Nonlinear System Analysis
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Lecture - 15
Phase Portrait
Part 03

Welcome everyone. Today is the 3rd lecture on non-linear dynamical systems. This is between Madhu Belur that is me and Harish K Pillai. So, we had just begun with phase portraits of second order systems last week.

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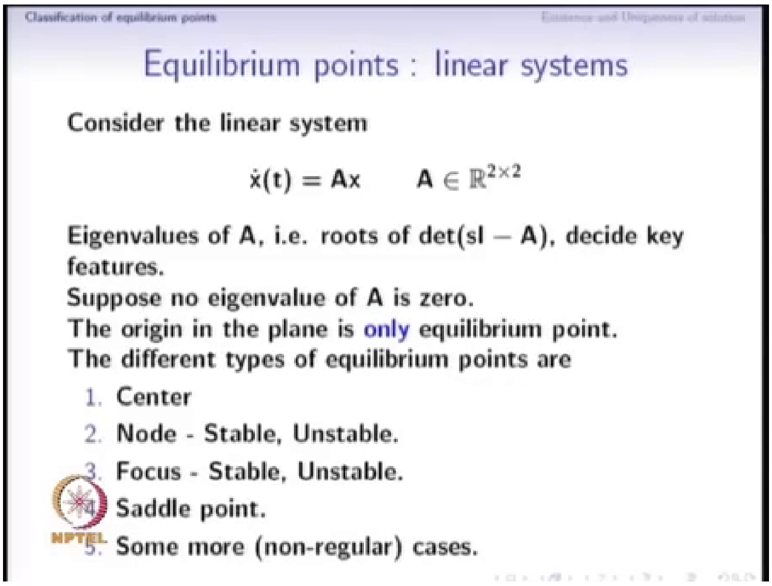

So, consider this differential equation x dot is equal to Ax in which A is a 2 by 2 matrix and now we are trying to see various situations, various situation that arise depending on whether the eigenvalues of A are real or complex whether that repeated or distinct whether A is singular or non-singular. So, the eigenvalues of A let me recap are the roots of the determinant of sI minus A and this decides eigenvalues decide the key features.

So, we will begin assuming that A has no eigenvalue at zero which means A is non-singular in such a situation the origin in the plane is the only equilibrium point. The different types of equilibrium points for this situation are center which we had just begun seeing. The node in which case it can be a stable or unstable node, then there is a focus, a stable or unstable focus, a saddle point and some other situations which for example, when there are repeated roots and when there is one or more eigenvalues at zero; those are the situations we will see separately.

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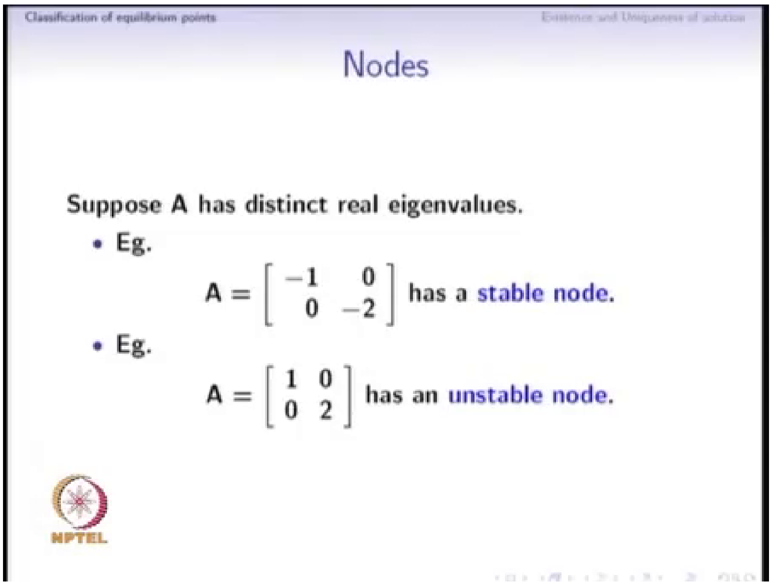

So, a stable node, a node can be stable or unstable. So, what is a node? It is a situation when A has two distinct eigenvalues and both are negative. In such a situation it is called as stable node. The other situation when A has both real eigenvalues and positive is called an unstable node. To analyze this we will quickly see how the vector field looks for this particular A.

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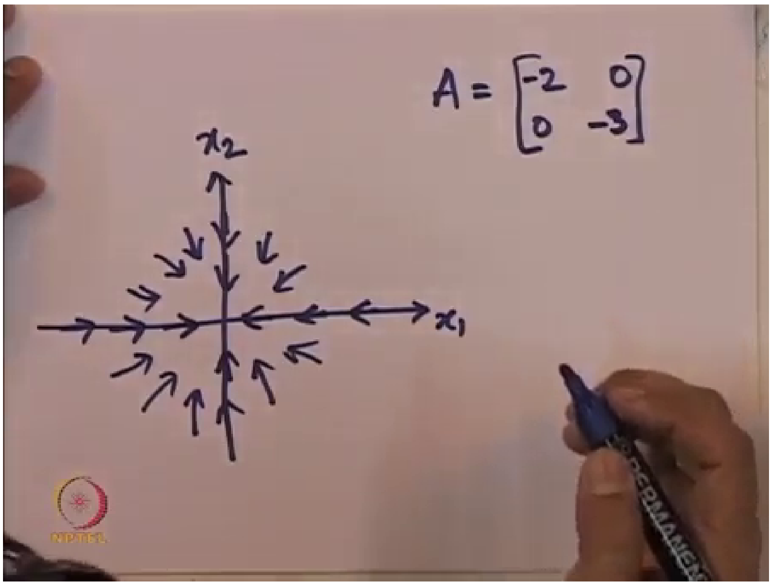

So, look at this figure, this is not the same example that is there on a slide, but it explains what is an what is a stable node. This is the x 1 axis, this is the x 2 axis. What this says is if you are along the x 1 axis, then because x 2 component is 0, when A acts on such a vector again the x 2 component is 0 that is the significance of a diagonal matrix A. And, similarly along the x 2 axis x 1 component is 0 and this diagonal entries being negative imply that that is also along the x 2 axis the arrows.

The relative distance, the relative length of the arrows certainly depends on the x 1 and x 2 components, but then as far as this picture is concerned as far as qualitative study is concerned this explains how the various arrows are. So, the origin, the unique equilibrium point appears to be a stable node. It is a node all arrows are directed towards it, there is no rotation involved because the off diagonal elements are equal to 0 and all arrows are directed towards the origin. This is what we saw as a stable node, we will later see that it is asymptotically stable in the sense of Lyapunov.

