Nonlinear System Analysis

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Lecture - 17
Limit Cycles

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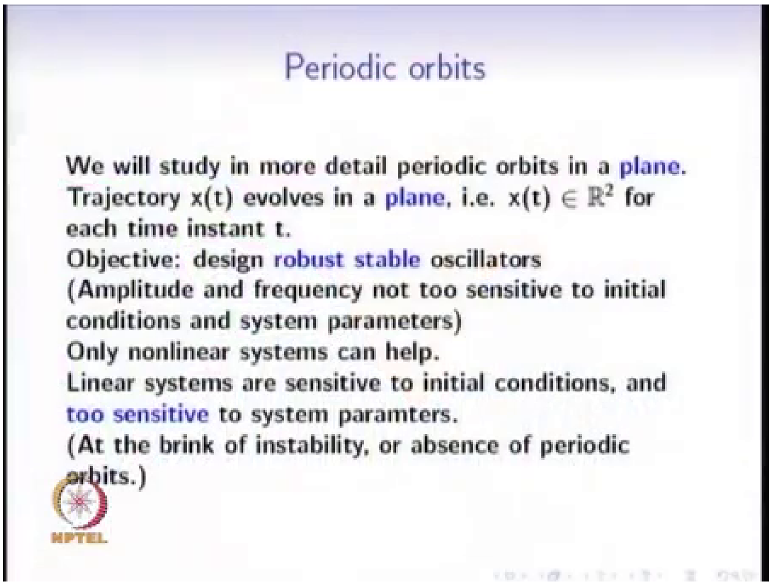

Now, we come to the other topic about periodic orbits. For this purpose, we will study periodic orbits in more detail for a plane where the trajectory evolves in a plane. So, in other words at each time instant at each time instant x of t is an element of R 2, it has two components only.

So, what is the objective? The objective is to design robust and stable oscillators. So, what is robust about this and what is stable about this? We want that the amplitude and frequency of the oscillations are not too sensitive to the initial conditions and are not too sensitive to the system parameters. So, as we had noted at the beginning of this of these lectures only non-linear systems can help. Why is that, because linear system first of all are very sensitive to are sensitive to initial conditions in other words if you start with a different initial condition then the amplitude is different.

Of course for linear systems the frequency remains the same, but the amplitude is different. Moreover, the fact that there are periodic orbits is extremely sensitive to system parameters. In other words the eigen values around the imaginary axis, but small changes in the system parameters eigen values could be in the right half plane or in the left half plane, which means that we might have either no periodic orbits and all trajectories go to 0 or the trajectories could become unbounded, and there is again absence of periodic orbits.

In other words linear systems are at the brink of instability and hence periodic orbits are extremely sensitive to the system parameters. So, for non-linear systems, the question arises how to even claim that there are periodic orbits for this system of equations? So, one extremely important result in this context is the Poincare Bendixon Criteria. So, what does a criteria tell?

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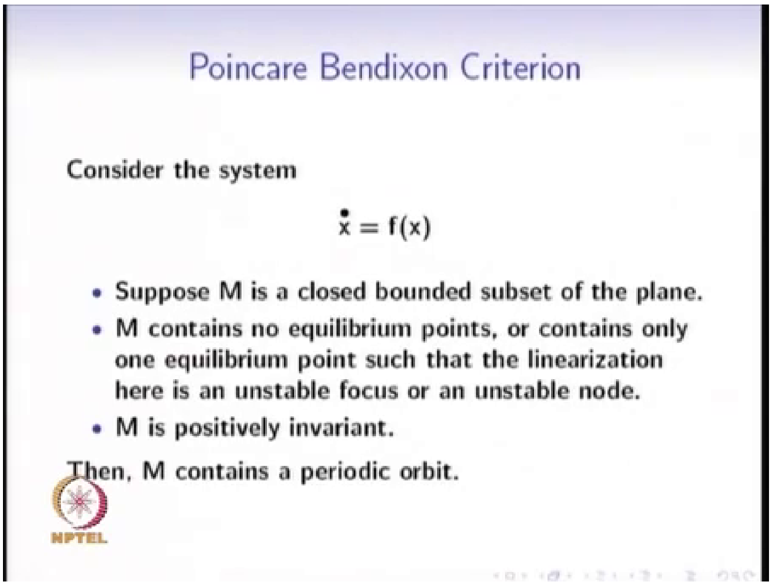

So, consider the system x dot is equal to f of x in which note that x has only two components; x here has only two components. And suppose, the set M; suppose, there is a set M which is a compact set, it is a closed and bounded subset of the plane Suppose, M has the property that M contains no equilibrium points or M could contain an equilibrium point such that that equilibrium point is either unstable focus or an unstable node.

