Nonlinear System Analysis

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Lecture – 21
Introduction to Bifurcation Theory - 2

Hi everyone. Welcome to this lecture number 2 of week 7 on the course on non-linear systems. My name is Ramakrishna. I am a faculty member at IIT Madras and this is a series of lectures or course that we are offering on a non-linear systems analysis.

So, in lecture 1 we started defining the basic notions of bifurcation and essentially this analysis had to do with what happens to the stability or some qualitative properties of a system as a certain parameter is varying. So, far we were interested in existence and uniqueness of solutions. We were interested in stability. We had different notions of stability why are the linearization the Jacobian linearization and then we characterize equilibrium points as a stable and unstable, nodes, focus, center and so on right.

So, last time we or in the previous lecture we had few notions of bifurcation starting with.

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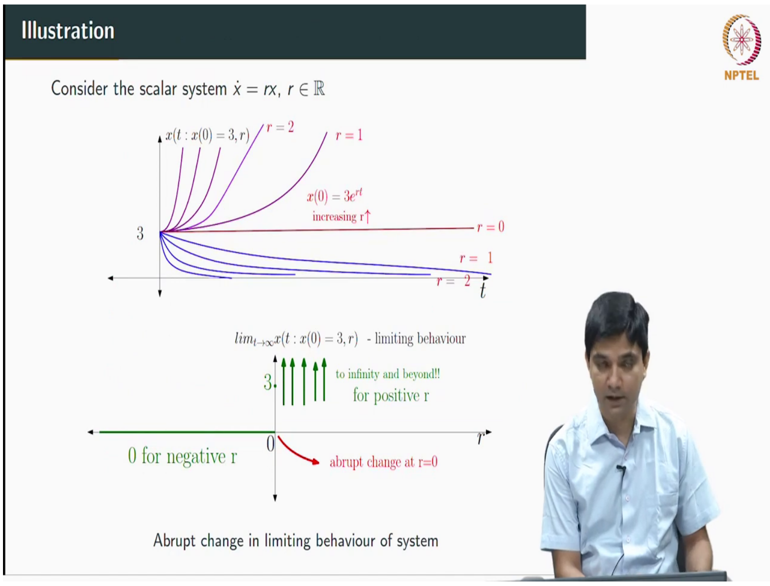

So, the linear system was pretty clear right. You either move so the only change that could happen or a qualitative change that could use could see drastically is when your poles move from the stable region to the unstable region.

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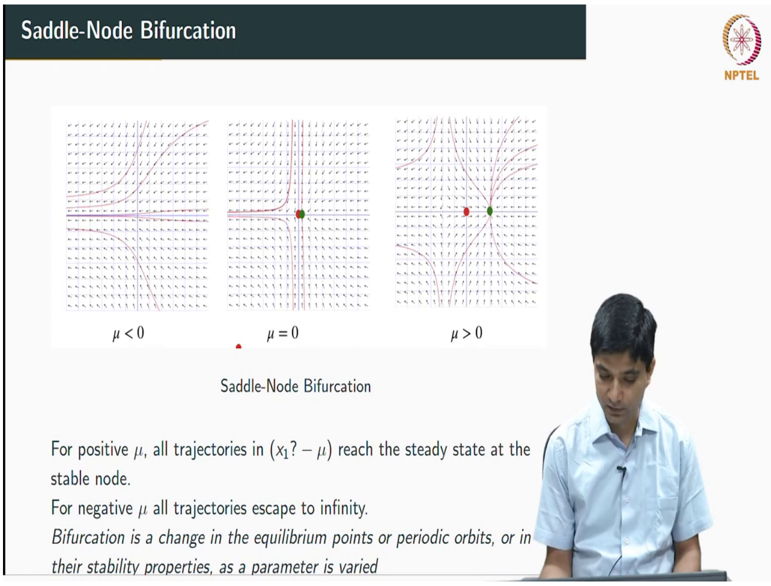

Whereas in the non-linear case you can have a bunch of a things happening right. For example, in the saddle node bifurcation you have an equilibrium point which was a stable focus a saddle as the parameter varies at mu equal to 0 and for mu less than 0 this equilibrium point just disappears. So, you had 2 equilibriums one of which was stable one of which was unstable. As the parameter value of the parameter mu changed to a negative value this equilibriums disappear.

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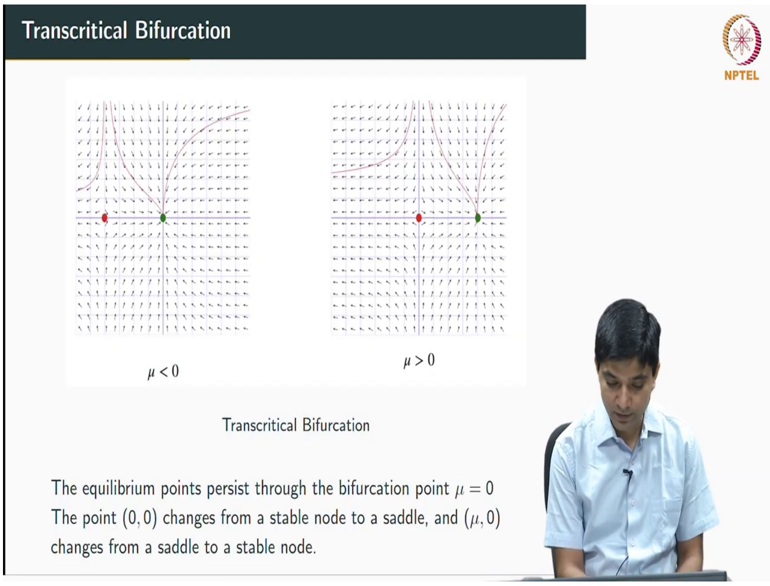

Next we had things where you had a 2 equilibrium points one saddle and one stable focus. And as mu goes from less than 0 to mu greater than 0, they just interchange their properties. So, the stable node becomes a saddle and vice versa the saddle point becomes a stable node.

