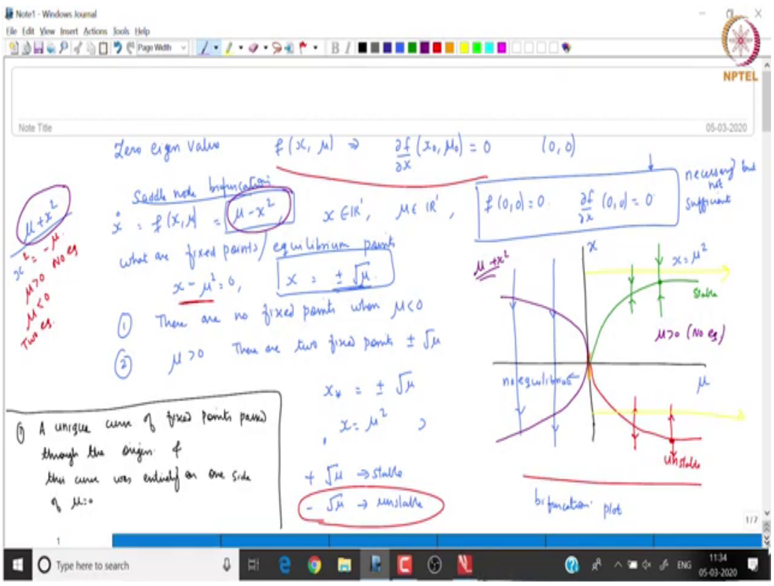
Nonlinear System Analysis

Dr. Ramakrishna Pasumarthy
Department of Electrical Engineering
Indian Institute of Technology, Madras

Lecture – 22
Necessary and Sufficient Conditions for Local Bifurcation

(Refer Slide Time: 00:14)



Hi everyone. Welcome to this lecture number 3 of week 7 on the course on Non-linear Systems. My name is Ramakrishna Pasumarthy, I am a faculty member at Electrical Engineering IIT Madras. So, in this lecture we will derive certain conditions which will guarantee existence or conditions which will establish whether or not there exists certain kind of bifurcation points.

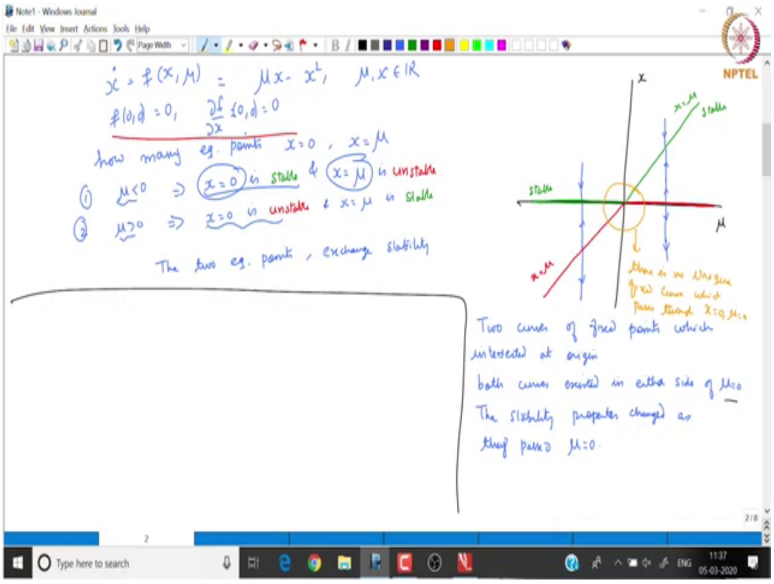
So, far we have looked about how what is bifurcation and what happens qualitatively. What is the qualitative change in the system we see as the system experiences as a bifurcation or there is a bifurcation point right. So, let us revisit and do some observations of all the bifurcation diagrams before we derive certain Necessary and Sufficient Conditions right ok.

So, let us start with the saddle node bifurcation ok. So, and I will use mostly the bifurcation diagrams to arrive at conditions ok. So, first if I look at this picture, I can make two observations. The first observation is a unique curve of fixed points or equilibrium points right pass through the origin. And not only that this curve was entirely on one side of mu equal to 0 right.

