Nonlinear System Analysis
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Lecture – 18
Limit Cycles- Examples
Part 01

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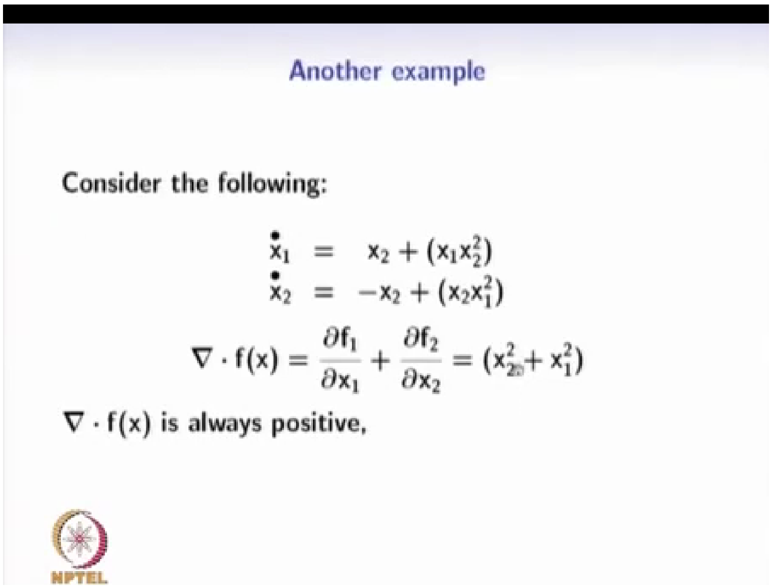

Now, let us take some other examples of dynamical system and check whether there are periodic orbits or not. So consider this example, so in which x 1 dot equal to x 2 plus x 1times x 2 square x 2 dot is equal to minus x 2 plus x 2 times x 1 square. So, we differentiate the first in order to use Bendixson criteria, this particular quantity that we were to evaluate is nothing but divergence of f yeah divergence of f is also denoted as dot product of this operator with f. So, when we evaluate this we get x 2 square here and so there is something wrong here this x

2 should have been x 1. So, please note there is a small mistake here if we have x 1 here then we get this.

So, now we have that this is always positive yeah after you substitute x 1 here you get that this one is positive. So, if the lotka volt this is very similar to lotka volterra predator prey model. After we have x 1 here this is very similar to the lotka volterra predator prey model. If it had depended or not the product of x 1 and X2, but some higher order power of one of the species then it is possible to show that the Bendixson criteria says that there are the no periodic orbits. Why because divergence of f is always positive except at the equilibrium point.

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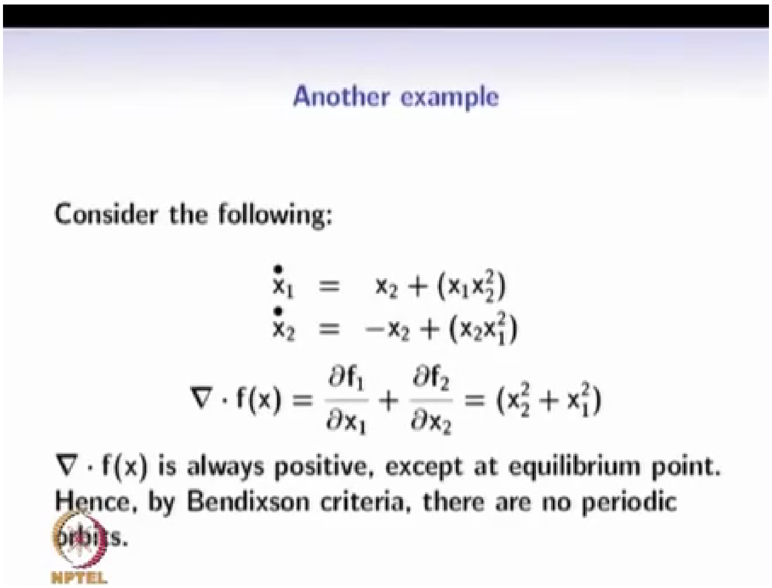

So, hence by Bendixson criteria there are no periodic orbits.