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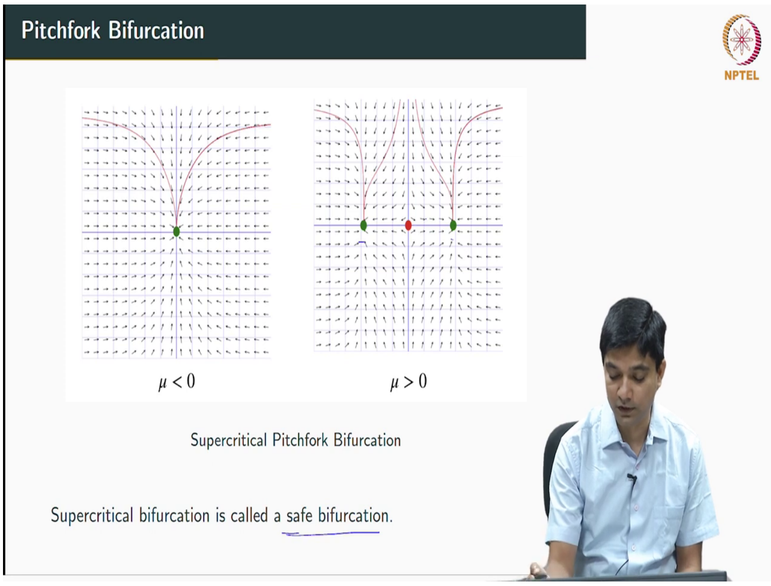

Third which checked how in the super critical Pitchfork bifurcation that for mu less than 0 we had a stable equilibrium point and as mu changed its value to greater than 0 we had this stable node becomes a saddle and then you have 2 other equilibriums which emerge out of out of for almost no (Refer Time: 02:47). And these are this is good bifurcation or a safe bifurcation to happen because your system trajectories even though after the bifurcation point they will just settle down to some points close to the equilibrium point for small values of mu.

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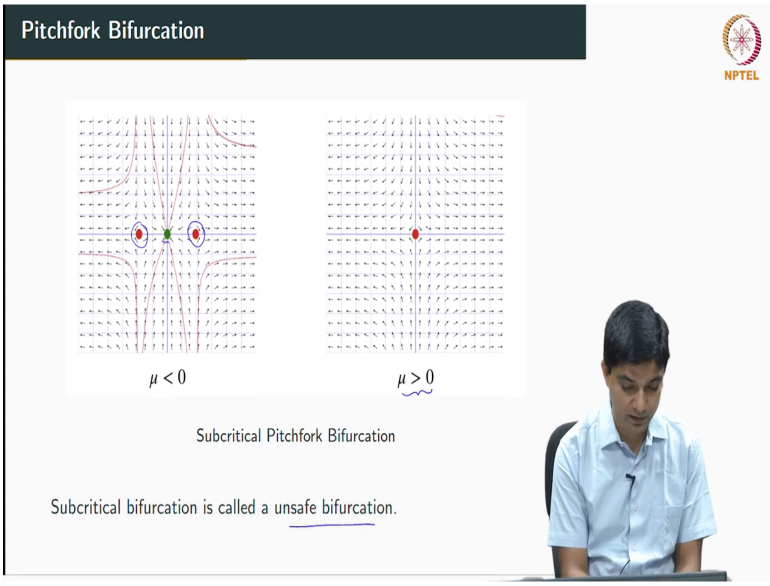

And then you had the reverse happening where you had stable node here and then for mu less than 0 you had 2 saddle points here. As mu transition has faces a transition from minus from negative value to a positive value this actually becomes an unstable point and this is typically called an unsafe bifurcation.

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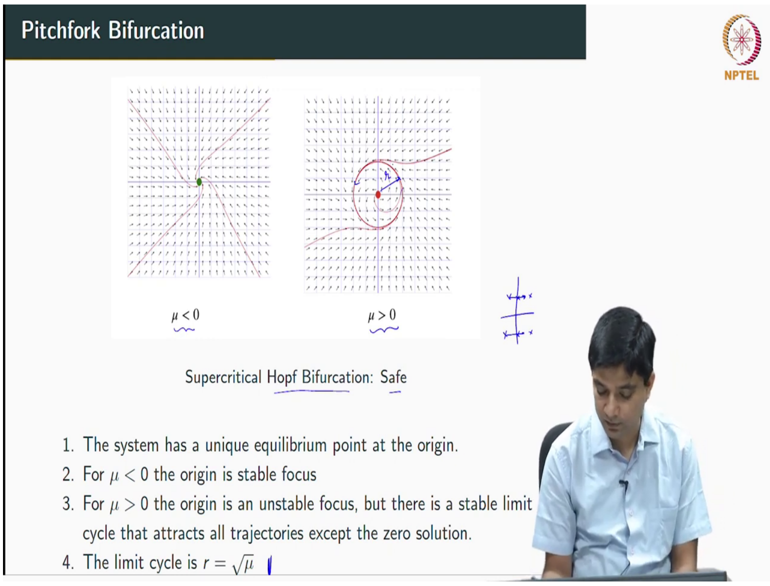

Last we also show showed an example where for mu less than 0 we had a stable focus and then for mu greater than 0 this stable focus became unstable. But, all the trajectories converge to some limit cycle here right and this was there was something nice that we actually saw. So, this is called Hopf bifurcation and this is safe and ok. There are other examples of unsafe Hopf bifurcations, but we will skip those examples for now, but.