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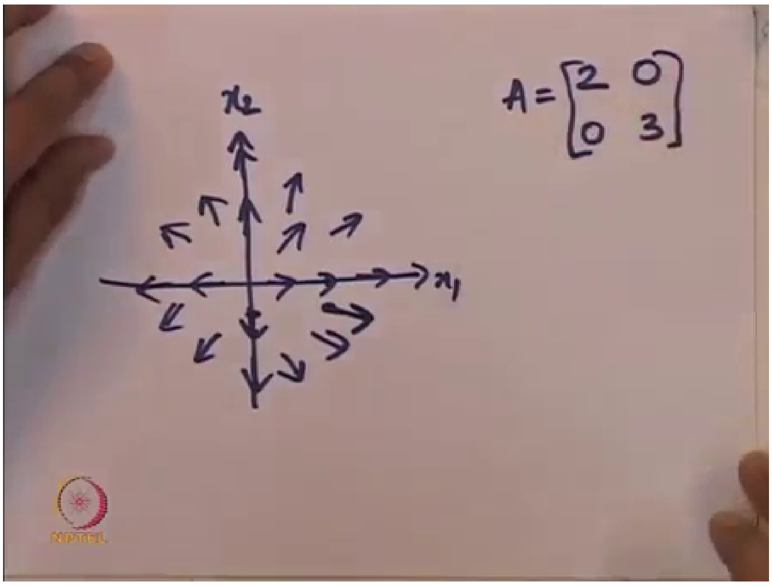

Let us quickly see what an unstable node is, take the same A except that the diagonal elements have sign opposite. Again, because of the diagonal nature of A along the axis the arrows are parallel to the axis themselves with careful attention to the arrows whether they are in the positive direction of x 1 or negative direction of x 1. It will be away from the origin because of the positive sign of the diagonal elements.

The off diagonal the for points which are not along the x 1 axis by just superimposition, because this is a linear vector field by superimposition. For example, if this point the x 1 component of this arrow can be obtained by this point and the x 2 component of the arrow can be obtained by the arrow at this point. This is a net arrow. So, this is what we can obtain by superimposition because, A is a linear map because, we have a matrix that decides the vector field at different points.

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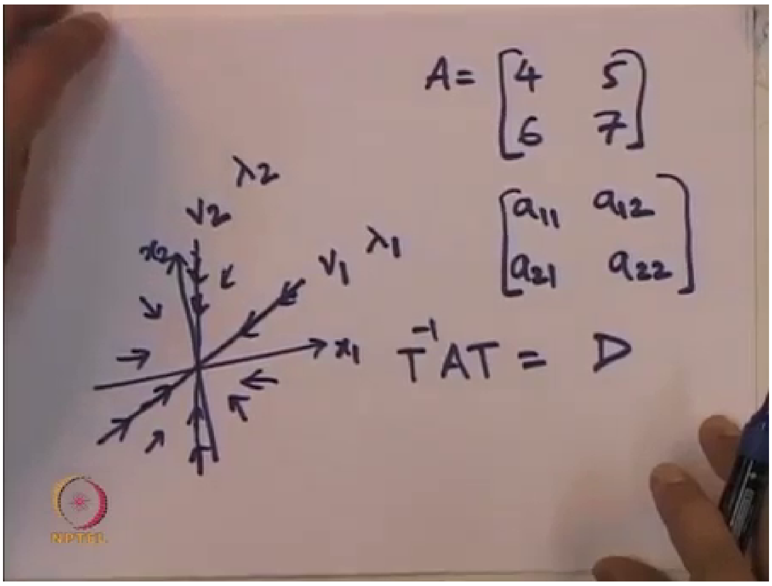

So, before we go to stable and unstable focus we will quickly see what diagonal has got to do with what we are studying. So, if we are given with a general A let us say 4 5 6 7. So, I am just guessing the various elements and suppose the entries are such a 11, a 12, a 21, a 22, suppose the entries are such that this matrix is diagonalizable, it may not be diagonal itself. In other words a 12 and a 21 might not be 0, but if it is such that there exists a non-singular matrix T such that T inverse AT is equal to a diagonal matrix.

Then by choosing the columns of T as a bases, we still have this decoupled vector field, decoupled vector field like we saw for x 1 x 2 we can see it is not along x 2 x 1 x 2 axis anymore. But, suppose this is 1 column of T 1 and suppose the other column of T 2 is like this, in general the 2 columns need not be perpendicular to each other. Suppose, this is eigenvector v 1 this is eigenvector v 2 and suppose this eigenvector corresponding corresponded to the eigenvalue lambda 1 and this correspondent to eigenvalue lambda 2.

These are the x 1; x 1 x 2 axis these are not the eigenvectors, more generally eigenvectors are vectors v 1 and v 2 which may or may not be perpendicular to each other. These eigenvectors are corresponding to eigenvalues lambda 1 and lambda 2. So, if lambda 1 is negative, then we can draw the arrows just like we have drawn for a stable node. And, if lambda 2 is also negative these arrows also can be drawn towards the origin and other places the arrows can be filled again as I said by superimposition. So, more generally if A is diagnosable we have 2 directions called eigenvectors along which we can draw the arrows either towards the origin or away from the origin; depending on whether lambda 1 is negative or positive respectively.

In which case again we are able to decide whether the node is a stable node or unstable node. Our assumption till now has been that both the eigenvalues are of the same sign. When they are of different sign that is the next thing we will see, sorry before we see the situation when the eigenvalues have opposite sign we will start with what a center is.

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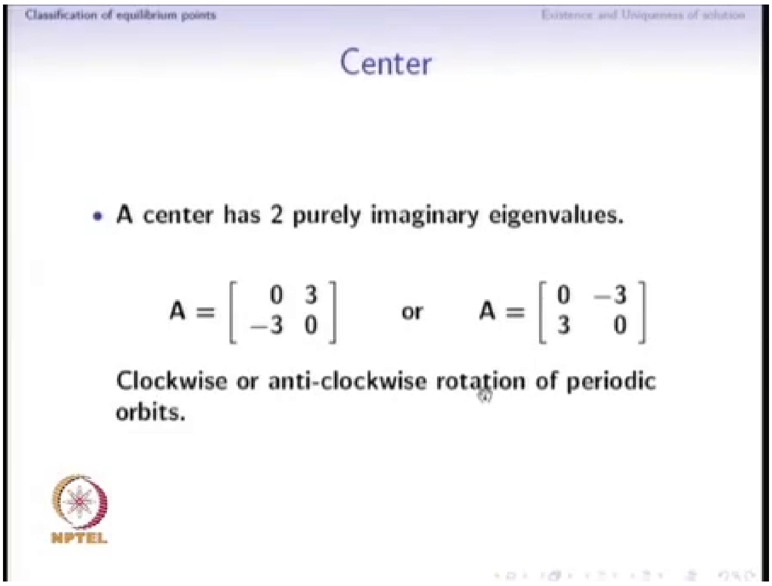

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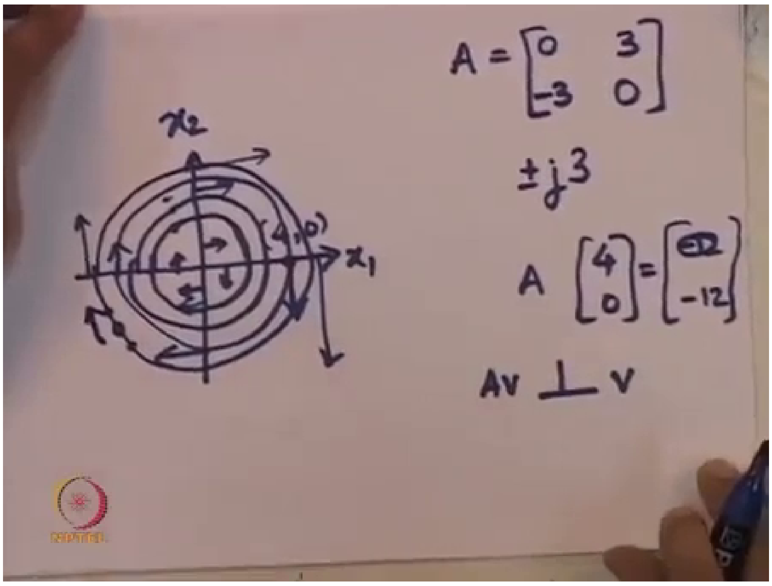

So, this is the situation when A has purely imaginary eigenvalue; take for example, this because of this particular form and in which the diagonal elements are equal to 0, the eigenvalues are plus minus 3 times j. So, this corresponds to as I said rotation about the origin either in the clockwise or the anticlockwise direction which we will decide very quickly. So, take a point along the x 1 axis, suppose this point is equal to 4 comma 0.

