Nonlinear System Analysis
Prof. Madhu Belur
Department of Electrical Engineering
Indian Institute of Technology, Madras

Lecture – 16
Phase Portrait of Nonlinear Systems: Examples

Welcome everyone to lecture number 10 on Non-linear dynamical systems. We will continue with the Bendixson criteria and the poincare Bendixson criteria. In particular we will see important examples one is the Lotka Volterra predator prey model and also the van der Pol oscillator.

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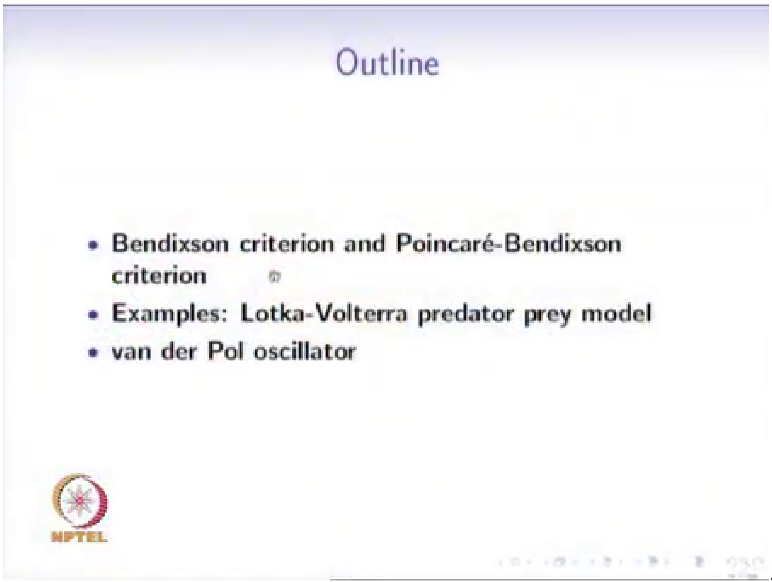

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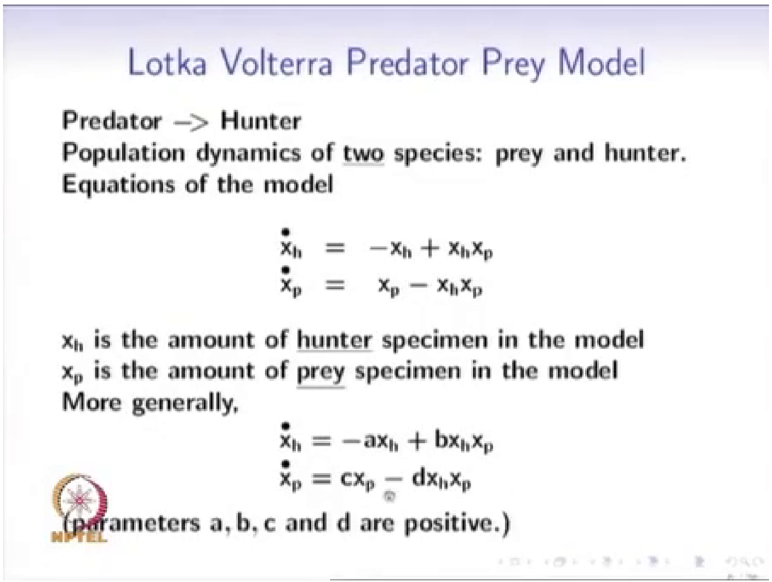

Let us start with the Lotka Volterra predator prey model; this is studying how the population of two species vary as a function of time, these two species are classified into prey and hunter.

So, there is one specie that is a prey another specie that is a hunter and we will study the model of this prey and hunter species. Of course, we are studying a simplified model; let x h be the hunter specimen in the model and let x p be the prey specimen in the model.

So, what does this equation say? x dot is equal to x dot h equal to minus x of h plus some quantity that depends on both x h and x p and x dot p equal to x p minus x h; x p. So, the first term in each equation is how the particular specie would evolve, if there were no others specie yeah.

So, the first system first equation says that if there were no prey that is if x p were equal to 0, then x h would just decrease as a function of time it would decrease exponentially because there is no food. So, left to itself the hunter specie would just decrease, but for each interaction between x h and x p; the hunter eats the prey and hence this extra, this extra the next term the second term in this right hand side is causing a increase in the hunter population. So, the hunter population decreases because of its own population and it increases because of its interaction with x p.

So, the rate of increase is proportional to both x p and x h population. So, it is bilinear in the two it is equal to the product that is the increasing increase causing term. On the other hand the prey itself is just going to multiply; it is going to increase exponentially when left to itself; if there had been no hunter specie and interaction with the hunter specie causes x p to decrease.

So, quantities x h and x p are all positive and whether the increase or decrease depends on its own population and also population of the other specie. So, this is a reasonable model for how dynamics of two species that interact with each other evolves as a function of time.

Of course, we have simplified most importantly in the sense that more generally there would be some constants x dot h would be equal to minus a times x h plus b times x h x p and x dot p is equal to c times x p, the rate of increase is proportional to some c times x p in general and decrease the interaction causes are decreased with this multiplication with d.

So, this is how one could study a general model, but one can consider that we are choosing a different unit for x h and x p, so that these constants become equal to 1, also there is some normalization that has been done so that we are studying this model. So, of course, this itself is a simplification, this model is also simplification because there might be some higher order derivatives.

We have seen already how the population of just one specie can vary with resource availability, with the ability to reproduce depending on the interaction between species; all that has been ignored. We have assumed this first order dynamics with respect to itself and just the product; the interaction is just the product of the two species population.