So, there are two situations for the second bullet. The first case is M contains the equilibrium points; the second situation is that when we linearize at that equilibrium point. If there is any equivalent point then at most one equilibrium point is allowed, and at when we linearize at that equilibrium point then the linearized system has an unstable focus or an unstable node. In other words both the eigen values of the matrix A, which we get by linearizing at this equilibrium point are unstable, both the eigen values are in the open right half complex plane.

So, suppose M has this property, further suppose M is also positively invariant. Yeah, if M satisfies these three conditions that it is a compact set, it is positively in variant and either M has no equivalent points or at most one which is unstable. These three conditions are sufficient to ensure that M contains a periodic orbit, yeah. So, under these assumptions, the Poincare Bendixon criteria claims that there is M is guaranteed to contain a periodic orbit.

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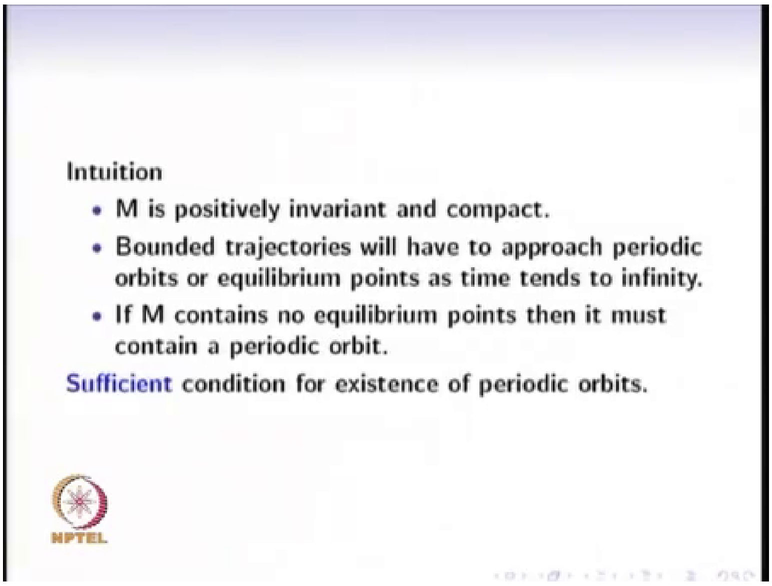

So, what is the intuition behind this? So, M is positively in variant and compact, in other words trajectories at start inside M remain inside M for all future time and since the M is compact. These trajectories are all bonded, they cannot become unbounded because they do not even leave M, and M is bounded.

Further, these bounded trajectories will have to approach periodic orbits or they can approach equilibrium points. As t tends to infinity what happens to all these bounded trajectories, they either approach the equilibrium points or they approach periodic orbits, these are the only two possibilities. Why, because the trajectories are all bounded and they exist for all future time.

Now, if we rule out existence of any equilibrium points inside M, then we are forced to have a periodic orbit, this is what Poincare Bendixon criteria says. Secondly, even if M had a periodic even if M had a equilibrium point, but if that were unstable and the trajectories could not be converging to them or trajectories could only be converging to the equilibrium point. So, we would have an periodic orbit, even if M had an equilibrium point which was unstable in that case.

So, these three conditions on M ensure that there is a periodic orbit. So, please note that this is only a sufficient condition for existence of a periodic orbit; of course, they can also be a continuum, they can be non-unique periodic orbits, they can also be a continuum of periodic orbits which we will see very soon.