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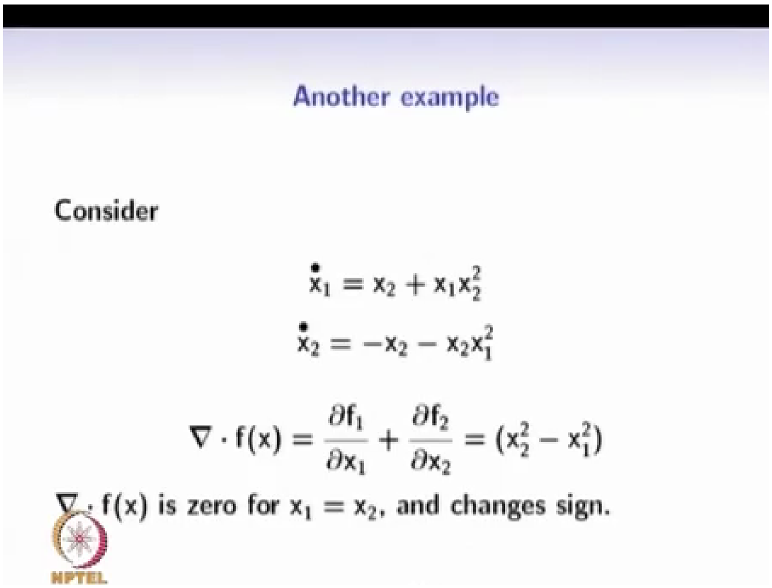

Now, consider this example this example also has we should have a modification here, we should have x 1 here in place in the second equation x dot 2 equal to minus x 2 minus x 2 times x 1 square. So, here we see that this changes sign this only means that inside the region where it has a same sign, there are superiority orbit contain inside that region.

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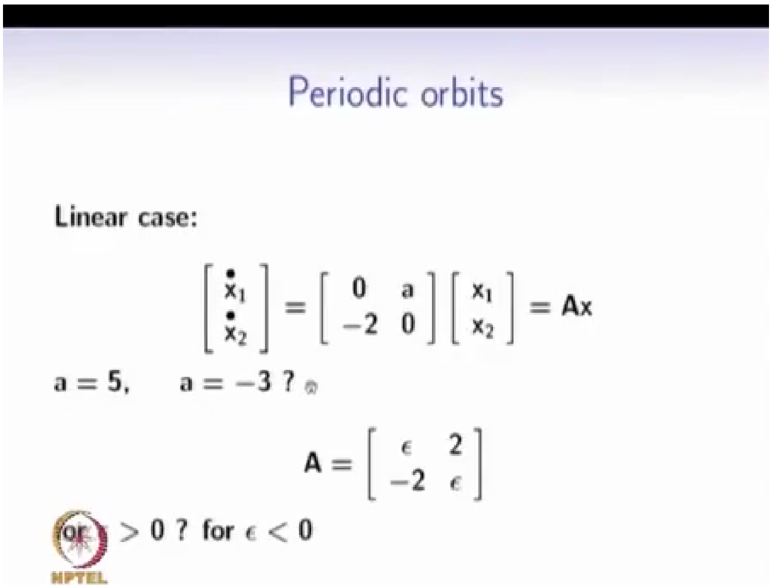

This is an example that we have already seen. Now, we will study an important case where we have an epsilon here along the diagonal entries yeah. So, for this particular A in which we have epsilon along the diagonal. What can we say about the equilibrium point? So, when epsilon is greater than 0.

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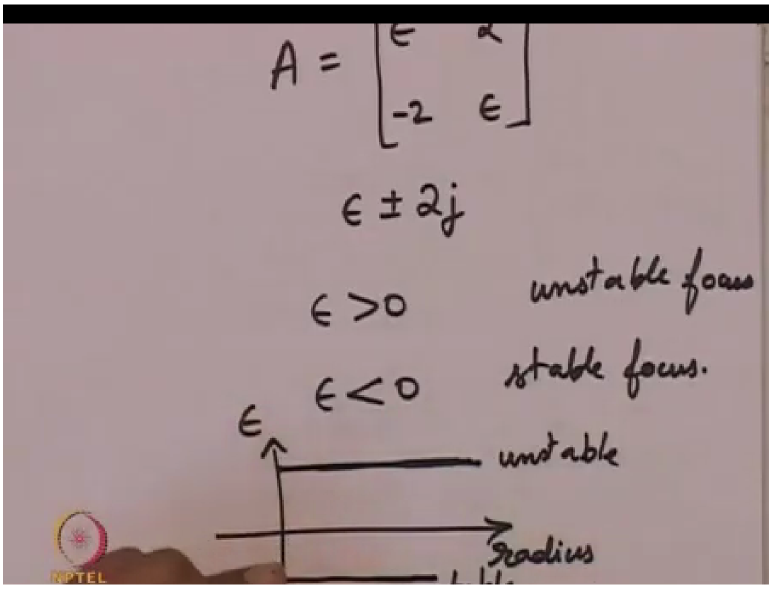

Then the equilibrium point 0 comma 0 is unstable focus this is something that we have already seen. What happens when epsilon is less than 0? When epsilon is less than 0 eigen values of this matrix are epsilon plus minus 2 j. So, epsilon greater than 0 implies unstable node or focus it is unstable focus, because imaginary part is non zero and for epsilon less than 0 we have a stable focus.