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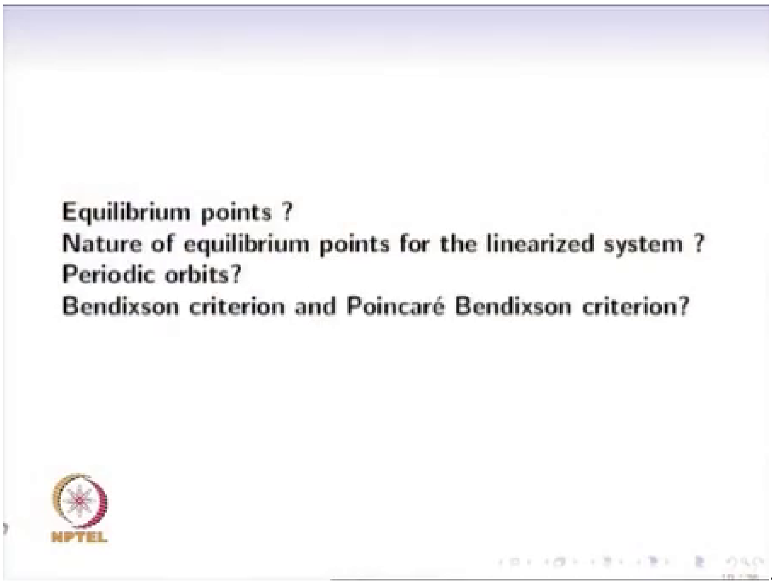

So, the questions that we can ask for this particular model is what are the equilibrium points? What is the nature of the equilibrium point of the linearized system? Are there periodic orbits? These are the questions that we will ask. So, let us go back to this particular model and we will find the equilibrium points for the system.

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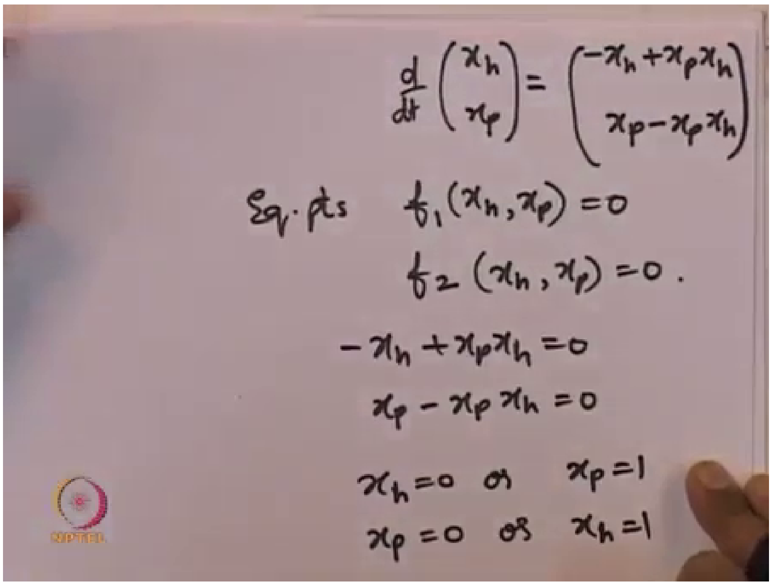

So, d by d t of x h and x p is equal to minus x h plus x p times x h and this is x p minus x p times x h yeah.

So, this is our f; so equilibrium points; points are those values of x h and x xp, where f 1 of x h comma x p equal to 0 and also f 2 of x h comma x p equal to 0. So, what do we get by equating x h minus x h plus x p; x h equal to 0 and x p minus x p x h equal to 0. For a particular value of x h and x p to be in equilibrium point, these two equations are to be satisfied.

So, let us see what are the values for which is this equations are satisfied; first equation says x h equal to 0 or x p equal to 1, second equation says x p equal to 0 or x h equal to 1. So, this gives us how many pairs of equilibrium point and equilibrium point has an x p and x h coordinate.

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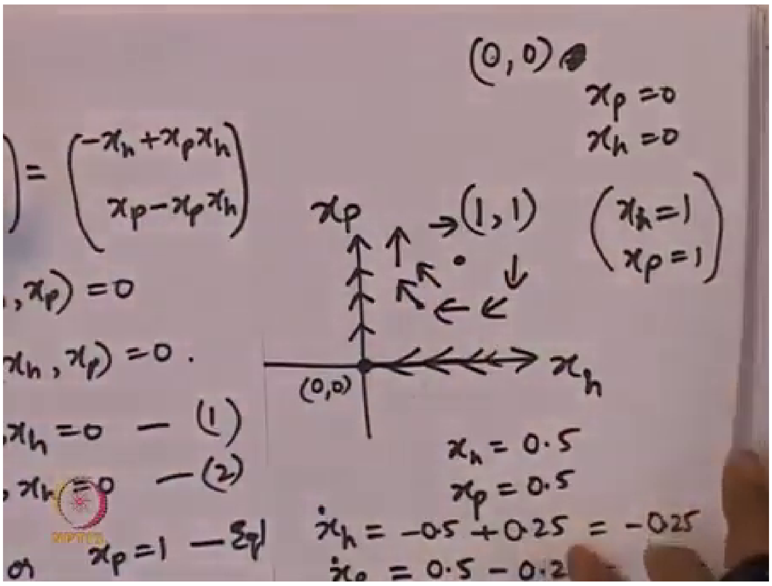

So, let us see what all possibilities are there for equilibrium point. See both equation have to be satisfied, then one can have 0 comma 0 yeah this is nothing, but x p equal to 0 and x h equal to 0. The first component in this is x h specie value; second is the x p population value.

So, both equal to 0 is one equilibrium point that is what we get from here and both equal to 1; 1 comma 1, yeah which means x h equal to 1 and x p equal to 1 this is another value for the equilibrium value point.

You see notice that other this if x h is equal to 0, you cannot have x p equal to 1 right because for the other for both the equations; this is equation 1 and this is equation 2; this is 1, is 2.

Equation 1 says that anyone of these two possibilities; equation 2 says any one of the these two possibilities.

And when we combine them we get that these two equilibrium points these two points; these two values of x p and x h are situations where the population specie does not change as a function of time. So, this is where x h, this is x p. So, one equilibrium point is here, another equilibrium point is here; this is the equilibrium point 1 comma 1; this is equilibrium point 0 comma 0. As I said the first component denotes the x h value.

Let us see what happens if x p is always equal to 0; x p equal to 0 means this is a hunter population yeah. So, our dynamical equation system says that if x p is equal to 0 which means that the second term is always equal to 0. And if you put x p equal to 0 here, then x h is just decreasing that is how we have drawn this arrows and if x h were equal to 0. So, this is sitting on the x p axis then x p just goes on increasing this is how the arrows look, but more generally it is a combination of the two.