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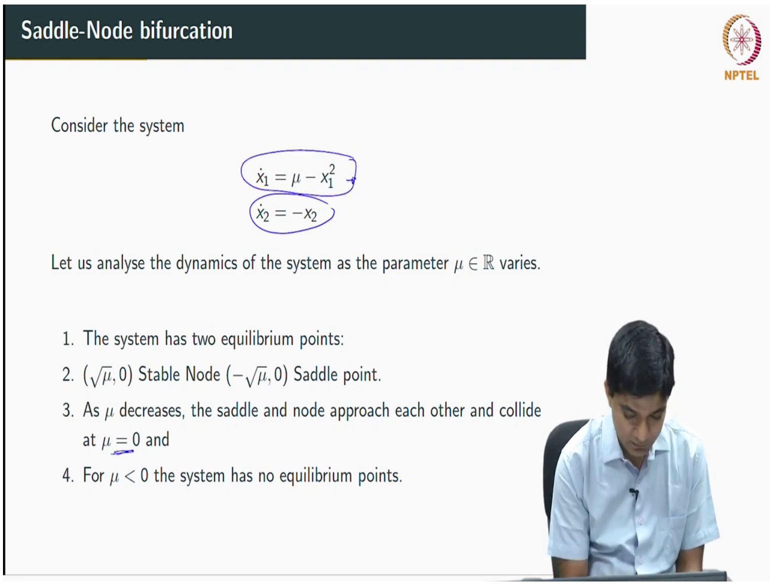

So, what is the common point here if we start from the saddle node bifurcation is when a mu equal to 0 you experience a 0 eigenvalue similarly as mu changes signs you the system or the linearized version is essentially associated with a 0 eigenvalue. That is true even in the transcritical bifurcation it is true also for the Pitchfork bifurcation.

So, the only thing that changes when you look at how bifurcation is that your eigenvalues are such that they pass through the imagine that like a pair of eigenvalues which were originally here. They will pass through the imaginary axis this way. So, you will have a roots on the imaginary axis and possibly they could go to the unstable region and then we also had had things to derive on how what decides the radius of this limit cycle or the properties of the limit cycle based on this parameter value mu ok.

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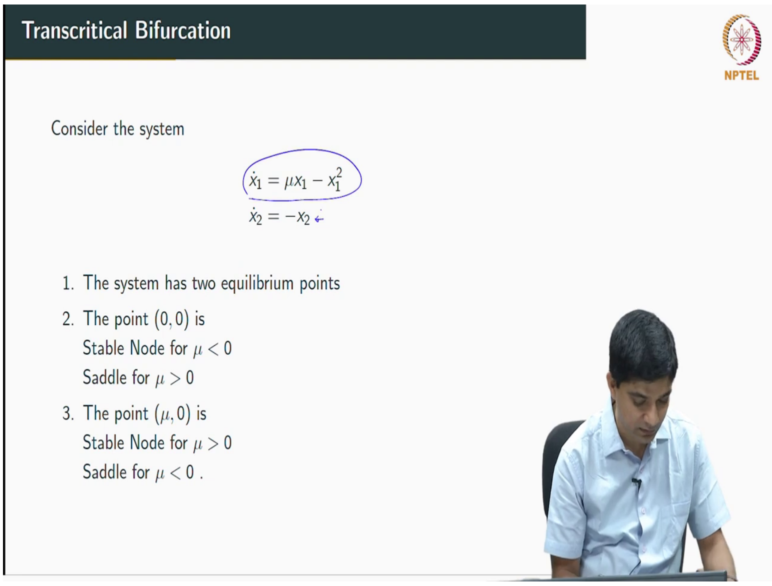

So, in much of this if you look at examples everything was depending only on x 1 right. So, the parameter mu had nothing to do with x 2. In this equation it was true for saddle node bifurcation and so on except when you are looking at the how bifurcation case. So, all these essentially characterize 1D bifurcation set. So, I can just study little more extensively just based on these the first equations on a on x 1 dot here, this equation here, this equation here and so on.

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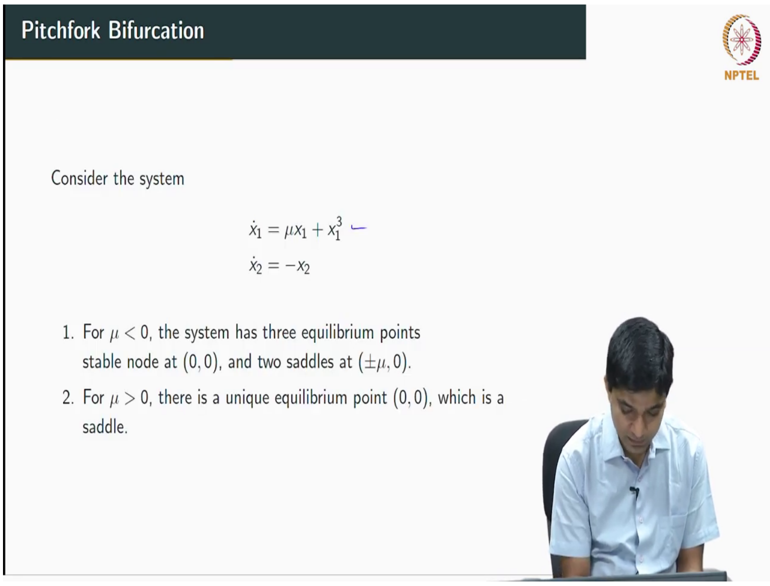