So, this is what we will say that as a if this epsilon this is our epsilon value and this is let us say radius distance of the point from the origin. So, if epsilon is some positive quantity, so epsilon itself is not dependent on x 1 x 2 yeah. So, it is just the same positive number positive means unstable focus and if it is some same fixed negative number, then it is stable. So, this is unstable and this is a stable focus. So, how about modifying this epsilon as a function of radius? So, that we have trajectories that converge to a periodic orbit now this is what we will see in detail.

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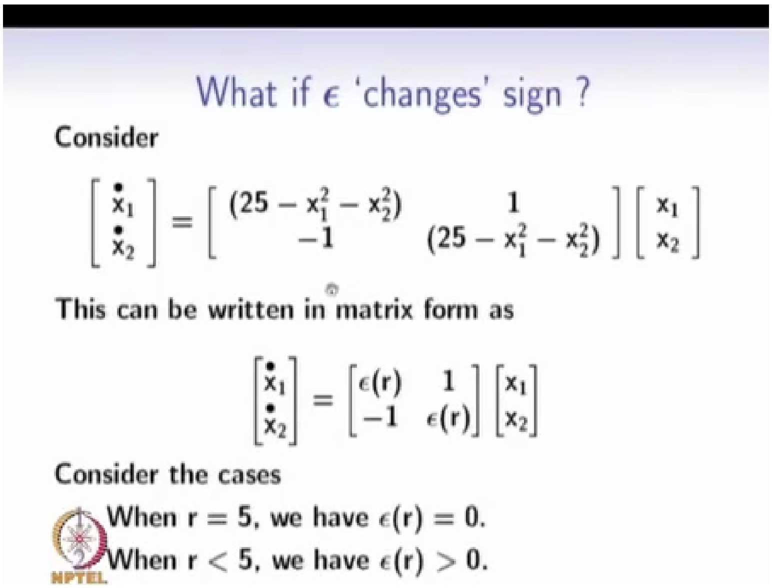

So, what if epsilon changes its sign depends on the distance from the origin and changes its sign. So, consider this differential equation in which along the diagonal we have put 25 minus x 1 square minus x 2 square along a diagonal and the off diagonal term we keep constant does not depend with change with radius. So, this is nothing but writing it in this form in which along the diagonal we have some function that depends on the radius, depends on the distance from the origin. Now, we consider the case when r is equal to 5 for that case we have this epsilon of r is equal to 0

We can consider the case when r is less than 5 for r less than 5 the diagonal entries are positive and when r is greater than 5 the diagonal entries are negative are both negative. We cannot speak of Eigen values of this diagonal is of this matrix itself depends on x 1 X2, we speak of Eigen values of only constant matrices.

So, it appears like if we make this radius, if we make this diagonal entry depend on radius, then we will have trajectories either coming towards origin or going away from the origin, depending on whether we are inside a particular circle. Whether we are inside the circle of radius 5 or outside and on the circle itself we are going to be remaining on a circle.

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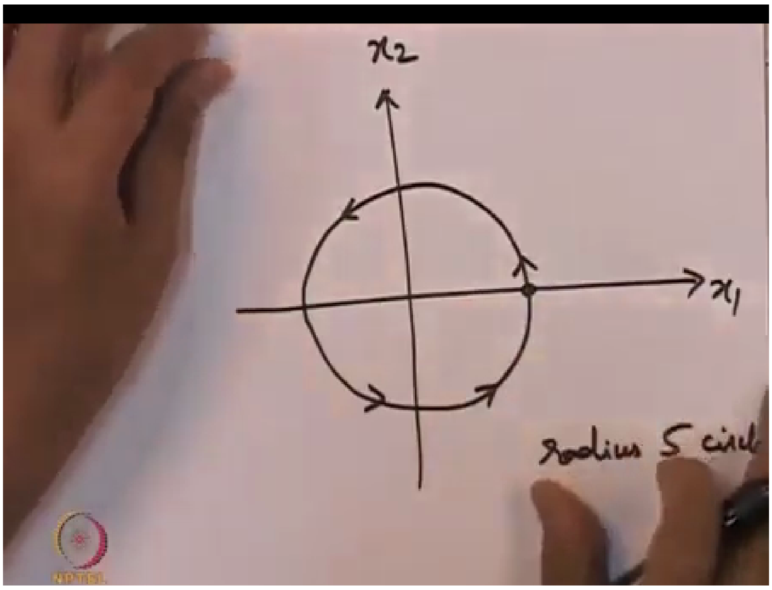

So, let us check it is a circle of radius 5. So, when x 1 is equal to 0 so this is radius 5 circle ok. So, when x 1 is equal to 0 and x 2 is equal to 5, the time x 1 dot x 1 equal to 0 sorry this orientations we will start again.