So, for example, let us take what happens at 0.5; 0.5 let us draw the arrow at this particular point which corresponds to point 0.5 and 0.5. So, at x h equal to half and x p equal to half; we get x dot h equal to. So, this is just substituting 0.5 in place of these two. So, we get minus 0.5 plus 0.25 which is equal to minus 0.25 and x p dot is just 0.5 minus 0.25 which is equal to 0.25.

So, this is the vector whose x h component is negative, but x p component is positive. So, this is an arrow that looks like this. So, that its x h component; the horizontal component is x h it is decreasing, but x p component is increasing. So, like this we can draw arrows for all the points; one can check and this is how we get yeah.

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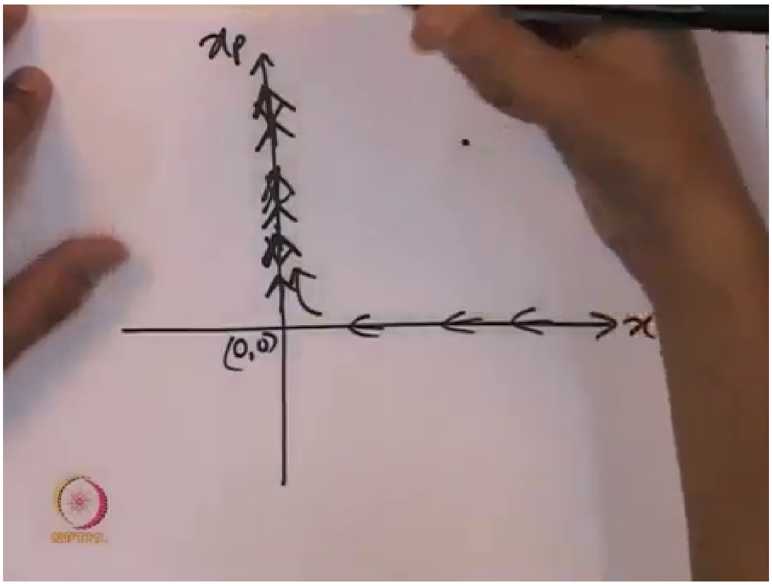

Let me draw a bigger figure, only the first quadrant is reasonable because of population do not become negative. So, this is a point 1 comma 1, this is 0 comma 0. As I said, x h population is going to decrease if x p is equal to 0; x p equal to 0 corresponds to this x h axis and x p axis corresponds to x h equal to 0.

So, I am sorry; so xh; x p when left to itself, the prey population is going to increase that is why the arrow should all be in the direction of increasing xp.

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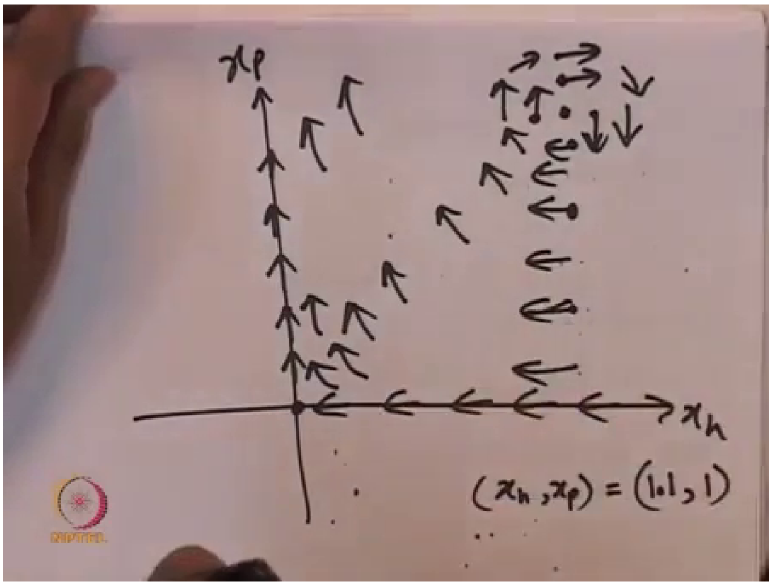

So, the correct figure should be and this is a point 1 comma 1 and you already checked that at intermediate points; at this point it is like this; one if it is a little higher.

Let us verify this that this is how it looks. So, this itself is an equilibrium point, if it happens to be at the point 1 comma 1; if the hunter population is equal to 1 unit and the prey population is also equal to 1 unit, then it remains constant, but for small perturbations above that point the arrows I have drawn like this, but this requires verification.

So, let us take a sample point; this particular point has x p coordinate equal to 1, but x h coordinate slightly more than 1. So, for example, let us think of consider the point x h comma x p equal to 1 point 1 comma 1 yeah; let us see what happens for this particular point.

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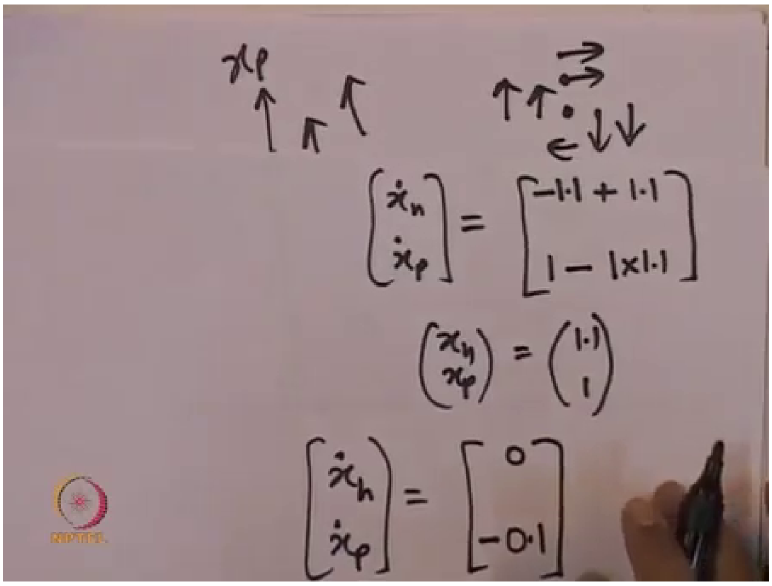

For this particular point we have drawn the arrow like this but let us check whether it indeed is like this. So, x h dot x p dot equal to where evaluating at the point x h comma x p equal to 1.1 and 1. So, this is minus 1.1 plus 1 times 1.1; so, this is equal to 1.1 and x p population rate of change of the prey population is equal to 1 minus 1 into 1.1 yeah.