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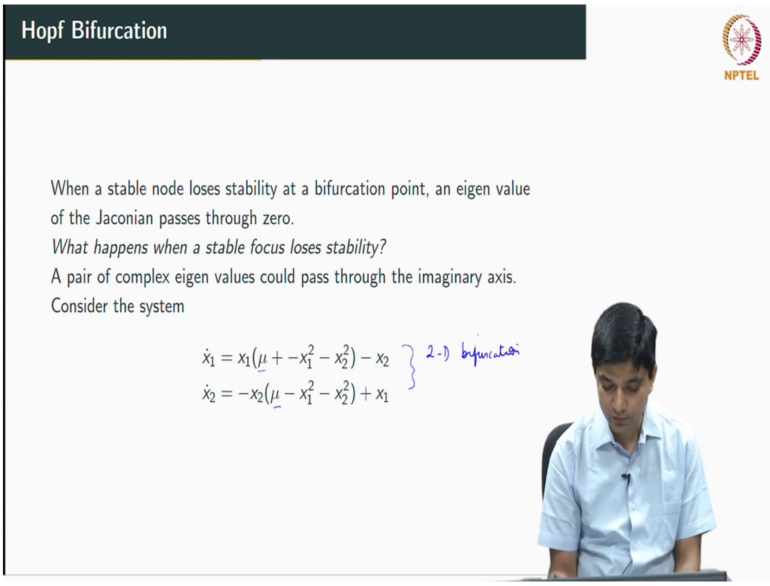

Except here where I am just looking that at a system where the mu changes or mu or x 1 and x 2 both depend on this parameter mu. So, this is a 2 D bifurcation. So, what we will do now is to look at properties purely from the first equation that we had right.

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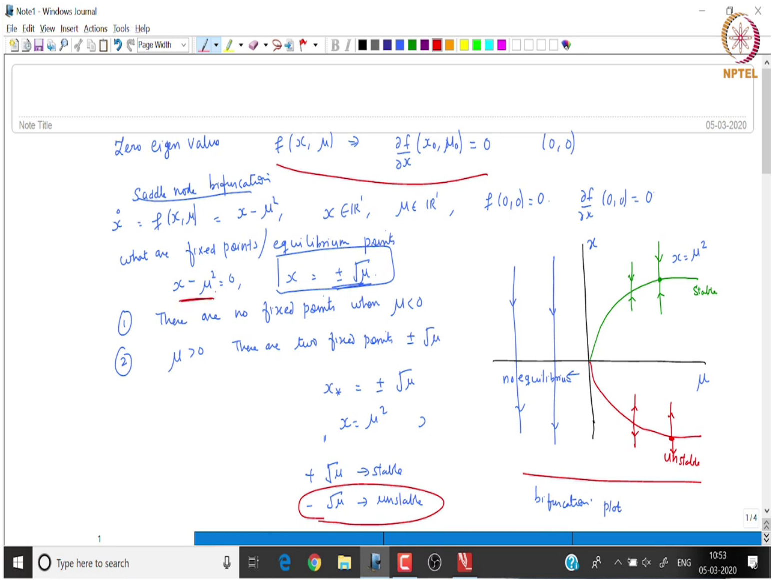

So, the first property that we see which is generic across all the bifurcations is that of a 0 eigenvalue. So, essentially this means that if I have my system f depending on x and mu this essentially means that df of the partial of x f by partial of x at some parameter value x as some state value mu 0 sorry x 0 and mu 0 this will go to 0 right.

Or in our case this just turns out to be that the origins. So, and then for mu being actually being equal to 0 ok. So, this is this we will see of how we can relate to each of this properties now or can we derive some patterns and in general conclude given a system what kind of bifurcation is likely to occur if at all ok. So, we start with the first example of the saddle node bifurcation ok.

In this case what we have is f x comma mu. This was x dot was of the form x minus mu square and we are talking about bifurcation. So, x and y both will belong to the real line ok. So, first it is easy to check that f at 0 0 is equal to 0. Next it is also equal easily easy to check that the partial of f with the partial of x at 0 0 is also equal to 0 ok.

Now, next is first what happens what are the fixed points and also call this as what are the equilibrium points and it is easy to check that ok. This occur when x minus mu square equal to 0 or the fixed points of the system are x is plus minus square root of mu ok. So, first thing is one that there are no fixed points when mu is less than 0 and second when mu is greater than 0 there are 2 fixed points.

This is this one is plus or minus mu plus or minus square root of mu ok. If I were to just draw a graph which tells me some relation between x or the or so around. So, when I met at equilibrium or so my equilibrium points are given by this one right. So, x if I just call this x star this square root plus minus mu I am just having x equal to mu square something like I said.

So, all the all points on this curve are the equilibrium points. So, some of them are stable. So, this is so, let us check how. So, I if I were just to plot what the relation between mu and x. So, here I can easily find out right so which one which of the equilibrium points is stable and which one is unstable.