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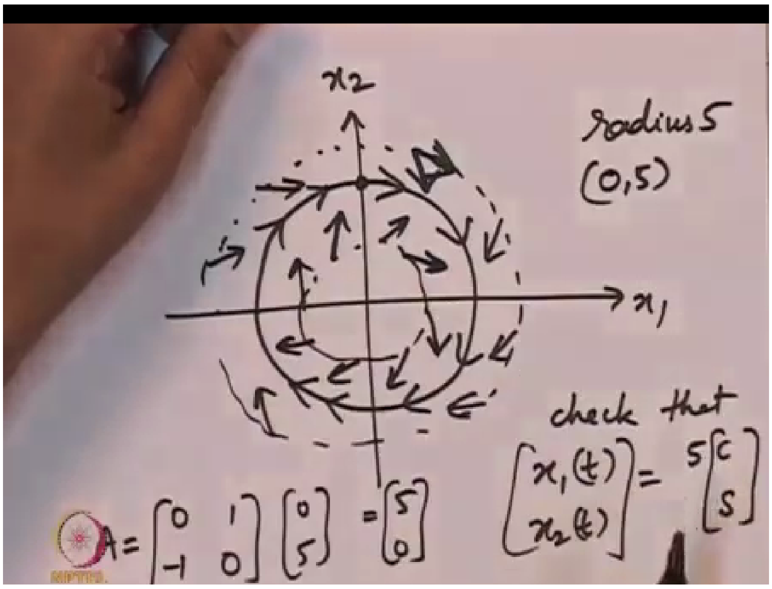

How do we get this orientation? we expect that for radius equal to 5 equal to 5 we have a periodic orbit. Why is it that we have a periodic orbit? You put r equal to 5 and you see that the matrix A the it looks just as if so check; check that x 1 of t x 2 of t equal to 5 times.

So, along the circle x 1 x 2 are like cos and sin a function of what frequency t just cos t and sin t, why because omega is equal to 1. For this particular A the solutions are x 1 x 2 are equal to cos and sin to the sin of the quantities and of radius 5 and why cos t sin t. In general it would have been cos omega t sin omega t and that omega is equal to 1, because off diagonal entries are equal to 1.

Now, we are going to decide why is it clockwise and not anticlockwise, that we can check by taking some sample points. So, consider this point this is x 1 component equal to 0 and x 2 equal to 5. So, consider the point 0 comma 5 add 0 comma 5, when this acts on 0 comma 5 when A acts on this matrix then we get that this is equal to 5 comma 0.

So, the x 1 component is increasing at this particular point, that is why it is along this direction yeah. So, by using the same argument we can decide where cos of t should come, where sin of t should come and whether this should be a negative sign to one of these. Now, the focus of this particular topic is to see what happens for radius larger than 5, it is larger than 5 and for radius smaller than 5. So, for radius larger than 5 there is the off diagonal term indeed cause some rotation, but the diagonal entries cause a decrease in the radius that is why it is coming inwards yeah.

So, we have these arrows coming inwards and for the circles inside the circle of radius 5, that is for circles of radius less than 5 there is some rotation causing caused because of the off diagonal terms. But the diagonal entries themselves are positive which is causing this radius to grow as a function of time, this is an important property that we will exploit to say that all initial conditions except the equilibrium point 0 comma 0 are all converging to this special periodic orbit, which periodic orbit? The periodic orbit of radius equal to 5.

So, let us check let us use Poincare Bendixson criteria and check that there indeed exists a periodic orbit. We are not able to say that this is a centre kind of arguments, because we can use that only for a linear system by linearising it in equilibrium point.