So, this turns out to be equal to x h dot; x p dot is equal to the top component is 0 and lower value is minus 0.1. This is what happens when x h is slightly more than 1, slightly more than equilibrium point, but x p is equal to the equilibrium point value that is equal to 1.

So, when we do this; then we are speaking of this point here, for this point we are getting that x h rate of change is equal to 0. So, the horizontal component is equal to 0 and the vertical component is equal to minus 0.1 that is why it is vertically downwards yeah.

So, similarly one can check for each of these four points; what is the property of this point? Its x p population the prey population is slightly more than 1, but x h population the hunter population is equal to 1. For each of these four points, one can verify and see that the arrows are indeed like this; suggesting that there is a periodic orbit; around this point so there are periodic orbits close to this, but this point on the other hand looks like a saddle point. So, let us verify this by linearizing the system at each of these two equilibrium points.

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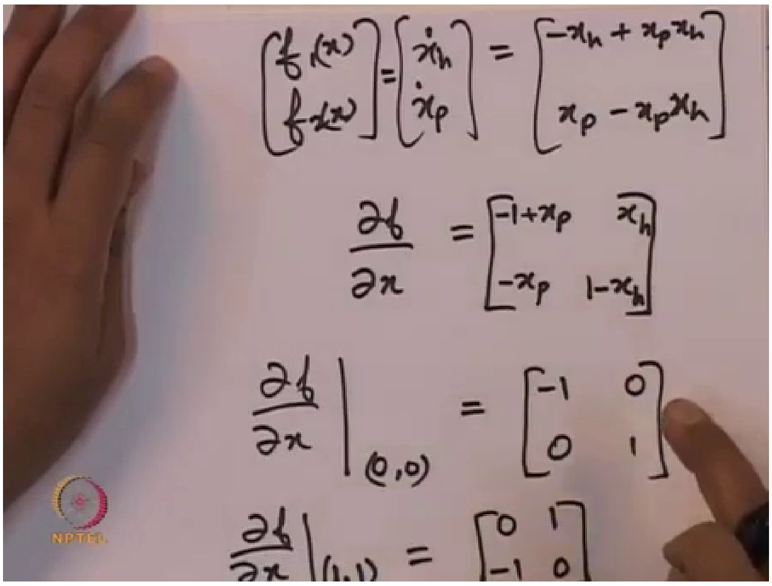

So, let us go back to the dynamical system; so x h dot x p dot minus x h plus x p; x h, x p minus x p x h. So, del f by del x; so this is equal to f 1 of x, f 2 of x equals this.

So, the first row the first function here is called as f 1 of x, the second function here is f 2 of x. Del f by del x is equal to a 2 by 2 matrix, the entry here is derivative of this with respect to x h that is equal to minus 1 plus x p. The entry here is derivative of this with respect to the second component of x that is x p; so this is equal to x h.

The entry that comes here is derivative of f 2 with respect to x h; here we get minus x p and the entry that comes here is the derivative of this with respect to x p, the second component of the state. For that we get 1 minus x h yeah. So, as expected this is a matrix, it is a 2 by 2 matrix which depends on x p and x h.

So, we are going to evaluate this matrix at the equilibrium point. So, del f by del x evaluated at the equilibrium point 0 comma 0, this is one of the equilibrium points and for this particular equilibrium point; we get minus 1, 0 0 1 and del f by del x; evaluated at the other equilibrium point, 1 comma 1 we get equal to by putting x p and x h both equal to 1; we get 0, 1, minus 1, 0.

So, we have these two a matrices; one a matrix for the equilibrium point 0 comma 0 and the other a matrix for the equilibrium point 1 comma 1. So, it is not difficult to see the eigenvalues of these two matrices.

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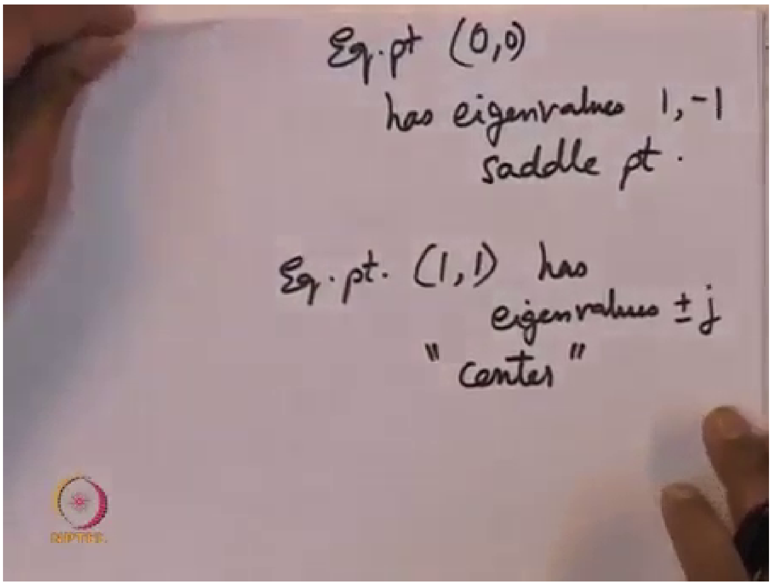

So, equilibrium point 0 comma 0 has eigenvalues for a diagonal matrix; the eigenvalues are nothing, but the diagonal entries, equilibrium point 0 comma 0 has eigenvalues 1 and minus 1.

So, we already saw that this is an example of a saddle point and the equilibrium point 1 comma 1 has eigenvalues. What are the eigenvalues of the matrix of which matrix? Of this matrix eigenvalues of this matrix are plus minus j.