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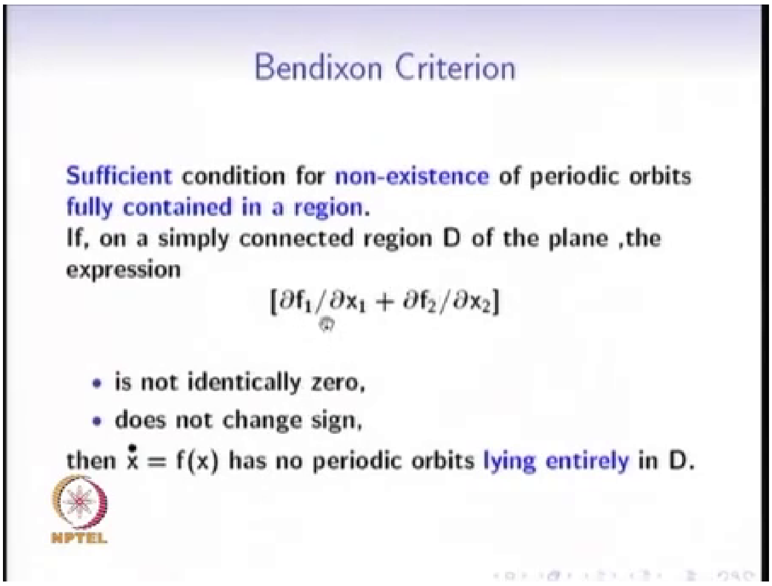

Another important criteria in this situation is, the so called Bendixon criteria. So, what does this criteria say? It is a sufficient condition please note which conditions are necessary, which conditions are sufficient. The Bendixon criteria is a sufficient condition for non-existence of periodic orbits; yeah, it is a sufficient condition for non-existence of periodic orbits that are fully contained inside the region.

So, what is the criteria? If on a simply connected region D of the plane so, we will quickly see what is simply connected region is. So, on a simply connected region of the plane if this particular expression here, yeah satisfies the condition that it is not identically 0 and it does not change its sign.

If these conditions are satisfied, then the system of equations x dot equal to f of x has no periodic orbits lying entirely in D. So, inside the region D, we check that this quantity is not always equal to 0 and it also does not change sign inside D. If those two properties are satisfied by this particular function then they cannot be any periodical orbit lying completely in D. So, please note that it is only non-existence fully contained inside D of periodic orbits that is being guaranteed by the criteria.

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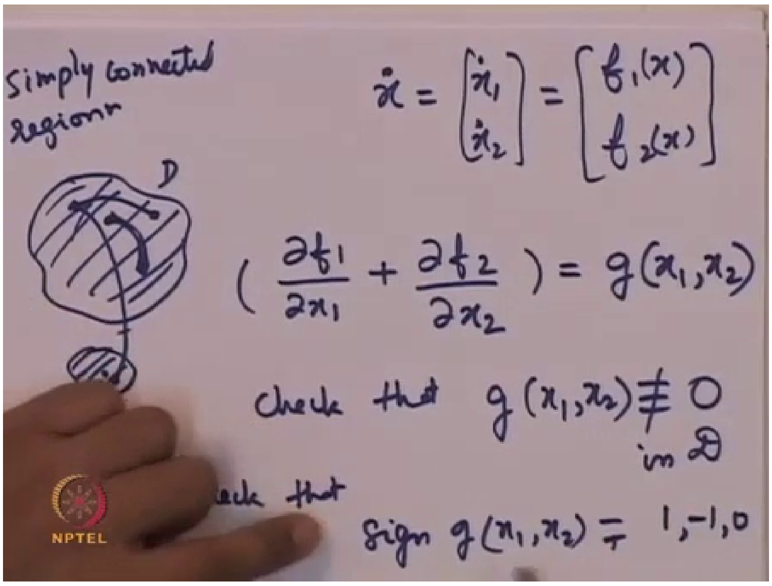

So, x dot is equal to x 1 dot x 2 dot which is equal to f 1 of x, f 2 of x. So, this is our dynamical system, as I said we are considering evolution of trajectories in a plane. So, x has only two components; x 1 and x 2 and hence this differential equation has only two equations inside this. So, now, we differentiate the first one, we differentiate f 1 with respect to x 1 and to that we add the partial derivative of f 2 with respect to x 2. So, note that f 1 depends on x which is x 1 and x 2.

So, f 1 can depend on x 1 also x 2 also and similarly f 2 can depend on x 1 and x 2, hence we have partial derivatives of f 1 and f 2 here, partial derivative of f 1 with respect to x 1, partial derivative of f 2 with respect to x 2. This particular quantity is sum function yeah, suppose that function is called g which depends on both x 1 and x 2. So, what does the Bendixon criteria say that, this particular quantity you check that g of x 1 comma x 2 is not identically 0 in D. So, D is a region inside this region, this quantity is not identically 0.