The point on the x 1 axis has x 2 component equal to 0, when matrix A acts on this we get minus 12 sorry 0 comma minus 12. We see that we get a vector which is parallel to the x 2 axis and in the negative direction. So, this is the arrow at the point 4 comma 0. Similarly, when we draw these arrows at different points we see that, we have a rotation in the clockwise direction. Every point except the origin is, if we start at any point then we are continuously rotating.

And, it turns out that the vector A times v is perpendicular to the vector v itself. So, if we are at any point v then the arrow at that point Av that is perpendicular to this. And, we see that this is nothing, but what corresponds to pure rotation in which the velocity is perpendicular to the radius vector. The clockwise or anticlockwise just depends on whether the sign whether we have a plus sign here or a plus sign here. So, this other example that is there on the computer corresponds to an anticlockwise rotation because, we have a negative sign here and a plus sign here. We have a anticlockwise rotation for the second example of A and both the A's correspond to periodic orbits, with the number 3 indicating the frequency.

But, since we are interested only in a qualitative study, the precise value of the frequency is not significant. Another important point to note here is we have a collection of periodic orbits. For each initial condition the radius, the distance from the origin decides which periodic orbit it is. The x 1 x 2 space itself is made up of periodic orbits which are all very close to each other which form a continuum. From each initial condition x 1 x 2, there is a periodic orbit, unique periodic orbit going around it.

And, if we go a little away or little closer to the origin then we have another periodic orbit. So, for the situation that A has imaginary axis eigenvalues, we have a continuum of periodic orbits and for a linear system it is not possible to have isolated periodic orbits. For a as we saw in one of our introductory lectures that we can have isolated periodic orbits for a non-linear system. But, for a linear system when we have periodic orbits it appears that we have a continuum of periodic orbits. In other words, if we start from a slightly different initial condition then we very unlike, it is very unlikely to be on the same periodic orbit.

If we are on this periodic orbit starting from this initial condition unless we are perturb, unless we perturb the initial condition to another point on the same periodic orbit, the periodic orbit is going to be different. If it is from this initial condition, then this initial condition corresponds to a different periodic orbit which means a different amplitude, even though it is a same frequency. So, this is a inevitable situation with linear systems when we have periodic orbits.

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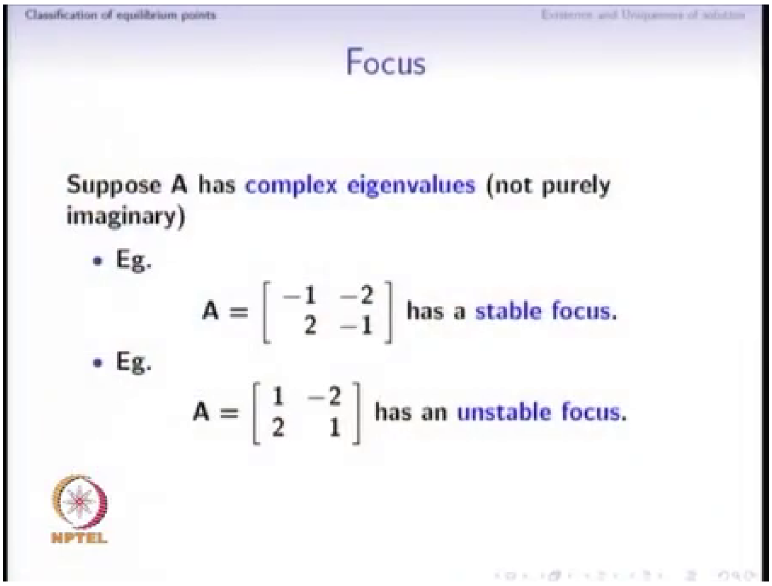

The next type of equilibrium point we will see is when A has complex eigenvalues and these eigenvalues are not purely imaginary.

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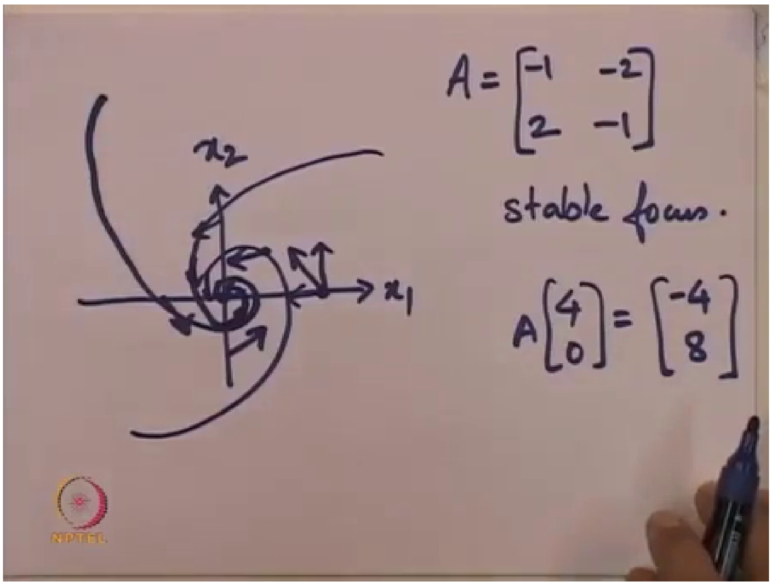

So, take for example, A equal to so, in which the diagonal elements are equal to minus 1 and the off diagonal elements have opposite signs one is plus 2, one is minus 2. So, this we will call is a stable focus, as I said the off diagonal elements cause rotation about the origin; each of these cases A is non-singular. Hence, the origin is the unique equilibrium point. So, let us take an example of a particular point and decide where the arrow is when we are at this point.

So, this point for example, is 4 comma 0 when A acts on this, we get minus 4 and in the and below we get plus 8. So, this is a vector which is like this, there is minus 4 component towards origin and 8 component along the positive x 2 direction because of which we have this. So, when we take different points we see that it is no longer perpendicular to the radius vector, but it is directed inwards. So, every point it turns out that we have some rotation and eventually the trajectories come to the origin.

For example, at this point if you draw the arrows at different points all trajectories seem to be approaching the origin, even though they do not approach the origin in finite time. So, each trajectory these trajectories do not intersect, but they all approach the origin and they reach the origin only asymptotically. So, this is a stable focus and unstable focus is also very easy to see, only that the diagonal elements have positive sign.