So, we just one on this side of mu equal to 0 right. So, there is nothing on when mu is less than 0 ok. So, this is one observation here, a general observation which we conclude I had last time was these conditions. So, f 0 0 is 0, df by dx at 0 0 equal to 0. What we saw is that these were necessary conditions, but not sufficient for any bifurcation.

We had one example x what is negative of x cube or some or mu minus x cube, we will again revisit that example. So, these are necessary, but not sufficient. So, we will derive conditions which what we need in addition to this to observe or conclude any kind of bifurcation in the system ok.

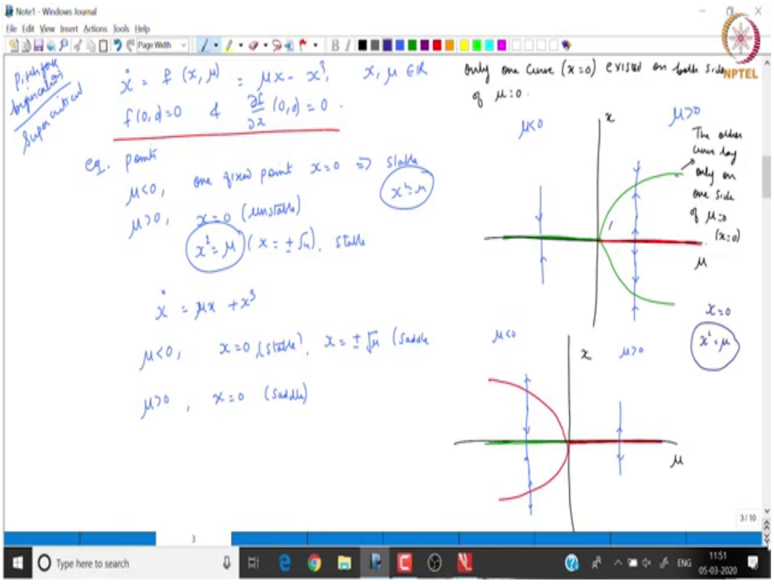
(Refer Slide Time: 03:15)



So, this was in example one. In the example two or in the second case we had two curves of fixed points and then which intersected at the origin. The two curves are x equal to 0 and x equal to mu, this one and this one ok.

Now, contrast to the previous case both the curves existed in either side of mu equal to 0, here the curves did not exist to the left of mu equal to 0, here you have on both sides ok. Only thing is that the stability properties of this curves changed as they passed mu equal to 0 ok.

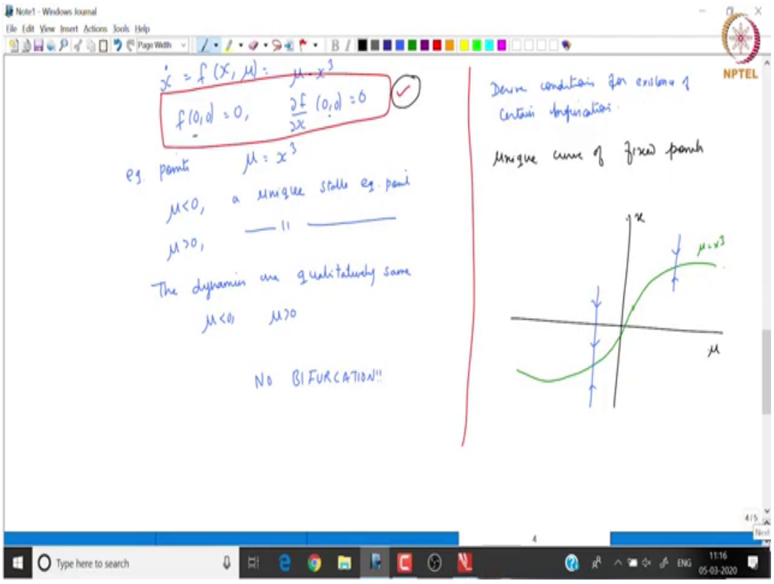
(Refer Slide Time: 04:53)



Now, the third one in this case; so what we see is that in this example, two curves of fixed points intersect this plane only one curve existed on both. So, you have two curves right; one curve was x equal to 0 second was this one, x square equal to mu ok. So, the distinction here was that only one curve and this for both the subcritical and super critical pitch for bifurcation.

Only one curve namely x equal to 0 existed on both sides of mu equal to 0 and the other curve; this one lay only on one side. And this has a stability property which is opposite to x equal to 0 ok. Now these are the main observations.