In other words there is at least 1 point x 1 x 2 where this is not equal to 0, as soon as soon as this is not equal to 0 at least at a single point, it means this is not identically 0 in D. It is allowed to be 0 at a few points at several points. However, it should not be equal to 0 in at all the points in D, that is that is a statement that this is not identically 0. Also check that the sign of g; the sign of g can be equal to either 1 or minus 1 or 0 yeah, this is in general possible. But, we require that the sign of this should not change in other words it should not go from minus 1 to 1; from minus 1 to 1 or 1 to minus 1.

As long as it is always 1 or always minus 1, maybe at some points it become equal to 0, if the sign of this does not change when you check for different x 1 x 2 points in this region D. If these two conditions are satisfied then the Bendixon criteria says then D cannot fully contain a periodic orbit. There is no periodic orbit that lies entirely in D. So, another assumption that we had made was the D is a simply connected region.

So, what is a simply connected region? So, in the plane x 1 x 2, suppose this is a region D. We will call this region D simply connected; connected of course, means that D should not be made up of two such parts, this is in D and this is in D. So, this is not connected right, because to say that is connected we take any two points and there should be a curve, there should be a path between these two points, and the path the points on the path also should lie in D and this should be possible for every two points in D.

So, if D had two components while certain points are connected by a path lying in D every two points are not connected yeah, look at a point there a point here a path from there to here is forced to go outside the set D. So, this such as set would not be connected even. So, for a region that is connected what do we mean by simply connected?

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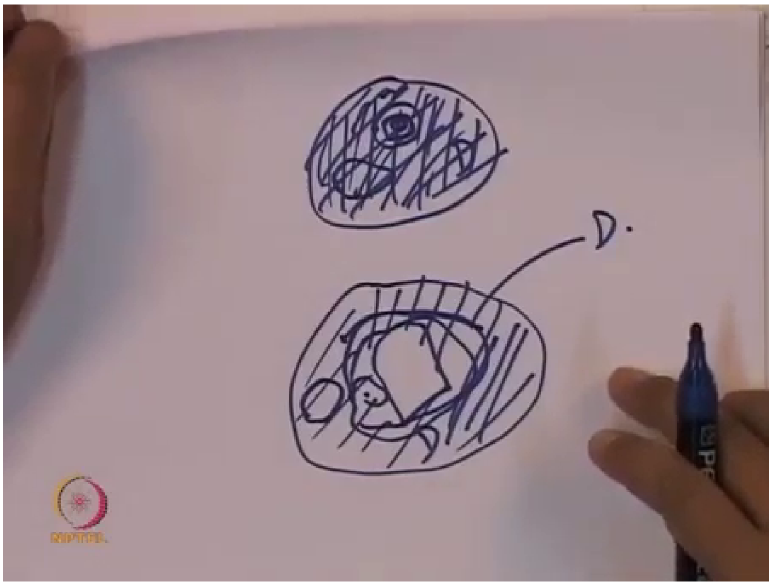

So, now we take this region D. So, we take a closed curve inside this, yeah it is just a closed curve and this closed curve can be shrunk to a point yeah, we can take a smaller curve slightly smaller curve. So, and this shrinking can eventually lead to a point and in the process of shrinking to a point inside at no situation does the curve have to leave the set D yeah. In other words every closed curve can be shrunk to a point while being inside D, if that is a situation that is the property that D has then we will say D is simply connected, yeah.

So, all these regions that we usually think of are indeed simply connected. So, an example of a set D that is connected, but not simply connected is a set with which has holes. So, take this and we rule out this particular case. So, what is our D? Our D is this shaded region and this shaded region without this particular place without this hole.

So, this; so, this sorry, this D with the hole, this is our D, this is an example for a region that is connected. Why it is connected? Because, we can take any two points on the in D and there is a path that connects these two and all points in the path are also in D. But, what about a closed curve yeah, notice that this closed curve cannot be shrunk, cannot be made smaller and smaller such that all the whole curve is in D; why because it cannot be shrunk to a point, yeah.