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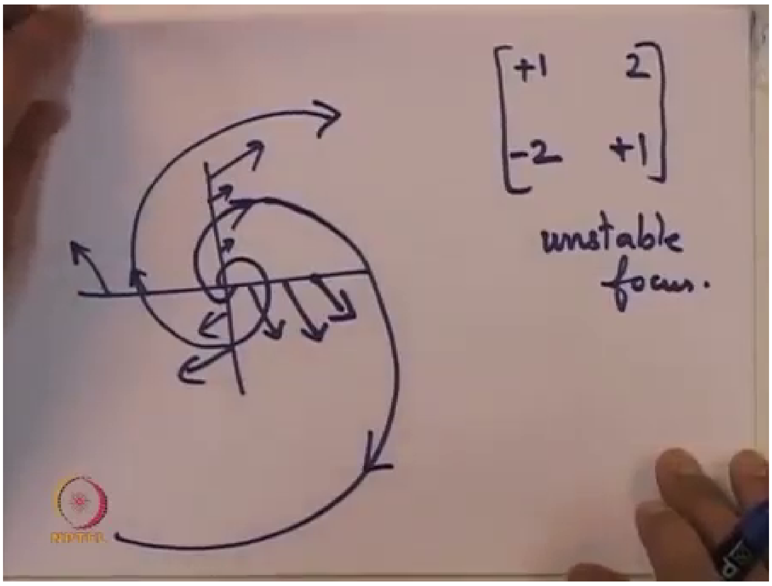

Now, all the arrows are directed away from the origin, also the rotation has been reversed because the signs of this and the previous example have been interchanged. So, here is an example where the arrows are all directed outwards. So, we have at any point we have trajectory that is going away, different points are all going away from the origin. So, this is what we will call an unstable focus finally, we will see what is a saddle point.

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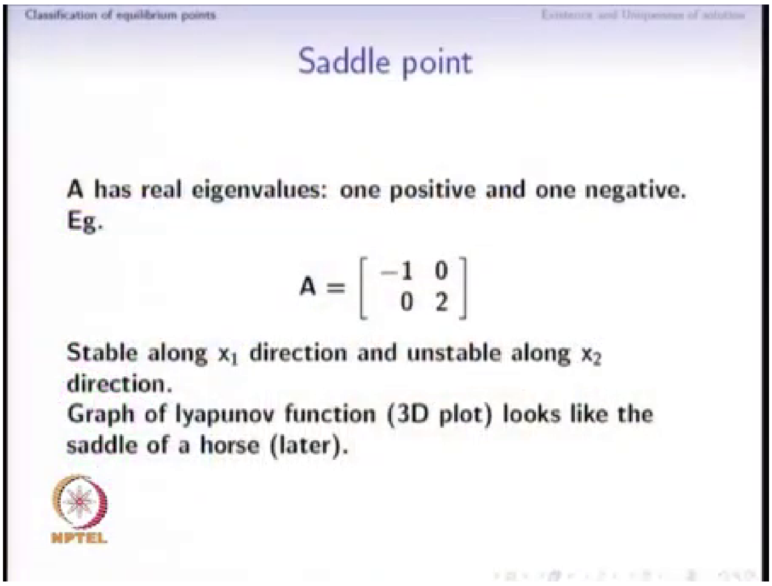

So, the situation when A has real eigenvalues one positive one, one negative; again for simplicity we will start with the diagonal case.

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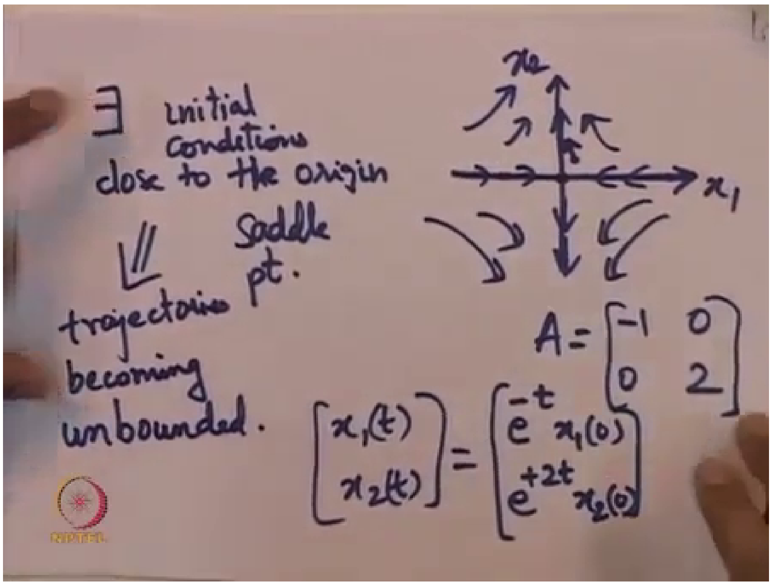

The time because, it is diagonal again we have a decoupled nature of the phase portrait. So, we see because A is equal to minus 1 0 0 plus 2 along the x 1 direction its approaching the origin, while along the x 2 direction is going away from the origin. And, any other point is a super imposition of these two features. So, we see that unless the x 2 component is equal to 0 which means we are along the x 1 direction all trajectories are coming towards the origin. Any other point where the x 2 component is non-zero, while the x 1 component is still decreasing the x 2 component is going to blow up.

Why? Because, the solution to this differential equation x 1 t, x 2 t because it is diagonal can be easily written as e to the power minus t times x 1 0, e to the power plus 2 t times x 2 0. So, unless the initial condition has x 2 component equal to 0, the x 2 as a function of time is going to grow exponentially. On the other hand, if the x 1 component is non-zero it is going to decrease and eventually become close to 0 asymptotically. So, this is what we will call a saddle point. The question arises is the saddle point stable or unstable equilibrium point?

We see that while the origin is an equilibrium point, for very small perturbations about the origin trajectories either come to 0; if they are along the x 1 axis or they do not come to 0, if they are not along the x 1 axis. In any case there are certain very small perturbations such that the trajectories when they begin from the perturbed initial condition, do not approach the equilibrium point. So, in other words there exists; so, this is the symbol for there exist, there exist initial conditions. These initial conditions are close to the origin; close to the origin.

What is significance of the origin? It is an equilibrium point, there exist initial conditions close to the origin such that the trajectories are not coming close to the coming back to the origin. So, we have in fact, the trajectories are growing, trajectories are becoming unbounded. This is precisely the property that decides that the equilibrium point, the origin is an unstable equilibrium point. So, the saddle point is an unstable equilibrium point, it is not an unstable focus nor an unstable node that equilibrium point is just an unstable equilibrium point.

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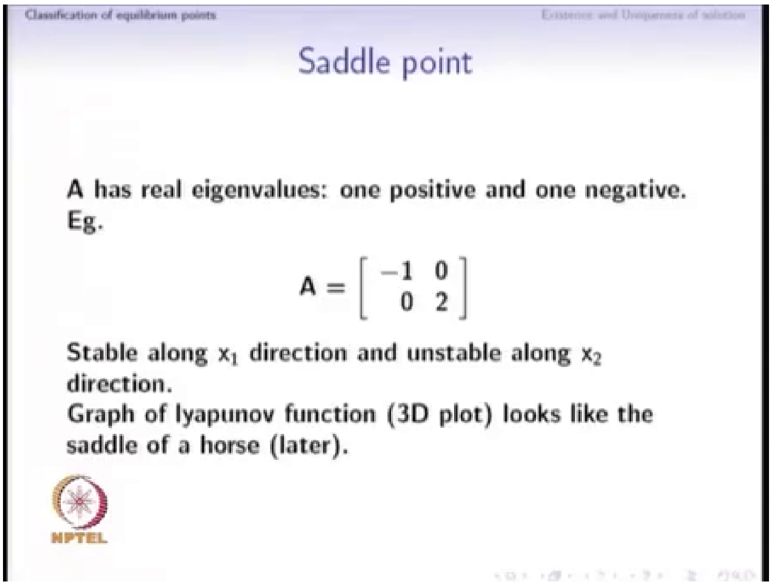

So, what is saddle about this? So, the graph of the Lyapunov function we will come back to this later. This graph in a 3D plot looks like a saddle of a horse, that is a reason that this equilibrium point is called a saddle point.