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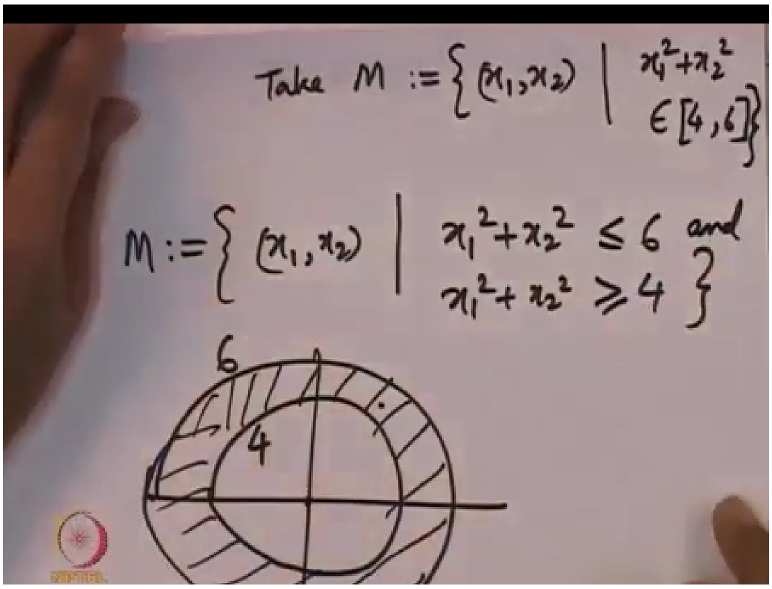

So, take M to be equal to the set of all x 1 comma x 2, such that x 1 square plus x 2 square lies in the interval 4 to 6 closed interval. So what is our set M, let me write again set M is a set of all x 1 comma x 2 points. Such that x 1 square plus x 2 square is less than or equal to 6 and x 1 square plus x 2 square greater than or equal to 4. In other words this is a circle supposed to be a circle, this is another circle this is a circle of radius 4 this is a circle of radius 6.

All the points in this ring these are all the points whose distance from the origin is greater than or equal to 4 and less than or equal to 6 also yeah this and this.

So, we will now check that this particular set M is positively invariant and has no equilibrium point inside it and its compact. The compactness is satisfied because this is a compact set and it is a closed set because these inequalities are non strict inequalities they are not strict equalities, but non strict inequalities. Because of the fact this is the closed set and it is a bounded set, because we see that all the points are at most distance 6 away from the origin, hence it is a bounded set.

So, in order to use Poincare Bendixson criteria we are going to check that the set M is a positively invariant set also. So, when would Poincare Bendixson criteria be applicable the set M should be a closed and bounded set should be positively invariant and there should be no equilibrium point inside it or at most one equilibrium point which is either an unstable node or an unstable focus.

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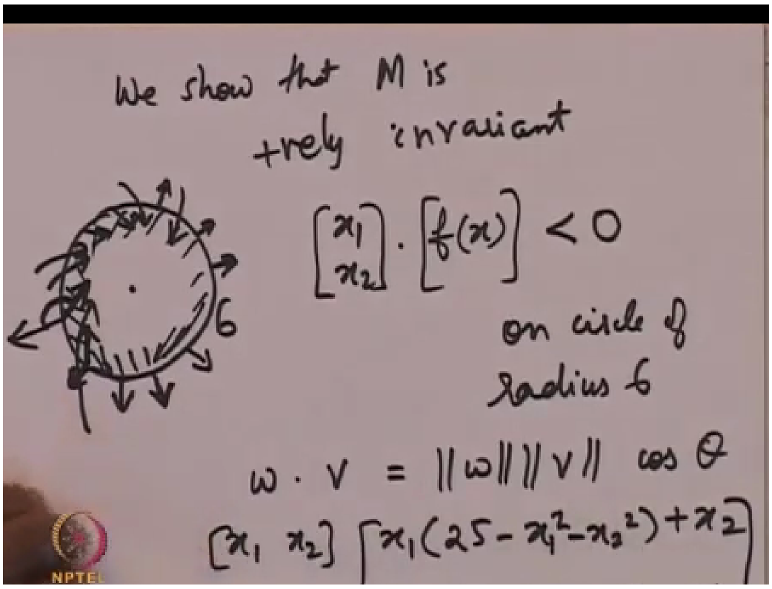

So, let us check what is the property of this particular M, we show that M is positively invariant. So, how can we show that this set M is positively invariant?

So, what we will do is we will take the circle of radius 6 and we will check that the circle of radius 6 the outward the vector that is perpendicular to this boundary is outward like this. This is the vector we will check what is the inner product of this vector with the vector field.

If at every point along the boundary this vector field is directed inwards, it means that all the points or trajectories are coming inwards yeah. What are these vectors? This is the unit vector perpendicular to the boundary and directed outside the region the region is directed. The region is inside this as far as the boundary 6 is concerned as far as this boundary of radius 6 circle of radius 6 is concerned. This is a vector that is directed outwards.