So, in the process of shrinking this curve to this point to any point, it turns out that this hole because it is inside the curve, but it is not inside D, inside the region D; we are not able to shrink this closed curve to a point. We might be able to shrink other closed curves to a point, but for it for the region D to be simply connected, every closed curve we should be able to shrink to a point. So, there are curves here which we cannot shrink to a point, hence this D is not simply connected, but this D is simply connected.

So, the Bendixon criteria requires that the region D for which we are checking is simply is a simply connected region. So, on this simply connected region we check whether the two functions, whether this function g; whether the function g here which is obtained from f 1, f 2 by doing this partial derivative operation.

This g should not be identically 0 on this region and it should also not change signs from 1 to minus 1. It is allowed to be 0 at a few points in which case the sign is equal to 0 that is not of concern, but it should not become from 1 to minus 1 or minus 1 to 1, if D satisfies these two properties at all points in D g satisfies these properties.

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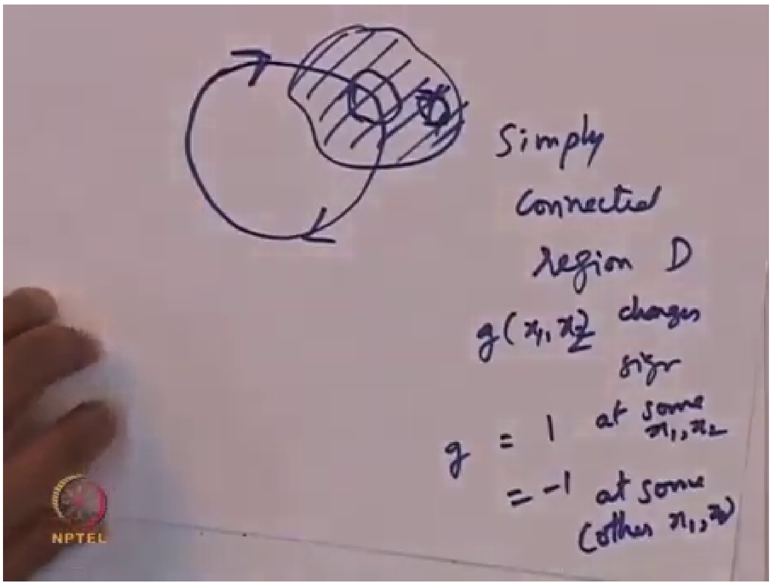

Then, the Bendixon criteria says that there is no periodic orbit lying entirely in D. What the Bendixon criteria does not say is that suppose this is a region g, it is simply connected; simply connected region D. And, suppose g was suppose g of the previous slide changes sign changes; sign means, what when we take different points x 1 and x 2 then it is equal to 1 at certain x 1 x 2 points and is equal to minus 1.

At some x 1 comma x 2 is equal to minus 1 at certain other points; yeah, at some at some other points; if g is changing its sign from 1 and minus 1 then the Bendixon criteria only says is what it says is that there is no such periodic orbit, there cannot be such a periodic orbit yeah. But, there could be a periodic orbit that is not lying entirely in D yeah, such a periodic orbit could be there which partly is inside D and partly outside D, such a periodic orbits could still exist. The Bendixon criteria does not rule out such periodic orbits existence, it only rules out any periodic orbit that lies entirely in D, this is ruled out yeah.

So, please note that there is a subtle difference in lying entirely in D and passing through D, and the criteria only says that if g does not change its sign, while being checked in D then there is a periodic orbit that is contained in D. So, let us take an example of a linear system.

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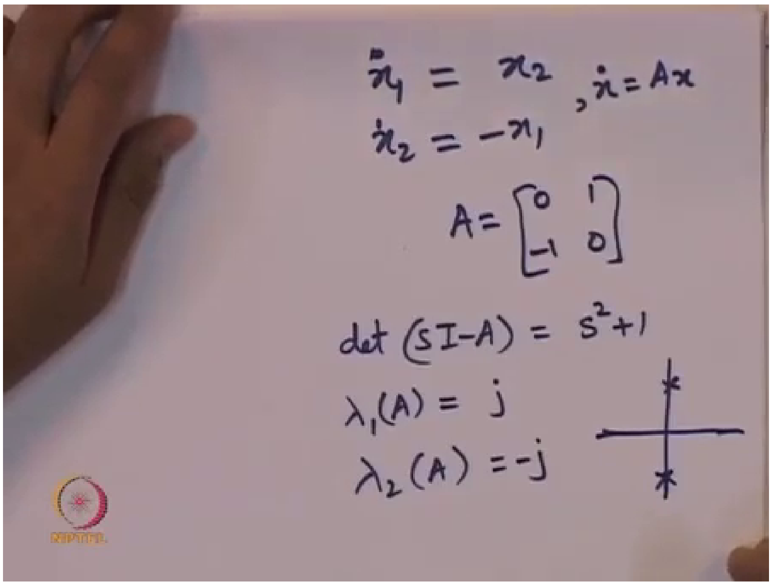