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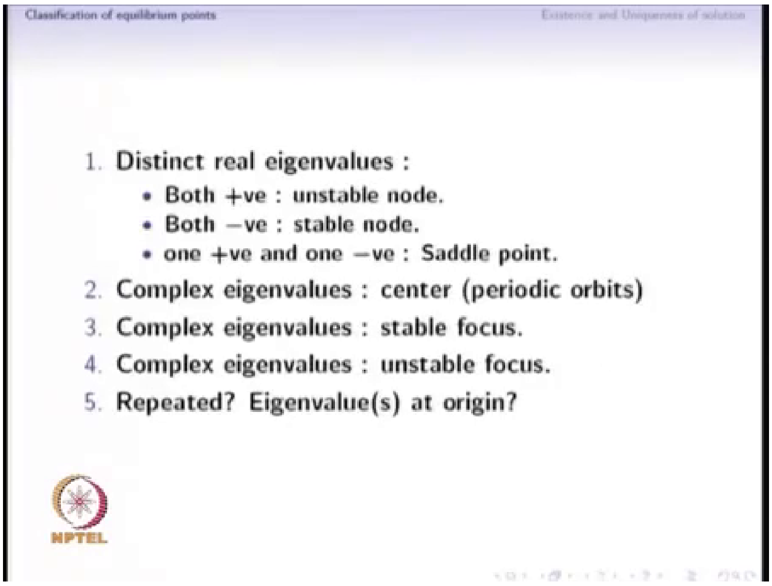

So, before we go to the other situation, where there are repeated eigenvalues or one or more eigenvalues at the origin; we will just quickly recapitulate what was done. So, we have seen the situation when there are distinct real eigenvalues. When both are positive or both are negative or when they have opposite signs, then we saw the situation when the eigenvalues are both complex. In which case, if they are on the imaginary axis we call it a center; this is the one that corresponds to periodic orbits and we saw that we will have a continuum of periodic orbits.

For the situation at the eigenvalues are complex, if they are on the imaginary axis then it is called a center; when the real part is negative we call it a stable focus and when the real part is positive we call it an unstable focus. So, whether it is stable or unstable depends on the real part of the complex eigenvalues. Now, the next situation, last situation that is remaining to be seen as when there are repeated eigenvalues and also the situation when one or more eigenvalues are at the origin.

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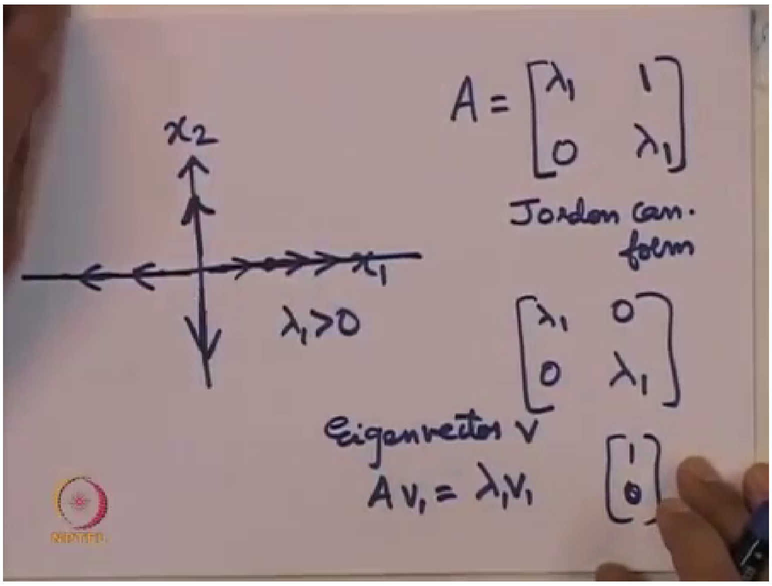

Coming back to the matrix A, when there are repeated eigenvalues that time the matrix A may or may not be diagonalizable. See suppose we have a repeated eigenvalue lambda 1 and if A is diagonalizable, that is in we will like to put a 0 here. And, if A is not diagonalizable then we put a 1, this is called the Jordon canonical form for the this case when there are repeated eigenvalues and A is not diagonalizable; we are restricting ourselves to the 2 by 2 case. And, this is the Jordon canonical form for the case when eigenvalues of A are repeated, but A is diagonalizable.

So, what is the significance of a diagonalizable matrix? We saw that the eigenvectors are so called invariant directions. In this particular example x 1 and x 2 directions are themselves eigenvectors. If we are along an eigenvector then lambda depending on lambda 1 being positive or negative, the arrows are directed either away or towards the origin. So, this is the case when lambda 1 is greater than 0. Let us restrict our study for that for that situation. The x 2 direction is also an eigen direction, is also an eigenvector and because lambda 1 was positive it is again directed away from the origin.

So, we see that the eigenvectors are the invariant directions. What is invariant about it? If the point starts along an eigenvector because, the arrow is also directed along the eigenvector we continue in that direction. So, we there is no tendency to move out of an eigenvector. Let me repeat eigenvector v is a non-zero vector such that Av is just a scaling of the vector v. So, we are interested in the first eigenvector v 1 which is nothing, but eigenvectors are not unique in magnitude. We can scale this vector to any number by any number and also get an eigenvector. So, it is a non-zero vector that satisfies this equation.

So, this v 1 if we are along this direction, if we are at a point v 1 then the vector is parallel to the vector v 1 because, of this particular equation. And, hence the trajectory will remain along that particular direction, if we start here then there is no reason to come out of the x 1 axis. Similarly, if we are here we will remain along the x 1 axis, similarly here x 2 also being an invariant direction; being an eigenvector it continues to be along the x 2 axis. So, we see that there are this particular complex plane contains certain invariant sets. What are those invariant sets?

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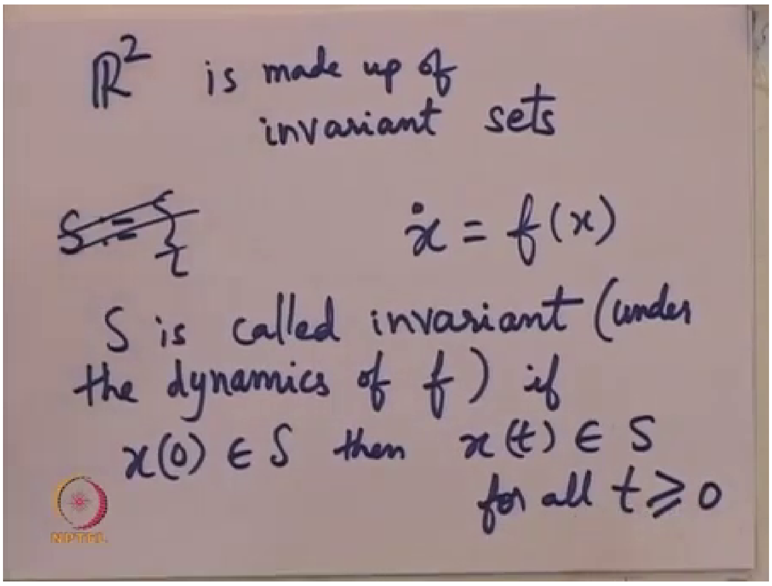

So, we will define, the plane R 2 is made up of invariant sets. So, what is this invariant set? A set S is called invariant, in this case it is invariant under the dynamics of the differential equation x dot is equal to f of x. If we start inside this set S then we will remain inside the set S for all future time is called invariant; invariant means under the dynamics under the dynamics of f if we start inside S, then x of t is also going to be inside S for all for all t greater than or equal to 0.