So, let us check whether what is this vector it is nothing but x 1 x 2 vector, it is inner product with f of x at that point f of x at each point again a vector of dimension 2. We will check whether this inner product is positive or negative. If this inner product is negative on circle of radius 6, it means that along the circle all trajectories are going inwards. Why is it inwards? Because this is a vector outward and this is the f of x if this particular angle is this dot product being negative means that the angle between 2 vectors is an obtuse angle.

Why because what is the dot product of w dot product with V this is equal to w norm times V norm times cos of the angle between w and V. So, if this quantity is negative it means these two quantities cannot be negative. So, this cos theta is negative and cos theta is negative only for theta beyond 90 degrees. Which means that this angle between these two vectors is greater than 90 degrees and since this vector is a vector that is directed outside the boundary. This angle being obtuse means the f is directed inwards. So, let us check whether this quantity is negative.

So, x 1 x 2 times f of x, so f of x is what we can see from this particular thing f 1 of x is first row times x 1 x2. So, this is nothing but x 1 times 25 minus x 1 square minus x 2 square plus x 2 this is the this is f 1 of x and f 2 of x is minus x 1 plus x 2 times 25 times minus x 1 square minus x 2 square. So, when we do the dot product of this that is nothing but this row vector times this column vector and we evaluate this. Let us see what we get.

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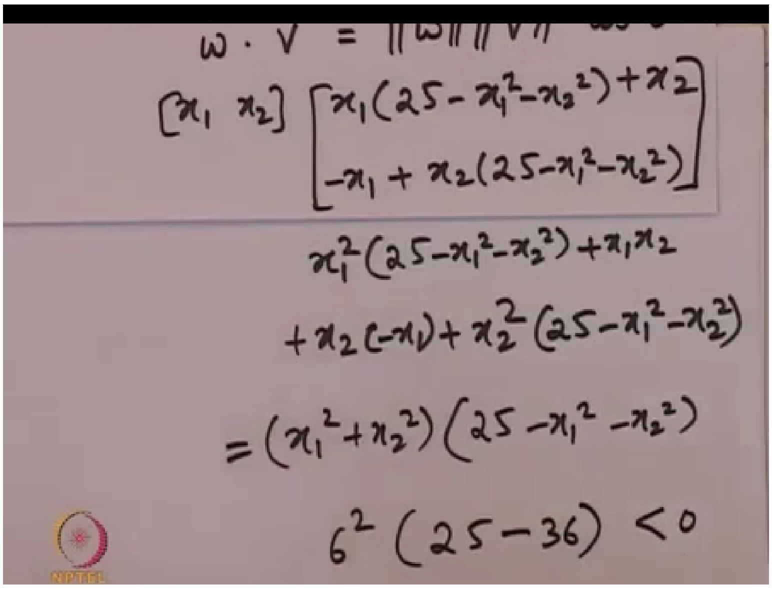

So, x 1 square times 25 minus x 1 square minus x 2 square plus x 1 x 2. This is just this quantity here that I have written is just x 1 times the first component, here plus x 2 times minus x 1 plus x 2 square times 25 minus x 1 square minus x 2 square. So, x 1 x 2 minus x 1 x 2 these both cancel, so we get x 1 square plus x 2 square in common 25 minus x 1 square minus x 2 square.

Now, are note that we are going to evaluate this along the circle of radius 6. So, we get 6 square times 25 minus 36 yeah. So, this is this is some quantity that is less than 0 that is all we needed. So, this proves that along the circle of radius 6, the arrow is directed inwards.

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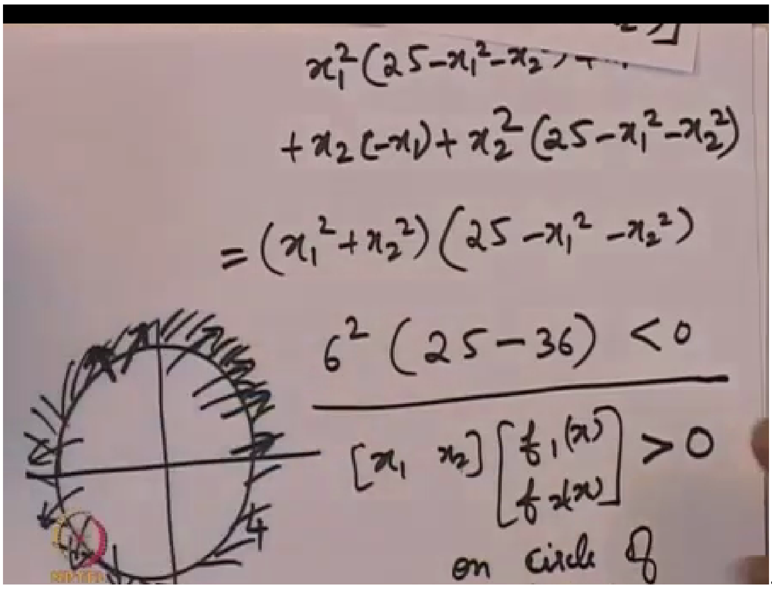