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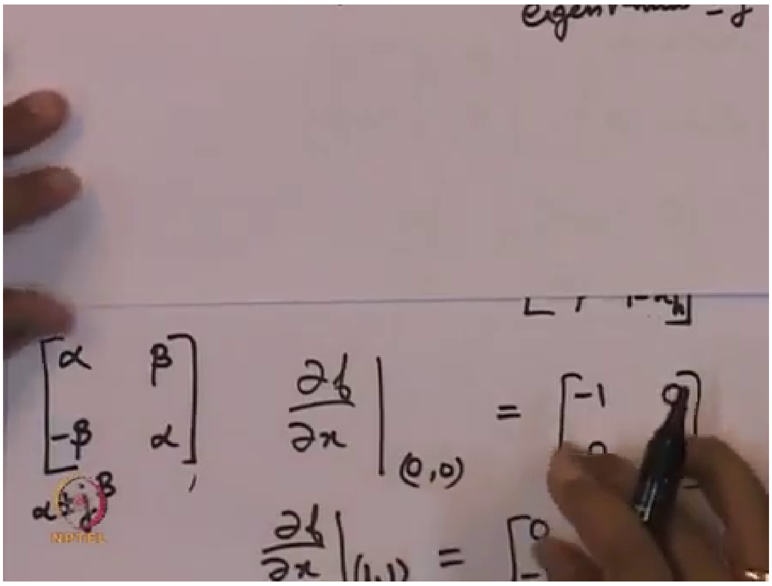

As we noted in one of the first few lectures that the eigenvalue of such a matrix; if beta is not equal to 0, then the eigenvalue of this matrix are equal to alpha plus minus j beta yeah. The eigenvalues of such a matrix are complex, precisely what complex values are the eigenvalues alpha equal alpha plus minus j beta; the diagonal entries are the real elements real part and the of diagonal entries with opposite signs correspond to the imagine part of the eigen value.

So, these are the eigenvalues even when beta is equal to 0. So, for this particular equilibrium point; equilibrium point we have this special case and so the eigenvalues are plus minus j which we know corresponds to a center yeah the equilibrium point is what we called a center. So, a center is 1 that has periodic orbits and we already saw that for this particular plot indeed this particular equilibrium point equilibrium point has periodic orbits and this is a saddle point.

So, the linearized system is a center which is nothing, but a continuum of periodic orbits very close by different initial condition correspond to different periodic orbits; they all corresponds to periodic orbits and different periodic orbits. Is that the same for the non-linear system also? This is the topic that we will see in detail today. So, please note that we have investigated the Lotka Volterra predator prey model; for convenience, the predator we have called as hunter, so that we can use a subscript h and the prey we continued to call p; x p.

The simplified model shows two equilibrium points; one equilibrium point the linearized system is a saddle point and the other equilibrium point of the Lotka Volterra predator prey model correspond to 1 comma 1 corresponds to a center after linearizing.

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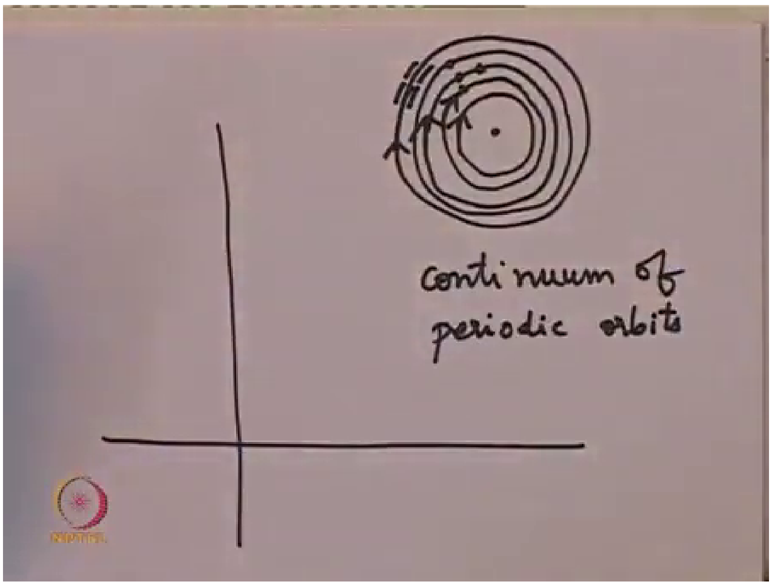

So, what is important is that this particular equilibrium point is the center. We have already saw that these arrows are suggesting like this.

If it were a linear system then when we go close this is a periodic orbit; when we go close to this and another initial condition, it may or may not be a periodic may or may not be different periodic orbit. The linearized system says so, but it need not mean for the original non-linear system also. For example, if these are two different initial conditions the correspond is same periodic orbit, but different initial conditions like this may correspond to different periodic orbits or might converse to the same periodic orbit.

This is a subject that we will see in detail today. So, if all these initial conditions correspond to different periodic orbits; then we will like to say that there is a continuum, continuum of periodic orbits. These periodic orbits are not isolated, but very close to each periodic orbit; there is another periodic orbit in a very close vicinity. Suppose, this is a periodic orbit the initial condition starting from here correspond to periodic orbits also; in that sense there is a continuum periodic orbits.

So, it is a very well known important fact that for the particular Lotka Volterra model that we have taken for; let us go back here. For this particular Lotka Volterra predator prey model for constants a, b, c, d we have two equilibrium points; 0 comma 0 and 1 comma 1, when you assume a, b, c, d equal to 1, but for a different point.

When a, b, c, d are some positive constants; possibly not equal to 1, there are two equilibrium points; while the 0 comma 0 is a saddle point, the other equilibrium point is a center and moreover for the non-linear system; for this Lotka Volterra predator prey model there is a continuum of periodic orbits. This particular fact for this particular model when you have these two models is a very important fact and one can modify this model suitably so that we have isolated periodic orbits.

So, today we are going to see a different example where there are indeed isolated periodic orbits.