So, that is the significance of an invariant set that a set S which could be a subset of the plane R 2 or it could be the plane R 2 itself, it is said to be invariant if, if the initial condition is inside S then the entire trajectory is inside S for all future time. Hence, this is also called a positively invariant set. What is positive about it? Because, we are interested only for positive values of time t, x of t is inside S.

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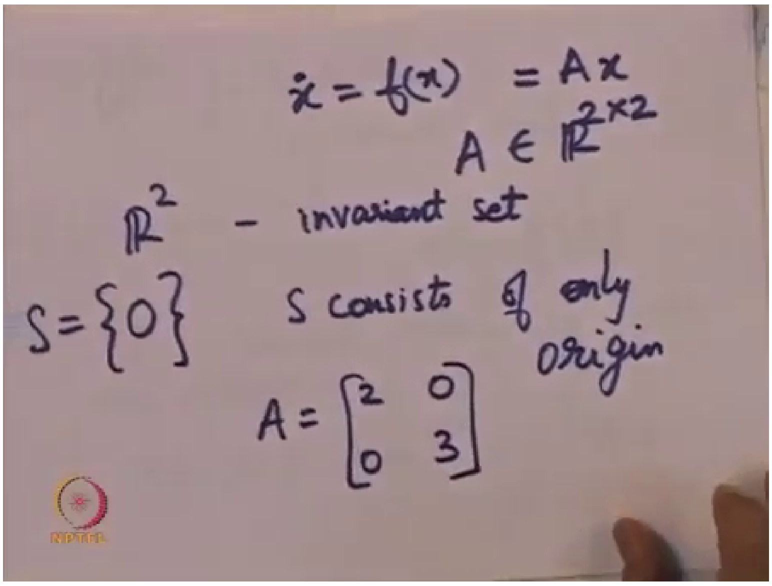

So, what is what are the invariant sets inside R 2? If we have this differential equation and let us take the special case when matrix A acts on the vector x and A is 2 by 2 which means x has 2 components. So, of course, R 2 plane itself is an invariant set. Why? Because, if it begins inside the set R 2, there is no reason it will leave the plane R 2. If the origin, the origin is an equilibrium point 0 the set S consisting of of just the origin yeah; S consists of only origin.

This is also an invariant set. Why? Because, if it begins inside this set S because it is an equilibrium point, it will remain at the equilibrium point for all future time and hence the set S is also an equilibrium is also an invariant set. So, all equilibrium points is an invariant set. For this particular case when A is a diagonal matrix, for this particular A there are some more invariant sets.

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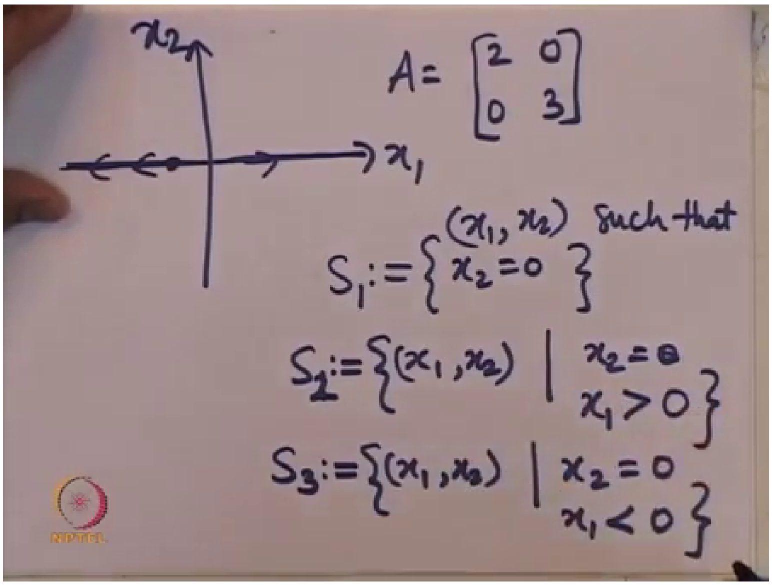

So, take the set S which is defined as all the points where, x 2 is equal to 0, the set the set of all points x 1 comma x 2 such that x 2 is equal to 0. This set is also an invariant set. Why? Because, if we are along the x 1 direction because A was a diagonal matrix; because it was a diagonal matrix x 1 axis itself being eigenvector we see that the set S 1 which is defined to be the x 1 axis is also an invariant set. Of course, the S x 1 axis itself contains the origin, it is also an invariant set.

In other words another let us call the set S 2, defined as all points x 1 comma x 2 such that x 2 is equal to 0 and x 1 equal to 0 which is nothing, but the equilibrium point is an invariant set. But we are interested in some nontrivial invariant sets for example, we could take x 1 positive. This particular situation is along the positive x 1 direction excluding the origin, this is also an invariant set. If it is once inside the set S 2, it remains inside the set S 2, consider S 3 which is the same x 1 comma x 2 except that now x 1 is negative.

This is another invariant set which corresponds to the negative x 1 axis, if the point starts here then it is going to always remain on the negative x 1 direction. So, these are different invariant sets. So, we are usually not interested in the equilibrium point as an invariant set. We are also not interested in the plain R 2 as an invariant set, because these are the trivial invariant sets; we are interested in some more sets which are larger than the equilibrium point and smaller than the set R 2 which are invariant under the dynamics of f.

And, the eigenvectors are examples of such invariant sets, eigenvectors the entire null, entire direction except the origin is also an invariant sets. And, the two sides of this eigenvector one on the positive side one on the negative side of the origin also form invariant sets.

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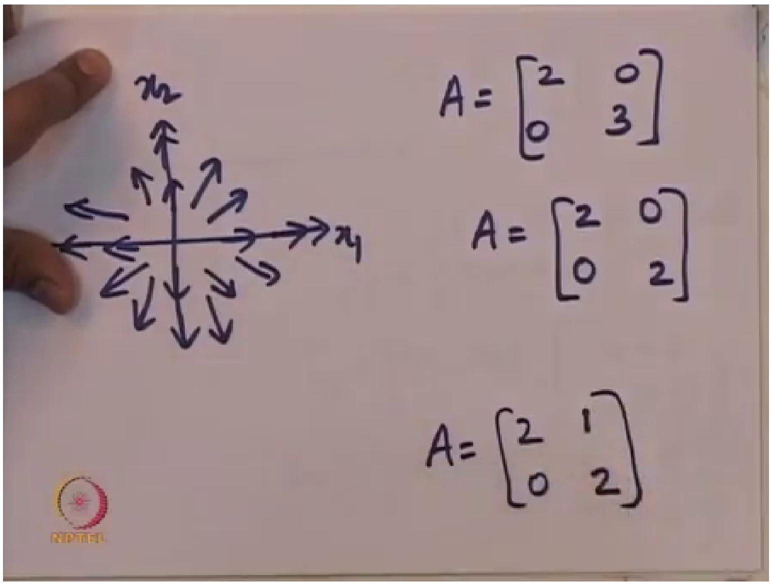

So, coming back to the case when A is diagonalizable, for that situation as we saw in some basis A already looks diagonal. So, we have x 1 axis which is an invariant direction, x 2 axis which is also an eigenvector; hence that is an invariant direction. And, it turns out that this invariance, these two direction been invariant is not particularly related to the eigenvalues being distinct. For the case when A has repeated eigenvalues, but if it is diagonalizable it is still is a unstable node.