Now, let us check what happens along the inner circle. This is a circle of radius 4 yeah and along this boundary this is a unit normal and the f itself. So, there are two vectors at each point along the boundary; one vector is the direction of f of x at that point and another vector is the direction of the unit normal. And in this case this vector says that it is directed inwards, why? because M lies to the outside of this region outside of this circle, M was the set of all points of radius greater than or equal to 4 and less than or equal to 6.

Since we are taking a circle of radius 4 the circle the region is to the inside. So, this vector is directed inwards of the region. So, at each point what is this vector it this x 1 x 2 again and the direction of f at that point is f 1 x f 2 of x. Now, because this vector is a vector directed towards inside the region along the boundary. This particular quantity being greater than 0, means that the angle is acute angle the angle between the two vectors is less than 90 degrees and then it would mean that the trajectories are all coming inwards into the region yeah; so that all these trajectories that are coming into the region as far as the boundary is concerned.

So, this boundary is concerned the circle of radius 4. So, by the same argument all we have to do is you have to now substitute x 1 square plus x 2 square is equal to 4 square and not 6 square. And this quantity becomes 4 square when we are looking at the circle of radius 4 and this quantity becomes 25 minus 16 which is now positive, so greater than 0 on circle of radius 4. So this proves that the set M is positively invariant.

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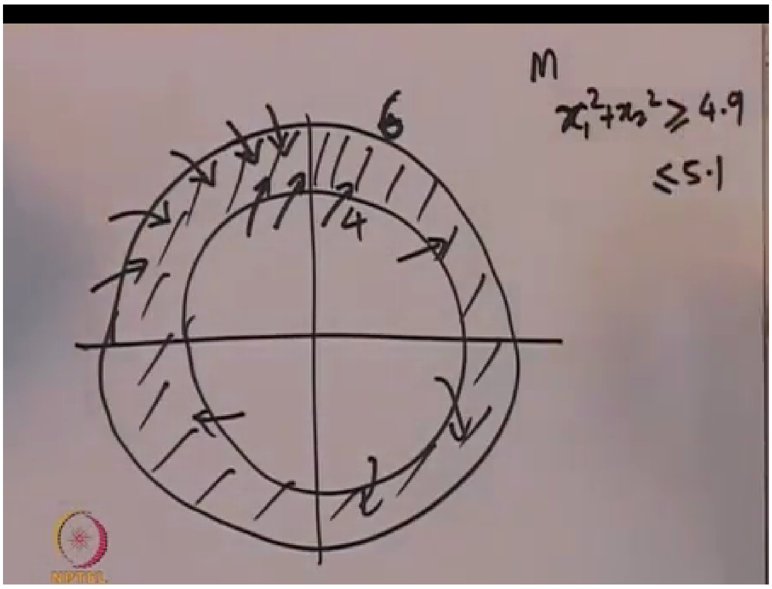

How have we shown that this that set is positively invariant we have said that this set M has boundary consisting of 2 circles. So, this is one boundary of the circle 4 the other boundary is circle of radius 6 and the region is like this, this is the region M. All along the outer circle the trajectories are coming inwards into the region this is what we checked, because the angle was obtuse there.

All along the inner boundary also the trajectories are coming inwards yeah. So, check that as long as the region M is defined as set of all points, where the radius of the where x 1 square plus x 2 square is greater than or equal to say 4.9 and lesser or equal to 5.1.

It will still be positively invariant that is only property that we used such an (Refer Time: 20:35) will be positively invariant and hence it will contain a periodic orbit. Is there any equilibrium point inside this region that is another thing that we are supposed to check before we use the Poincare Bendixson criteria, this is the last thing we will check before today's lecture ends.