So, the plus square root of mu is stable as we saw also in the previous lecture and the negative of square root of mu is an unstable equilibrium point right ok. So, how does this picture now look like. So, if I were to say let us give me a green color for the stable regions. This would be the stable region. This is the unstable region and everything is characterized by mu is x square.

So, what we see from this picture here is that when mu is less than 0 there are no equilibriums, for mu greater than 0 the equilibrium points mu move through this green line which is stable. So, if I were to draw the vector fields around this. So, the vector fields typically would be something like this. So, they would approach this point both this point and so on whereas, the red line or the red curve is it represents this part of the equilibrium.

So, for all values where x is the square root of mu with a negative sign you are looking at an unstable behavior which means if I plot the vector fields. So, all points in this are the equilibrium point.

So, if I plot the vector field around this they will just the trajectories will just move away. This is my equilibrium point the trajectories will just move into the equilibrium point right and for all values here the eigenvalues or the system will just go to infinity. So, there are no equilibrium points in this region. So, you have 3 regions right. You have no equilibriums, you have a stable region represented by the green line and unstable region represented by the red line.

So, at any point here my vector field should simply be just this way and here they would just move away from the equilibrium points. So, each point on this red line is an unstable equilibrium point. Each point on this green line is the it is a stable equilibrium point. So, these are these are typical bifurcation diagrams as they call them. So, bifurcation plot when you just check how the equilibrium points or the vector field changes with varying values of mu.

So, the next bifurcation. So, this was the simpler the first case of the saddle node bifurcation right. So, we had one stable equilibrium point and unstable equilibrium point.

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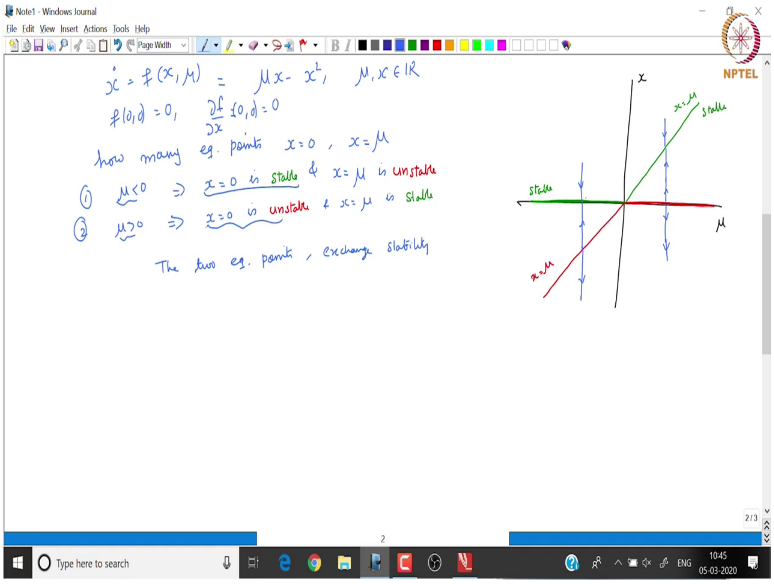

So the next one we will have is x dot is f x comma mu which now takes the form mu of x minus x square. Again mu and x both belong to the real line. So, one this is easy to check. This is also easy to check 0 comma 0 is 0 ok.

So, first is how many fix points? How many equilibrium points? Or fix points I will use this words interchangeably, but they actually mean is the same. So, I have x equal to 0 is one fixed point or equilibrium point, x equal to mu is another fixed point. Let us check the situation when mu is less than 0. In this case x equal to 0 is stable and x equal to mu is unstable.

You can check this by the Jacobian linearization or even what we did in lecture number 1, 2nd case is you have mu greater than 0 in which case x equal to 0 is now unstable and x equal to mu is now stable. Let me just draw the bifurcation plot of this.

So, again I am drawing curves between mu and x and I just want to check what happens as mu varies ok. So, first now x equal to 0 let us let us focus on this one. x equal to 0 is stable when mu is less than 0 which essentially means. So, this is x equal to 0 line right here. So, this is stable when mu is less than 0 ok. Now mu greater than 0 the same equilibrium x equal to 0 becomes unstable.

So, it is plotted this way and then you have the line x equal to mu right. So, how does this look like x equal to mu when mu is less than 0 is unstable. So, I am just in this region this is the line x equal to mu and the same line when x equal to mu for positive values of mu is a stable value it is like an x equal to mu.

So, all the green lines are stable and the red lines are stable. So, this is just show you at how. So, what happens in this case is there is just that you have the 2 equilibrium points. What I do it is x in stability. The stable one becomes unstable and the unstable one becomes become stable good. We go to the next one.

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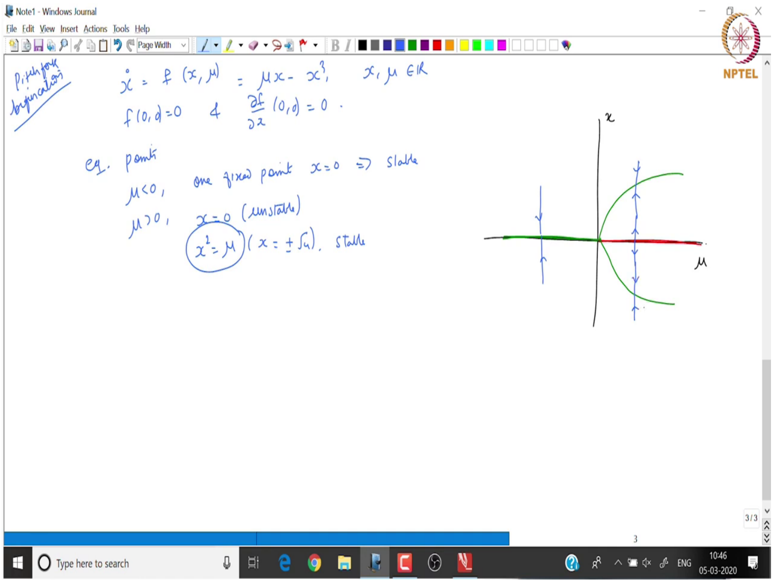