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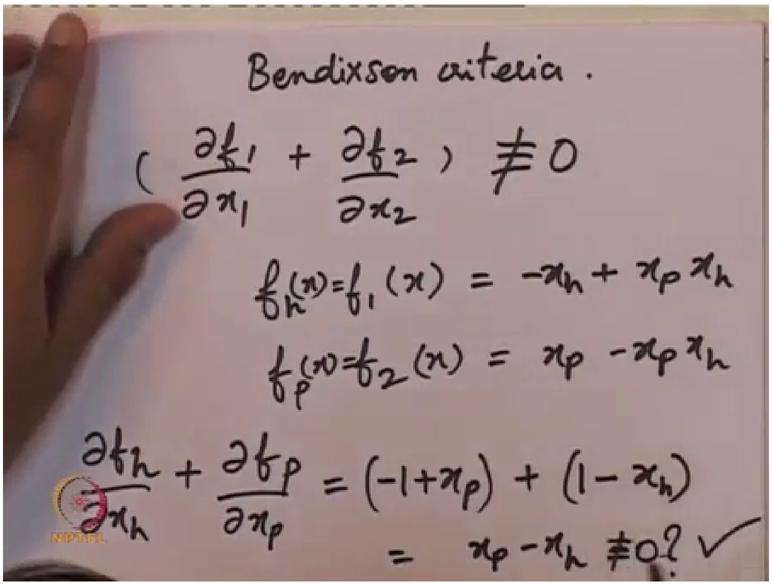

So, let us use Poincare Bendixson criteria and the Bendixson criteria to check if there are periodic orbits. Let us see the Bendixson criteria; what is this Bendixson criteria say? We will evaluate this particular quantity and check whether it is; whether this is identically equal to 0 or not. If it is not identically equal to 0, only then we can go ahead and apply the Bendixson criteria.

So, let us evaluate this particular quantity for our example; for our example f 1 of x was equal to minus x h plus x p times x h and f 2 of x is equal to x p minus x p times x h. So, this f 2 we also call as f p and this is equal to f h; f h denotes the rate of change of x h and f p denotes the rate of x p.

So, let us evaluate del f h by del x h plus del f p by del x p. When we evaluate this, we get derivative of this with respect to x h is equal to minus 1 plus x p plus derivative of this with respect to x p; we get this equal to 1 minus x h; so, this is equal to x p minus x h. So, is this identically equal to 0? No, it is not identically equal to 0 yeah. So, it is that is why we can go ahead and apply the Bendixson criteria.

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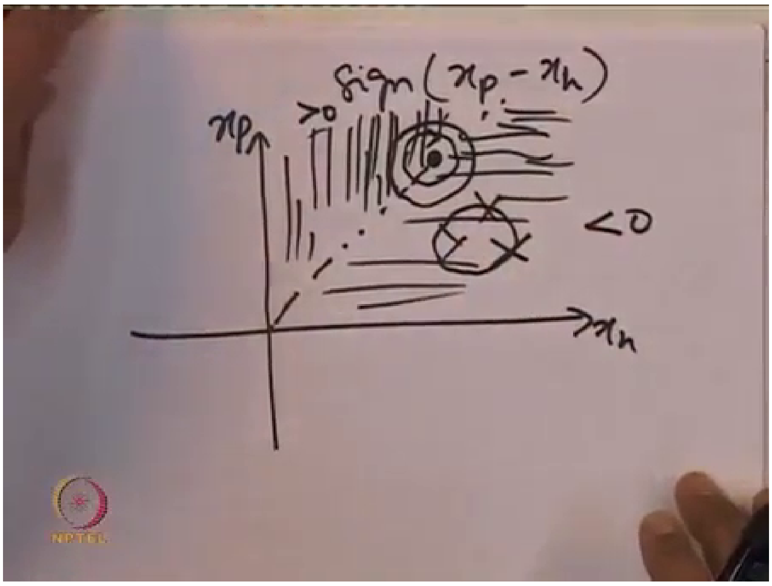

Let us now apply it and see x p minus x h; sign of this quantity, if the sign does not change over a region the Bendixson criteria says that if the sign of this particular quantity does not change on a region, then there are no periodic orbits contained inside that region.

So, when does x p; x p minus x h equal to 0? It is along this line. So, everywhere to the right of this line this; this is x h, this is x p to the right of this line, this quantity is negative and above this line or to the left of this line; this quantity is positive. So, the Bendixson criteria says that there cannot be a periodic orbit contained to the right of this line nor can there be a periodic orbit to the top of this line.

It does not; this is the equilibrium point 1 comma 1, the Bendixson criteria does not rule out such a periodic orbit, that does not lie entirely in this region nor does it lie entirely in this region. This is an important property to note that the Bendixson criteria only says that can such a periodic orbit exist inside this region? No, this is not possible can a periodic orbit lie entirely in this region where the sign of this is all positive that is also not possible. However, this particular periodic orbit could exist.

So, Bendixson criteria is only a sufficient condition for known existence of a periodic orbit lying entirely inside a region.

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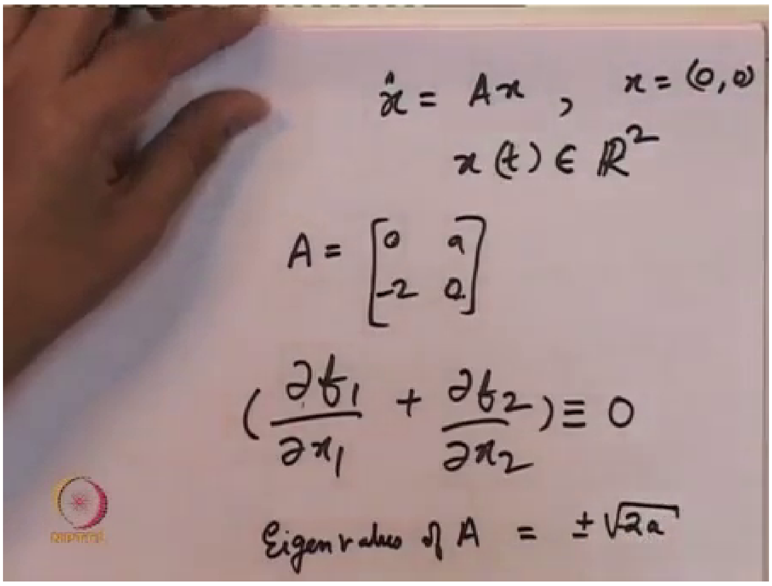

Let us now check what the Bendixson criteria says for a linear system x dot is equal to A x for its equilibrium point is 0 comma 0 yeah. So, Bendixson criteria is applicable when for the planar case; that is when x has two components, at each time instant x of t has two components x 1 and x 2. So, suppose A was equal to yeah maybe we see a slide about this.