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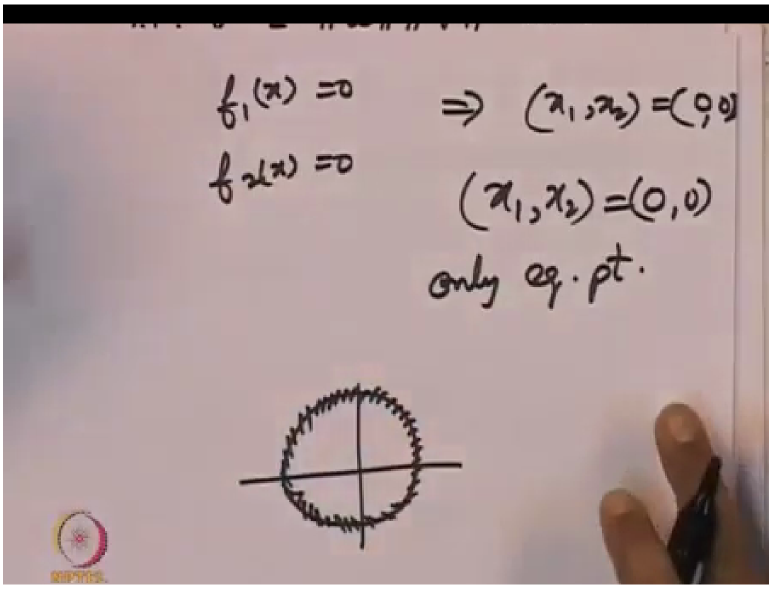

So, let us put f 1 of x equal to 0 and f 2 of x equal to 0. So, f 1 of x we had evaluated and we had got that equal to.

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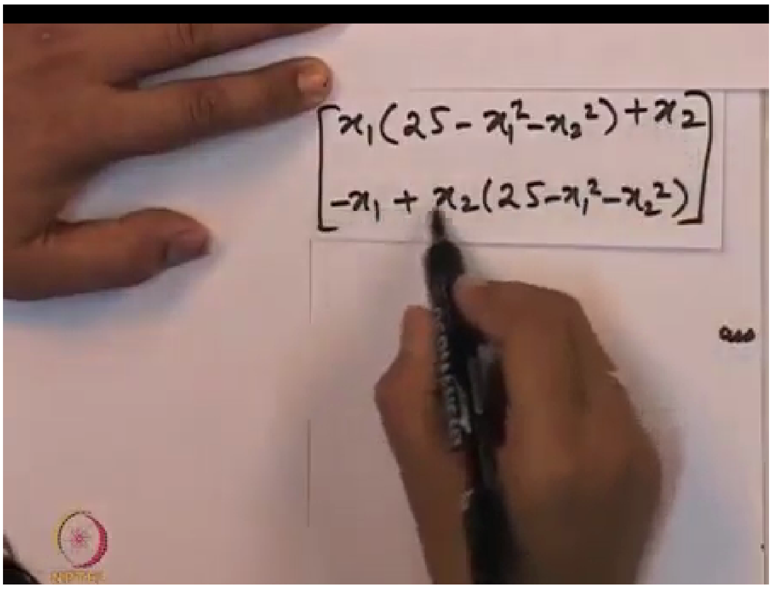

So, what is shown here is f 1 of x and this is f 2 of x we have to substitute both equal to 0 and find the values of x 1 x 2 such that both these functions are equal to 0. Those x 1 x 2 values will comprise the equilibrium points. So, one can check that the only equilibrium point for this is x 1 x 2 equal to 0 comma 0, in other words if M is a set of all points of distance greater than or equal to 0. If the radius is strictly greater than 0 then n will not have any equilibrium points, that is why we can so this implies that x 1 comma x 2 equal to 0 comma 0, x 1 comma x 2 equal to 0 comma 0 is the only equilibrium point.

The equilibrium point itself is stable or unstable one can check let this be as a homework that this equilibrium point is unstable. Why because at this equilibrium point we ensure that the diagonal entries of this particular matrix of which matrix. Let us go back to the slide of this particular matrix for x 1 equal to 0 and x 2 equal to 0 this particular matrix has diagonal entries positive and hence the equilibrium point is a unstable focus.

So, this is allows us to use Poincare Bendixson criteria. Since the region that we have considered set of all points of radius greater than 4 and less than 6, greater than equal to 4 and less than or equal to 6 is compact is positively invariant and has no equilibrium points and hence this is a periodic orbit.

By making the set M smaller and smaller such that it just contains the circle of radius 5. So, we can take the region of M to be set of all points of distance slightly less than 5 and slightly more than 5 and it will still the same argument will hold and there will be a periodic orbit. This shows that there is there is no continue of periodic orbits here there is an isolated periodic orbit for this particular example. We will consider modifying this example and obtain the Van Der Paul oscillator as a special case.