I have x dot. This is f x comma mu x minus x cube is again x and mu both being in the real line ok. So, this is easy to check f 0 comma 0 is 0 and the partial of f with partial of x 0 comma 0 is also equal to 0. What are the equilibrium points? Equilibrium points for this system are as follows.

So, for mu less than 0 there is one fixed point x equal to 0 which is stable. So, you can also compared with what the features we had earlier right. So, this one. So, I am now; I am I am now move over here in the previous case you can just check what is happening here right that the red dot becomes a green and the green one becomes red for the transcritical bifurcation. So, what I am; what I am drawing here is a different version of what we what I drew in the first lecture.

So, now so now we are in this Pitchfork bifurcation in case. So, mu less than 0 there is one fixed point x equal to 0 which is stable. Same that was the case here also like this one. So, this is the case of Pitchfork bifurcation and for mu greater than 0 you have x equal to 0 which is now unstable and in addition you have 2 mu fixed points which are given by x square equal to mu or x equal to plus minus square root of mu and both of which are stable ok. So, if I were to draw the bifurcation diagrams this is again mu versus x ok.

So, for mu less than 0 the point x equal to 0 which is this line x is stable this one ok. This is stable and then the vector field will be just pointing towards this one. I just if I were to plot the vector fields here they would just be. So, that will approach these points, approach this points, go away from this points, we will approach this points, approach this points, go away from this points.

So, whenever you have this unstable equilibrium the arrows naturally will point outwards and where there is a green line the arrows naturally will point to point inwards and you get this keep on drawing this. So, similarly here right. So, when x equal to 0 what I have is and for mu less than 0 x equal to 0 is stable for mu greater than 0 this becomes unstable. In such a way that the vector fields will just point outwards and then what else is; what else is the other equilibrium condition.

You have x square equal to mu, x square equal to mu we know looks something like this right and x square equal to mu which was a parabola like this. So, this is stable in both cases right. So, I will just have a green line here. So, we are not the best of drawings, but good enough for explanations here. So, this will converge here this and this. Well, a little contrast to what happened in the first case.

So, for in this region x greater than 0 we had a stable equilibrium points x less than 0 we are unstable equilibrium points whereas, here well on both regions I have stable equilibrium points. So, this is how we can look at Pitchfork bifurcation. So, the other case will be very similar x dot is fx u.

So, this was the super critical thing right. In the sub critical thing we had mu of x plus x cube in which case for mu less than 0 mu less than 0 we had 3 equilibrium points. One was x equal to 0 then we had this was stable and then you had x is plus minus square root of mu which were both saddle or unstable.

For mu greater than 0 we had only one equilibrium point that was x equal to 0 and this was saddle or inverse or in other words unstable. So, if I again draw the bifurcation diagram it will look pretty neat. So, mu x then I am looking at this a region. So, this region typically is system mu less than 0 region this is the mu greater than 0 mu less than 0.

So, x equal to 0 was stable. So, this remains (Refer Time: 24:31) it is and for x greater mu greater than 0 there is only one equilibrium point which looks like this and for mu less than 0 I am looking at this curve x square is mu and they will just be a series of unstable points right. So, these are so if I just go to plot the vector fields they will just go in here go away from the red line and here the just keep going on.

So, here say I just a changed from a stable to an unstable equilibrium points from mu less than 0 to mu greater than 0 ok. So, we will not do the Hopf bifurcation because that is that the 2 dimensional case. We will stick to one dimensional case and what we will try to find is given a set of equations x dot equal to f of x and mu can I derive certain necessary and sufficient conditions which will ensure or rule out the presence of certain bifurcation points.

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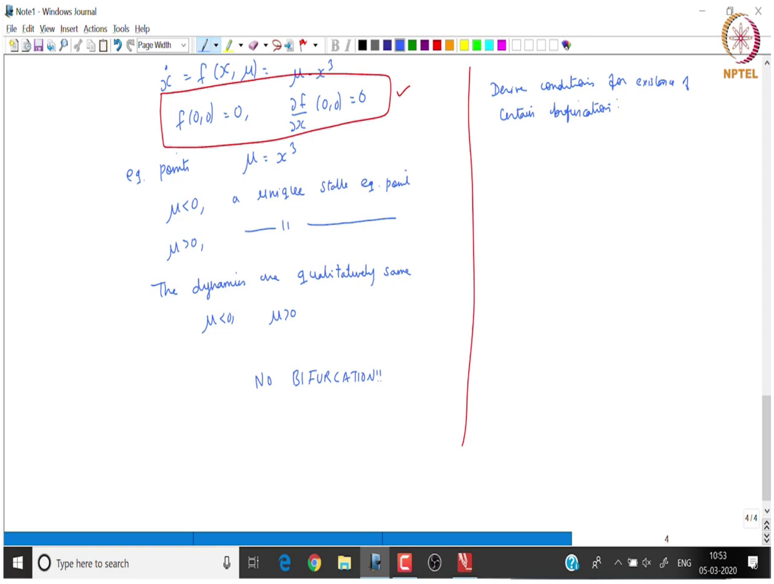

So, the last example that I would like to do here is this one x dot is f x comma mu. This looks like this. This is mu minus x 3 ok. So, the reason I am doing this example is I will be obvious. So, let us say I am looking at the situations again f 0 comma 0 is 0. The partial of f by partial of x at 0 comma 0 is also equal to 0 ok. What are the fixed points or the equilibrium points? The equilibrium points are given by mu is x cube.