Of course, in this case every direction is an invariant direction, is every line through the origin is a invariant set; because the two eigenvalues are repeated. But, for the situation when A is not diagonalizable; so, let us consider the case when A is equal to 2 0, but with a 1 here. This example of an A has only one eigenvector so, other eigenvector is what we want to call a generalized eigenvector.

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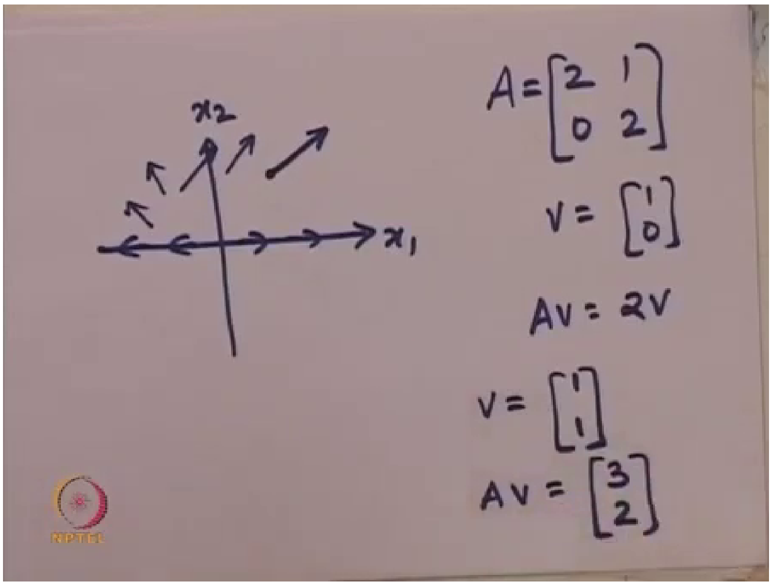

This A, which A are we dealing with now. We see that the x 1 axis, if we take the vector v equal to 1 comma 0, Av is nothing, but 2 times v. So, the x 1 axis is a invariant direction and all arrows are directed away from the origin. But, there is no other invariant direction, there is only one independent eigenvector. And, hence if you take an example let us say v is equal to 1 comma 1, when A acts on v we get 3 comma 2; let me check this yeah. So, for this particular vector at 1 comma 1, the vector has it has both x 1 x 2 components of that arrow non-zero.

So, we see that because there is only one independent direction, x 2 axis is no longer than eigenvector, but there are these other arrows that cut. How exactly they cut? They depend on the particular form of the Jordon canonical form, but along the independent axis there is only one x 1 direction. So, this is a significance of a non-diagonalizable A, that there is only one eigenvector x 1 and everything else is emanating out of this x 1 direction if it is very close to x 1. But, x if it is along the x 1 axis, then x 1 axis is being an eigenvector is a independent is a invariant set under the dynamics of f and hence, it does not leave the x 1 axis.

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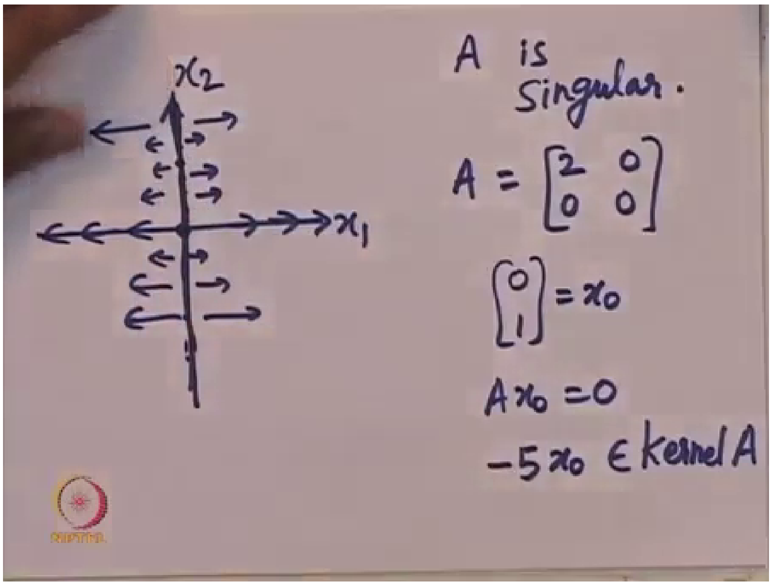

So, this brings us to the final case when A is singular, when A is singular there might be one or more eigenvalues at the origin.

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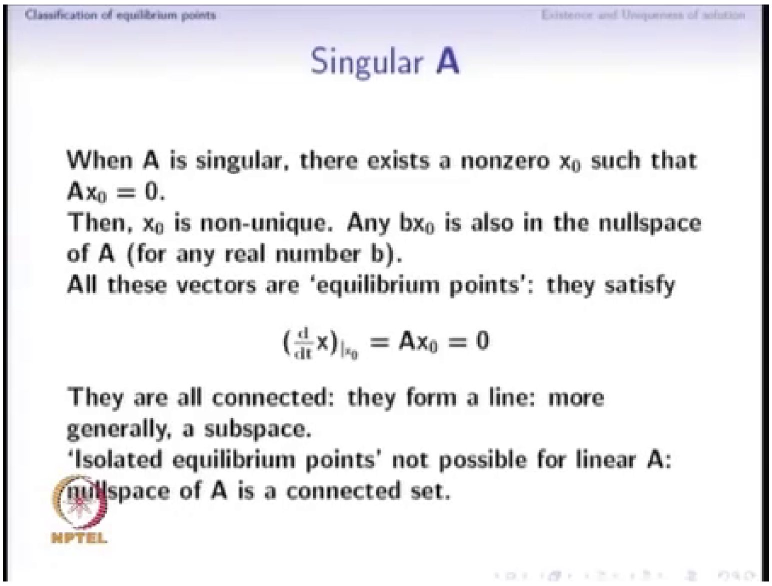

So, let us see the case when there is only one eigenvalue at the origin first. So, when A is singular it means that there exists a non-zero vector x naught; since that Ax naught is equal to 0. This x naught is also said to be in the nullspace of the matrix A, the origin is always there in nullspace. But, when A is singular there are some non-zero vectors also sharing the nullspace. And, such a non-zero vector x naught is non-unique.

Why? Because, if you are given with x naught then we can multiply x naught by a real number b and also get the b x naught to be in the nullspace of the matrix A. So, all these points x naught, b x naught any scaling of the vector x naught are all equilibrium points. Why? Because, they satisfy the derivative of x at that point evaluated at the point x naught is obtained by A acting on x naught which is equal to 0. So, we see that in this case all the equilibrium points are connected, they form a line.

The nullspace which is a linear subspace, in general they form a subspace and in our case because A has only one eigenvalue at the origin, they form a line. So, as we have seen in the beginning of the series of lectures, we saw that isolated equilibrium points is not possible for a linear system. For a linear system the equilibrium points as we have seen happen to be in the nullspace of the matrix A. If there are some non-zero vectors in the nullspace then they are all connected, they form a line. So, the isolated equivalent points is possible only when we have a non-linear dynamical system.