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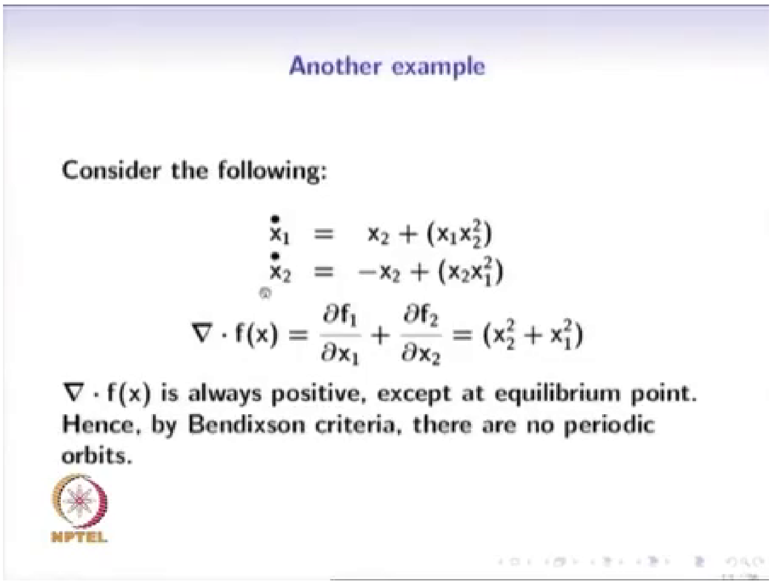

So, for the Lotka Volterra predator prey model; we have already seen this.

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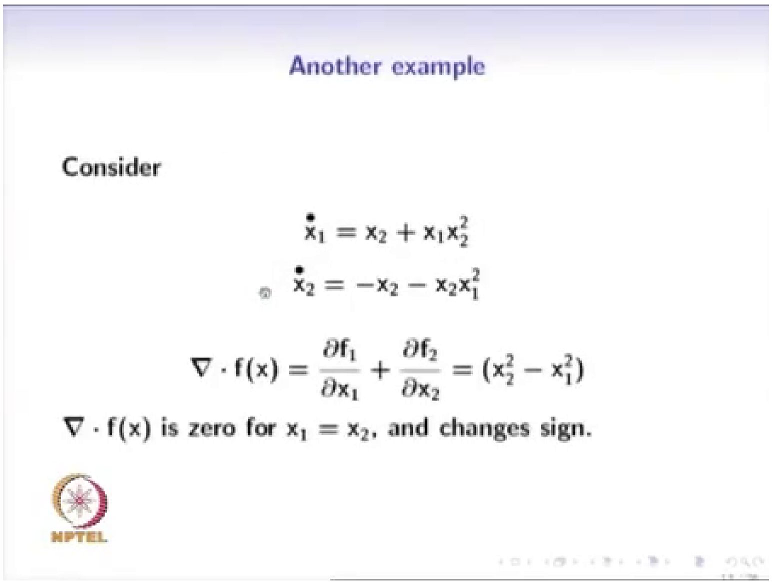

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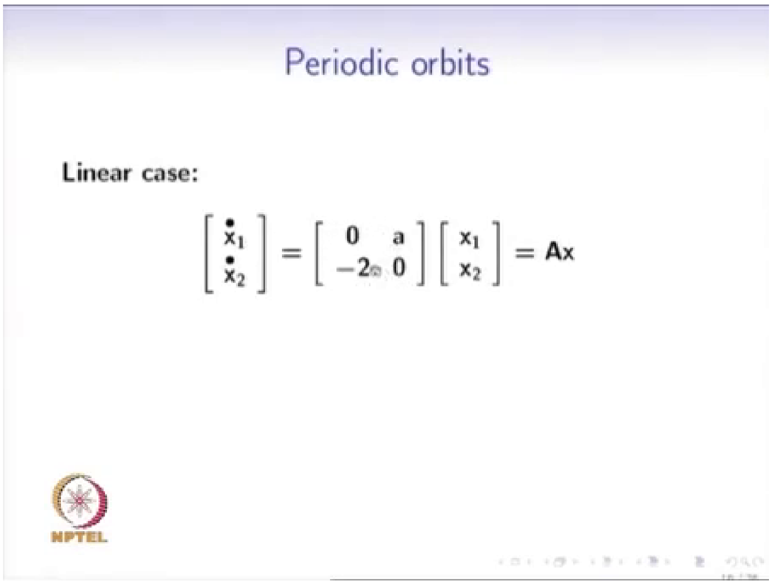

Before we see another example; let us see this particular case periodic orbit for A; that looks that is of this form.

So, our A, we have already assumed in the it is of this form and we have now we will do del f 1 by del x 1 plus del f 2 by del x 2; notice that these two terms are nothing, but the diagonal entries of this matrix A. So, for this particular A; the diagonal entries are both 0, so they add up to 0 also; so their identically equal to 0.

No matter which x 1, x 2 you check; this is going to be equal to 0. This particular quantity is expected to be independent of x 1 x 2 for linear systems. Why for linear systems; for linear time invariant systems? These four entries are all independent of x and hence you differentiate f 1 and f 2; f 1 with respect x 1, f 2 with respect to x 2 which is nothing, but just picking up these entries; picking the values at these two positions and they are going to be independent of x.

So, for this particular a; we get this identical equal to 0. So, do we say that Bendixson criteria is not applicable or do we say that there are no periodic orbits? Of course, we know that for this particular a; the eigenvalues of A are equal to plus minus square root of two times a. So, if a is positive, then the eigen values are plus minus 2 times minus of 2 times a; we will just verify this.