So, the corresponding matrix A, this can be also written as x dot is equal to A x, where A is equal to 0 1 minus 1 0. So, let us take the eigen values for this matrix A. So, determinant of S I minus A is equal to S square plus 1. So, please check that the determinant of this, the characteristic polynomial turns out to be this.

So, the eigen values of A are plus and minus j, one eigen value of A is equal to plus j, the other one is minus j. In other words there are two eigen values both on the imaginary axis which suggests that there are periodic orbits, this the equilibrium point 0 comma 0 is a center for this particular A. So, let us see what happens to the Bendixon criteria.

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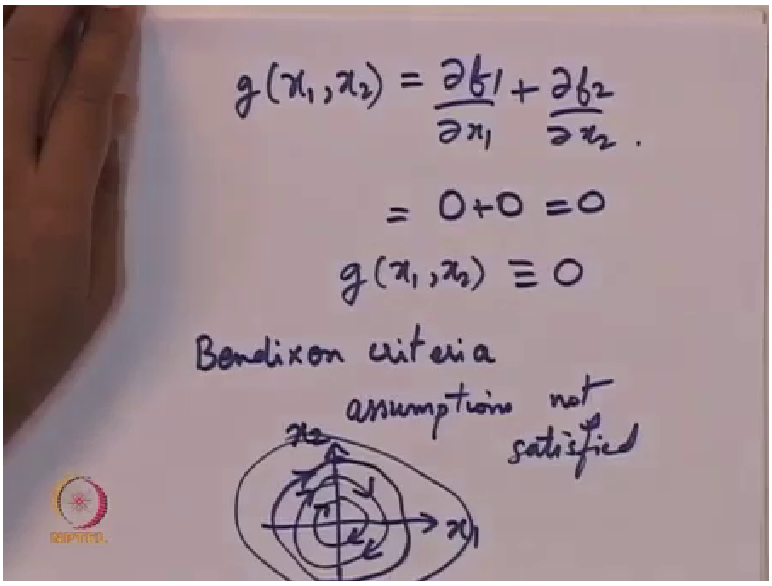

So, what is our g which we are defined as del f 1 by del x 1 plus del f 2 by del x 2. So, for this particular case, this is our f 1 and this is f 2. So, derivative of f 1 with respect to x 1 is equal to 0; derivative of f 2; x 2 does not even appear in f 2, only x 1 appears.

So, derivative of f 2 with respect to x 2 is also again 0, this is 0 yeah. So, no matter which region you take; no matter which simply connected region you take g of x 1 x 2 is identically equal to 0, it is equal to 0 without even having to specify at which point x 1 x 2 we are checking this. So, this is the situation where Bendixon criteria is not applicable, yeah.

So, Bendixon criteria assumptions not satisfied. The assumptions are not satisfied, does not mean that there are no periodic orbits lying entirely in the inside that simply connected region D? No, it does not mean that it only means that because assumptions of the Bendixon criteria are not satisfied. We cannot go ahead and conclude anything, because Bendixon criteria is not valid, the statement is not valid. However, we know in this particular case, that it is identically 0 and there are periodic orbits indeed, yeah. In fact, these are all the periodic orbits.

So, from any initial condition on this plane, there is a periodic orbit passing through that. In other words for every simply connected region that contains the origin as long as this region is some region like this, there are plenty of periodic orbits. However, Bendixon criteria does not tell us that, why; because, Bendixon criteria assumes that this g is not identically 0 and that situation is not satisfied here for the case of a linear system with periodic orbits, and hence we are not able to use Bendixon criteria here. We will see some more examples of where Bendixon criteria is applicable in the next lectures.

Thank you.