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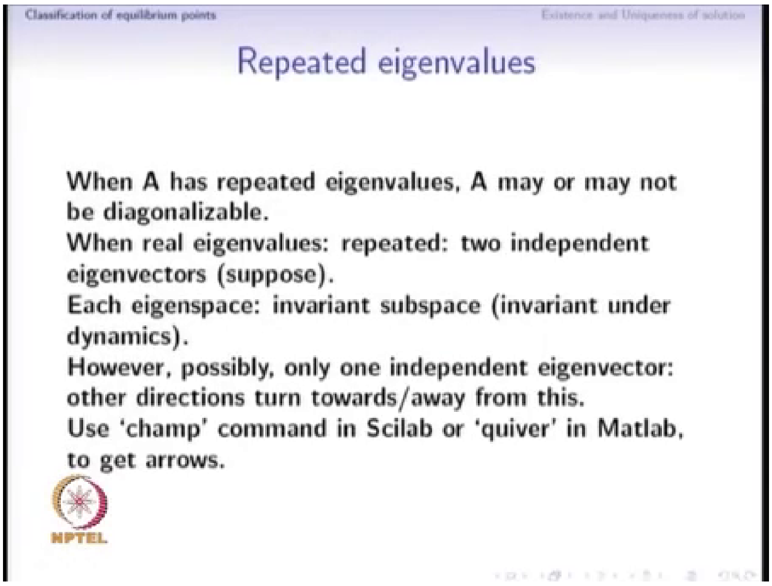

So, we have just begun seeing the repeated eigenvalue case. When A has repeated eigenvalues A may or may not be diagonalizable, we will quickly review this part. So, when the eigenvalues are repeated, then they have to be real for the case that A is 2 by 2 matrix. If they are if the matrix is diagonalizable then we have two independent eigenvectors, then each eigen space is an invariant sub space. Invariant meaning it is invariant under the dynamics of the system.

But, it is also possible that we have only one independent eigenvector in which case other directions either turn towards this or turn away from this depending on whether the eigenvalue is positive or negative. So, one can have a look at how the arrows look using 'champ' command in Scilab or 'quiver' command in MATLAB, when A is singular. So take for example, A equal to in this case this is a example such that A x naught is equal to 0.

This is of course, not the only vector x naught that satisfies Ax naught equal to 0, because any constant minus 5 times x naught also is in the null space; also in the null space is also said to be the kernel of the matrix A. So, what is the significance of this? We see that the x 1 axis is the eigenvector, but corresponding to eigenvalue is 2. And, hence we will draw the arrows away from the origin, but the x 2 axis are all equilibrium points so, each of the arrows I have length 0. So, if the x 1 component is non-zero, then we see that the trajectories are having that x 1 component increasing as a function of time, increasing with exponent equal to 2.

But, the x 2 component is always going to become equal to 0, when A multiplies to it. And, hence; and hence we see that these arrows are all parallel to the x 1 direction first of all. And secondly, along the x 2 axis because x 1 is equal to 0 along the x 2 axis all these points are equilibrium points, they form a connected set. The origin is not the only equilibrium point for this example, but each of these points are equilibrium points. So, this is what we see for the case when A has one eigenvalue at 0.

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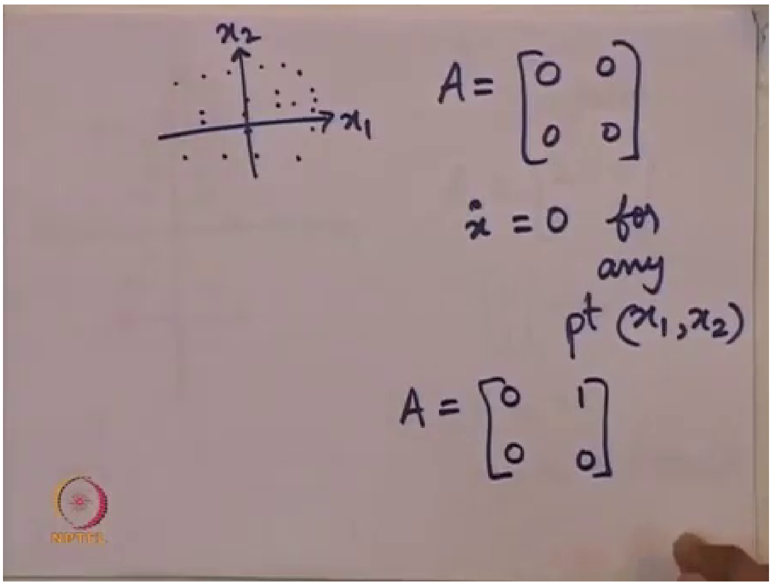

The next example is when A has both eigenvalues at 0, this is again an example of repeated eigenvalues. So, let us first take when A has two eigenvalues at 0 and when A is diagonalizable the reason we have 0 here. So, A is a 0 matrix so, the entire R 2 plane is made up of equilibrium points. Any point x 1 comma x 2 is an equilibrium point. Why? Because what does this matrix say? x dot is equal to 0 times x which is equal to 0 for any point; for any point x 1 comma x 2. So, this is the less interesting case, but still this situation is likely. The other situation when A has repeated eigenvalues at 0, but A is not diagonalizable is when we have this for example.

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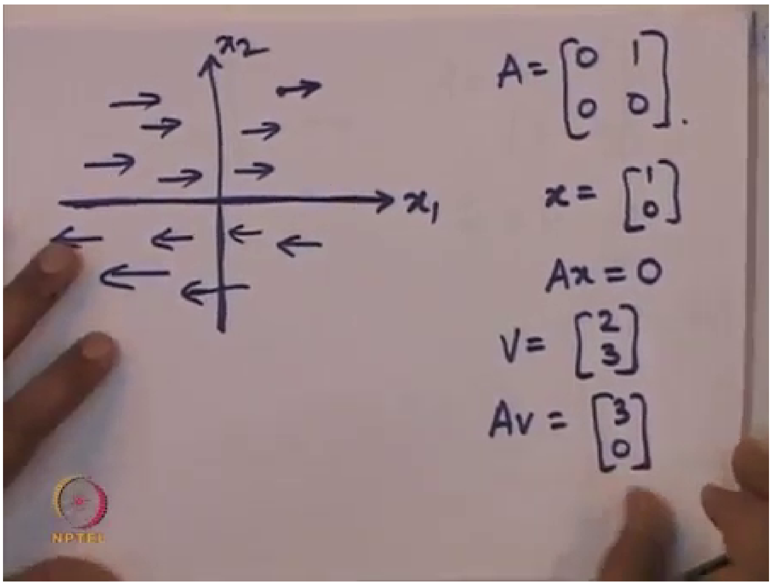

So, in this case we see that, if we have vector x equal to 1 comma 0 then Ax is equal to 0. So, the vector 1 comma 0 and all linear multiples of this x are in the kernel of the matrix A, they are in the null space of the matrix A. And, hence the x 1 direction is a set of equilibrium points. What is important about the x 1 direction? They all have x 2 component equal to 0, but if we take a vector v which is equal to 2 comma 3 in particular the second component x 2 component is not equal to 0. This particular vector here when A acts on v, we get something that is parallel to the x 1 axis.

So, we see that the arrows look like this, they are all in increasing direction of x 1; when x 2 component is positive and they are all along decreasing direction of x 1, if the x 2 component is negative. Why? Because, A is this matrix and A acts on a vector v it gives us the second component of v as the first component of A times v. So, this is an example where we have only one x 1 axis which is the equilibrium point, set of equilibrium points. And, every other vector is being turned towards either positive direction of x 1 or negative direction of x 1 depending on whether x 2 component is positive or negative.

So, this complete our study of equilibrium points for second order systems. We have seen the case when A has repeated eigenvalues, distinct eigenvalues and when A has real or complex eigenvalues.