Now, now for mu less than 0. This is the system has a unique stable equilibrium point. Similarly as mu is greater than 0 this is not be the same this will be a unique and a stable equilibrium point. So, what happens is that the dynamics are qualitatively the same. Both for mu less than 0 and mu greater than 0 which means there is no addition or deletion of equilibrium points.

If an equilibrium point was stable for mu less than 0 it continues to be stable for mu greater than 0. So, in this case it actually no bifurcation ok. So, it is so the reason I am doing this is one may think that whenever this happens right that you may think that there is actually a possibility of a bifurcation it was true here, it was true here. In the case of transcritical bifurcation it will also true in the case of the saddle node bifurcation.

So, whenever I satisfy these conditions it does not guarantee existence or not of bifurcation right ok. So, then we will look at what are the conditions then how does one derived conditions? For existence of certain bifurcation ok. So, before we do that I will just tell you some result that we will use from calculus and that has to do with the notion of what we call as the implicit function theorem.

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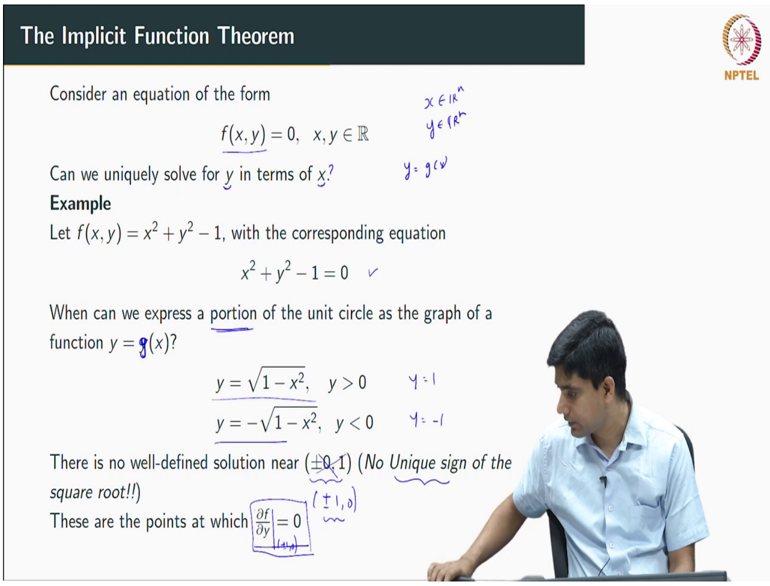

So, there are more general notions of this, but we will stick to f to functions f x y comma 0 where x and y both belong to R. In general x can belong to R n y can belong to R m and so on and the f and lie in R m. So, in so here I am just talking of a very simple function. So, the question that we will ask and it will be obvious why am I doing this before I derive in necessary sufficient conditions for existence of bifurcation is I just start by asking a question can I uniquely solve for y in terms of x right.

So, given this or other words can I write y as some function of x in this. If I can write can I write it in all conditions? If I cannot write it in all conditions what are those conditions? So, we will just derive that with the help of some small example ok. So, let us start with this simple example. It is like example which most textbooks teaching implicit function theorem would do. So, I have x square plus y square minus 1 equal to 0 and the reason I do this is it is nicer to interpret.

So, the question is can I write can we express a portion of the unit circle. It will be clear why I am saying the portion of the unit circle as a graph of a function this one you say not y of x, but say y equal to some g of x ok. So, what I know is that with this equations I can easily write these things like that y I can solve for y given x y are this expression whenever y is greater than 0 and for when y is less than 0 I can use this expression.

Additionally there is no well defined solution near these points plus minus it should be plus minus 1 comma 0. So, when x equal to plus 1 or x equal to minus 1 there is no unique solution because if when I say x equal to plus 1 I will get a 1 here. When x equal to my minus 1 and I will get a minus 1 here.

So, there is no unique sign of the square root ok. Now a close observation and which we will try to it also justify in more general setting is that at this and these are the points and where no unique sign of the square root happens is precisely the points when df or the partial of f with partial of y equal to 0 and this is at plus or minus 1 comma 0 ok. So, this is this is a little motivation for the implicit function theorem. What it also means is if I were to just draw a little schematic to explain from the circle yourself.