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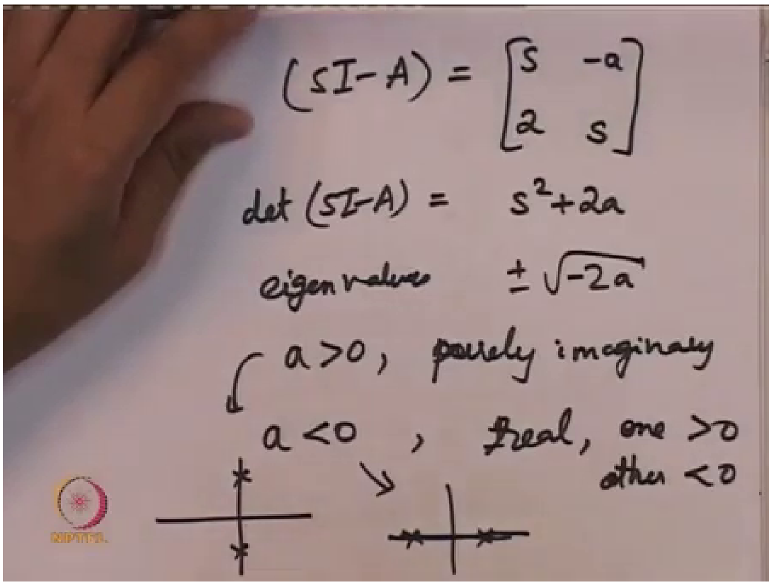

So, what is SI minus A; SI minus A is equal to; so determinant of SI minus A is equal to S square plus 2 a. So, eigenvalues; eigenvalues of the A matrix are nothing but roots of the determinant. So, roots are square root of minus 2 a plus minus. So, if a is positive a greater than 0, then complex purely imaginary in fact.

If the eigenvalues are purely imaginary; then we know for a linear system that are periodic orbits. And if a is less than 0, then eigenvalues are plus are real; real, one of them is greater than 0, other is less than 0. Why? Because this if a is negative this quantity itself under the square root sign is positive. So, we can take the square root and one is positive; one is negative.

So, for this case the eigenvalues are here and for this case; the eigenvalues are here and here. How far from the origin? Depends on the value of a of course, but whether they are depending on whether it is positive or negative affects whether the roots are purely imaginary or real. So, we know that for this case; the equivalent point is a center and there are periodic orbits. While for this case; the equilibrium point is a saddle and there are no periodic orbits.

So, the important case when this is identically equal to 0; that particular case could correspond to either there are periodic orbits or there are no periodic orbits. This is just to see that the Bendixson criteria is unable to say anything; when this is identically equal to 0 that is precisely the reason that Bendixson criteria assumed that this is not identical equal to 0 and then you start looking at whether the sign changes or not.

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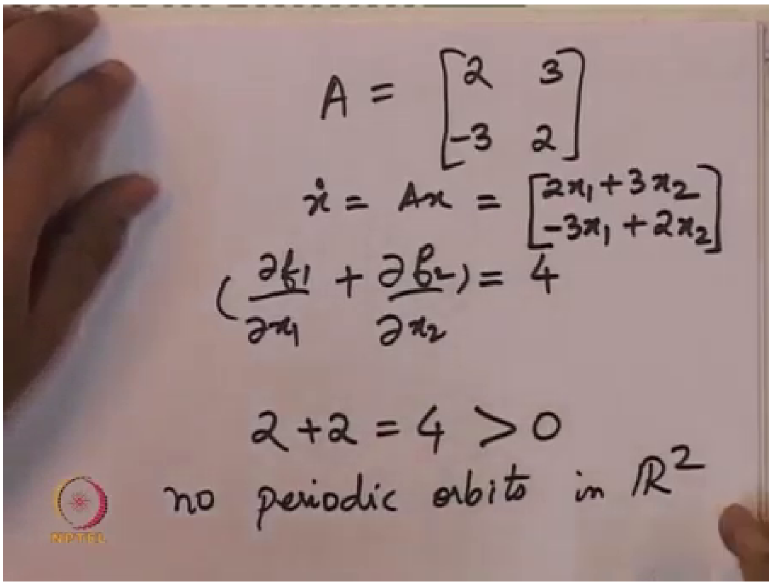

So, let us take a case where for x dot is equal to Ax; let us check what, what is del f 1 by del x 1 plus del f 2 by del x 2. What is the value of this? We will check that this is equal to 4 by calculation explicitly.

So, x dot is equal to Ax means 2 x 1 plus 3 x 2, that is a meaning of A acting on x and the second row of A will be used to multiply with x to get minus 3 x 1 plus 2 x 2. So, when we do this; then we differentiate the first component of x dot with respect to x 1 and we get this equal to 2, we are picking at just this entry. And the second component of x dot that is f 2 of x with respect to x 2; we are doing this.

So, notice that derivative of this with respect x 1 is just this component; this first one by one entry. And the derivative of this with respect to x 2 is just this entry that is the reason that I said that doing this particular to evaluate this quantity is nothing, but to add the diagonal entries for a linear system; for a linear time in variant system.

So, we get this equal to 4, this is greater than 0 and it is independent of x 1, x 2. For linear systems, we expect that this will not depend on x 1 x 2 and it is indeed independent of x 1; x 2. Since, it is greater than 0 for all x 1; x 2 we get that no periodic orbits; no periodic orbits in R 2. In the entire state space, in the entire plane there are no periodic orbits.

So, for linear systems we can check that as long as the diagonal entries do not add up to 0 yeah; as long as the diagonal entries do not add up to 0, this quantity will not be identically 0 and then we can see that periodic orbits are ruled out. When would periodic orbits be possible?

If the diagonal entries add up to 0; if the diagonal entries add up to 0 we cannot say that the periodic orbits exist because the Bendixson criteria is silent for that case; it does not say anything when the diagonal entries add up to 0 identically. We already saw that it is possible that there are periodic orbits; it also possible that periodic orbits do not exist when the diagonal entries add up to 0.

So, this is already the complication for linear systems. So, for the Lotka Volterra predator prey model; to show that there are periodic orbits, is a difficult thing and it is a important research topic; after which it has been concluded that there are there is a continuum of periodic orbits for the particular model that we studied.