 is equal to minus x_1 and \dot{x}_2 equal to minus 2 x_2 . So, we see that if we are along the x_1 axis to be along the x_1 axis means x_2 component is equal to 0, then x_2 always remain 0 it continues to be 0. The origin itself of course, is an equilibrium point because A is non singular the origin is a unique equilibrium point along the x_1 axis we see that all arrows are directed towards the origin when we are along x_1 axis that is what this differential equation says.

Similarly, along the x_2 axis also all arrows are directed towards the equilibrium point, moreover these arrows along x_1 and x_2 axis are parallel to the x_1 and x_2 axis themselves. And if we take a general point suppose we take the point $(3, 4)$ this general point how do we determine the arrow at this point? So, we take the vector $(3, 4)$ and make matrix A act on it when we make matrix A act on $(3, 4)$ vector we get exactly $(-3, -8)$.

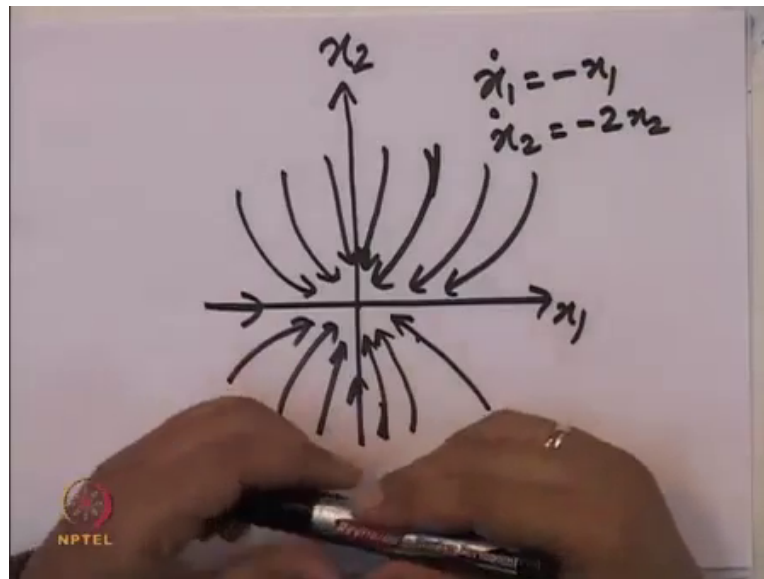
So, this point $(3, 4)$ to determine where the arrow should go towards we will draw vector towards $(-3, -8)$ starting from this point $(3, 4)$. So, the vector has it is some vector like this with 3 components in this direction along with x_1 direction and 8 components along the negative x_2 direction. So, when we resolve this we get this vector.

So, this is how all arrows are directed. So, at any point x_1 and x_2 whatever components we are given we can directly decide where the arrows lie and this is how we are able to fill the x_1 x_2 plane, fill the plane \mathbb{R}^2 with arrows and we see that for this example all the arrows are directed towards the origin, why they direct towards origin at any point, how to draw the arrow. The x_1 component can be determined by just the x_1 component of the arrow can be determined by just looking at this arrow here.

Similarly, the x_2 component of the arrows starting from this point can be determined by the arrow starting at this point and the net is just the sum of these both this is how the arrows are drawn along the x_1 x_2 plane on the x_1 x_2 plane. So, we see that all trajectories are directed towards the origin, whether they are all while the trajectories are coming towards origin, whether the arrows are directed towards the origin or not is an important issue.

For this example we saw that the arrow had more direction along the x_2 axis compared to the x_1 axis and hence it was not directed towards origin, but we see that the solutions will all converge towards the origin.

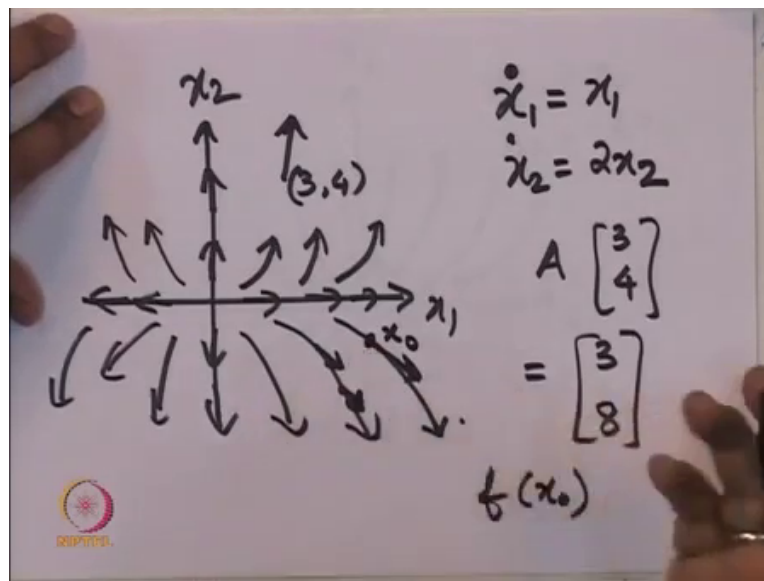
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So, for the same example for which \dot{x}_1 is equal to minus x_1 and \dot{x}_2 equal to minus $2x_2$, when we complete when we draw a curve that is tangent to each of the arrows we see that the curves look like this solution curve. So, this we will say the stable node, it is a node and all the trajectories are directed towards it, that is why it is a stable node. The other situation is where when A has both eigenvalues negative and it is diagonalizable.