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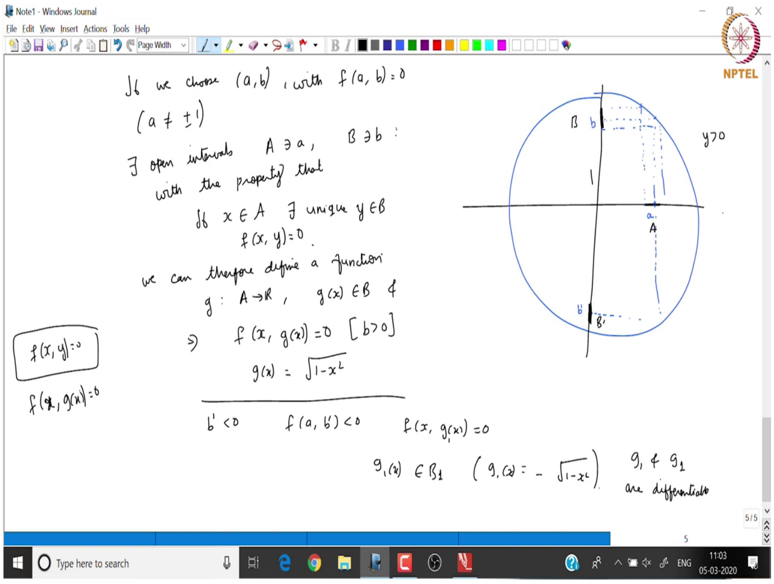

So, I just take this I just take a little circle ok. So, assume this looks like a circle and this side take a point a here right and a little area or a little neighborhood around at point a and I call. Sorry, this is a here and the little neighborhood around that point which I call oh sorry a say this is a around a point a this is point a.

Then I can (Refer Time: 33:29) then find a point b and for all of this region you can find a little region here called capital B. Let us similarly I can do it on the other side of the circle also. So, here I can find a point. So, b prime and some area are on the area b prime. This one I call this as b prime and this dotted this area darkened area here has b has (Refer Time: 34:16) not b prime this is b.

So, if I were to write it more formally I can say if we choose a comma b with I have a comma b equal to 0 again I rule out this points at a should not be equal to plus or minus 1 for a further reasons explained earlier. So, there are open intervals A which contained a and open intervals B which contain this small b such that or with the property that if x is in A there is a unique y will be such that f of x comma y is 0 ok.

Like if I look at is the upper portion of the circle where this is y is greater than 0 then give me an a I can find a unique b which will relate a and b and therefore, we can define a function g again from A to R since that g of x. So, this is all the B is a g of x g of x belongs to B and now the function where I started of saying that f x comma y is 0 can now be written as f x comma g x is equal to 0 right. This is when I am just let us say I am looking at the region b is greater than 0 ok.

So, when b is greater than 0 in this in this region then I can see that this g of x is can be uniquely identified by 1 minus x square. Similarly, if I am looking at say b1 where this is less than 0 such that f a comma b1 is less than 0. So, over here or b prime then I can I have another function f x comma g x equal to 0 where g1 we are call the g1 g1 of x belongs to b1 and this g1 of x is uniquely represented by square root of 1 minus x square and not only that this both g and g1 are differentiable.

So, again so the idea is given a function of this form when can I write y as a function of x and under what conditions I can write. So, let us go back to this conditions here ok.

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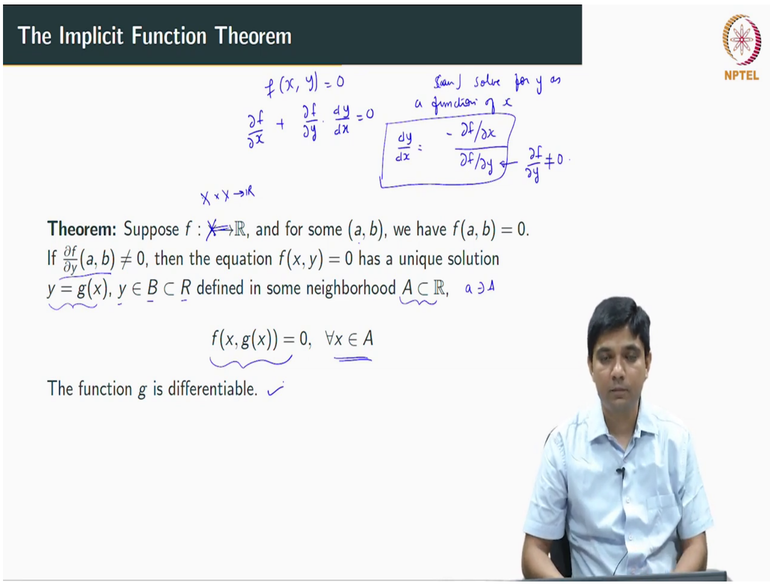

So, let us say ok. So, I am just looking at f x comma y equal to 0. Can you do some basic calculus and say right df by dx plus df by dy or the partial of dy dy by dx equal to 0. I can frequently write this as this. So, (Refer Time: 39:08) I was writing a very simpler crude version of what I am trying to say. So, I want to write y as a function of x or can I solve for y as a function of x (Refer Time: 39:27) means that I am looking at a solution to this equations dy by dx is minus partial of f partial of x partial of f by partial of y ok.

So, I can just by looking at these expressions just say that will I can solve for this assuming either denominator does not go to 0. It actually means that df by dy or the partial of f with partial y should not be equal to 0 right and now we are ready to state the general theorem now right. So, suppose f is from x to R for some a sorry this is belong (Refer Time: 40:08). This f should be x comma x to R and for some f a b and for some points a and b we have f a comma b equal to 0. There are bunch of points you can find on the circle.

So, if df by dy at that point is not equal to 0 then the equation f x comma y has a unique solution y equal to g comma g of x where y is defined in some neighborhood b of R right and then this a is. So, this A belongs to A as I explained previously then I can write that f x gx equal to 0 for all x belonging to A and not only that we find that the function g is differentiable ok.

I will not really do the proof of this that is not really important, but we will try to understand how to apply this particular proof to get to our results that will come up in the next lecture.

Now, thanks for watching.