So, let it go back to this example, when A has A is again a diagonal matrix and along the diagonal there are 2 numbers which are both positive this situation A is an unstable node let us see this example just like we saw the previous case.

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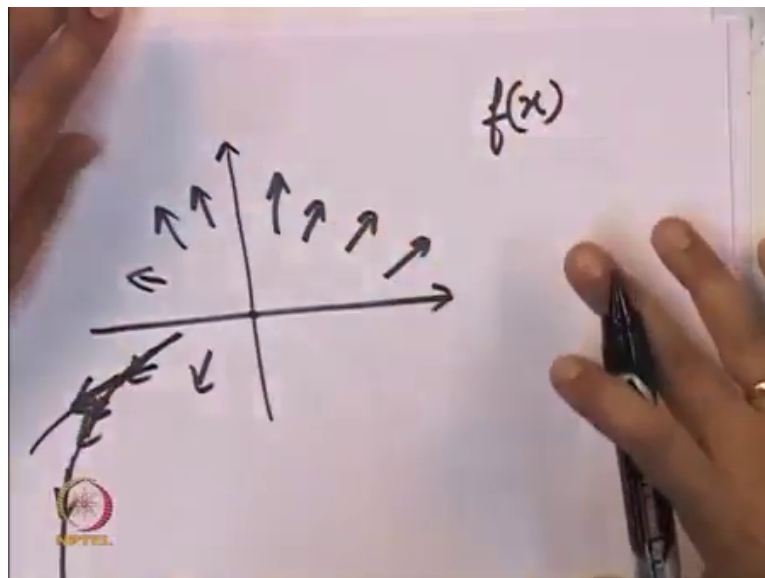


Because it is diagonal because the matrix A is diagonal x_1 dot is nothing, but x_1 and x_2 dot is equal to $2x_1$ $2x_2$. So, we have to decoupled set of equations that was advantage of assuming A is diagonal. So, you are along the x_1 axis all arrows are directed away, along the x_2 axis also all arrows a directed away and at any other point let us say 3 comma 4 we can see that we can draw the arrow.

How do we find the coordinates of the arrow? It is an arrow starting from the point 3 comma 4 directed towards which point that is to be evaluated by making the matrix A at on the vector 3 comma 4 at which point we are interested in finding the arrow. So, this as we evaluate in the previous example is just 3 comma 8 negative of the previous situation. So, this is how we are able to draw all arrows and then upon completing all these arrows into a solution we see that this is how the trajectories look.

So, what is a trajectory? Let us make note of an important property here. A trajectory is a curve such that at each point when we draw the tangent to this curve then we get an arrow and that arrow is precisely f of x at that point. So, suppose at this point we draw this particular tangent to this curve it is this particular arrow, then this arrow this vector starting at this point has some coordinates, what would the coordinates be? If this is a point x naught it is precisely f of x naught which gives the coordinates of this vector, but starting at the point x naught. Conversely somebody gives us the vector field if they give us arrows drawn at various points. So, let us see this example.

(Refer Slide Time: 49:10)



Somebody gives this arrows at at various points, then what is the solution. It is we just go along these arrows and we complete all these arrows into a curve that is how the solution to differential equation looks. We can now ask is it possible that 2 solutions intersect, when would they intersect? At that particular point there are 2 different continuations. If there are 2

different continuations at this point there should be 2 arrows and that is not possible for a function it appears.

Why because a function, at every point x takes only one value, one vector valued it could be vector valued, but it can take only a unique value for each value of x and hence it is not possible that at a particular point we have 2 arrows emanating out of that point. This already lays to rest the situation about intersection of trajectories.

However, from the origin there is no arrow it appears, but it turns out that we could have under non Lipschitz property on the function f we could have trajectories is emanating out of the equilibrium point. So, that is how we should be understanding solutions to a differential equation.

That they are various curves that are completion of that that is a if you are given with arrows at different points this curve is just connection of all these arrows to form a smooth curve a smooth curve such that tangent to the curve at any point is precisely the arrow that was specified.

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Classification of equilibrium points Existence and Uniqueness of solution

Center

- A center has 2 purely imaginary eigenvalues.

$$A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$$

Clockwise or anti-clockwise rotation of periodic orbits.



Navigation icons

So, with this we would ok, the situation about center the equilibrium point being a center is a example we were study in more detail from the next lecture onwards ok.

Thank you very much.