(Refer Slide Time: 06:38)

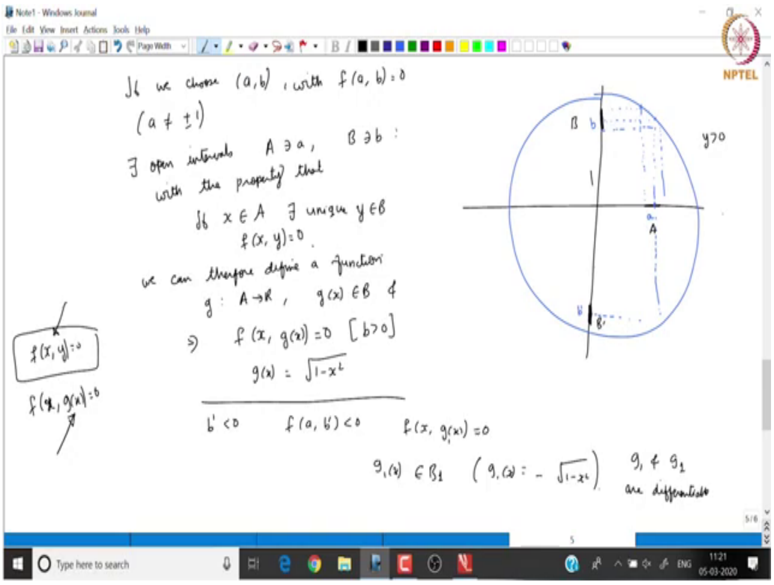


And, here in this example what we saw was even though this condition is satisfied f x, f 0 0 is 0 df by dx at 0 0 is equal to 0. What we see saw is that now this had a unique curve of fixed points.

If I were to draw it ok. Let me just spend some time drawing this ok. How does mu equal to x cube look like it is something like this ok. The vector fields are such that everywhere they are just to use. So, for mu here I am looking at positive values of mu and here negative values of mu. The stability properties do not change even though you had a 0 eigenvalue right.

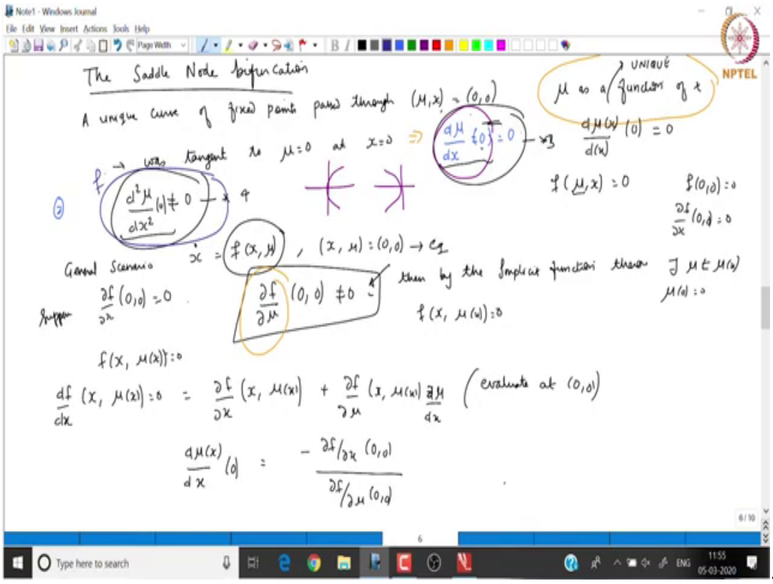
So, the stability properties do not change with as mu changes from changes sign from mu less than 0 to mu greater than 0 via mu equal to 0. So, these conditions do not say much. They do not really help me in deriving whether or not there exists bifurcations. What kind of bifurcation is still different question to ask.

(Refer Slide Time: 08:19)



So, what we will do now is just spend some time to derive this conditions for bifurcation points and we will make use of the implicit function theorem ok. So, let us start with the first one.

(Refer Slide Time: 08:34)



So, I will do this the first one in detail and then the next one I will just mentioned result, so that you could work all the steps by yourself. The saddle node bifurcation again so we will just read these two statements again a unique curve of fixed point pass through the origin and this curve was entirely on one side of mu equal to 0. In this case it was on the right of mu equal to 0.

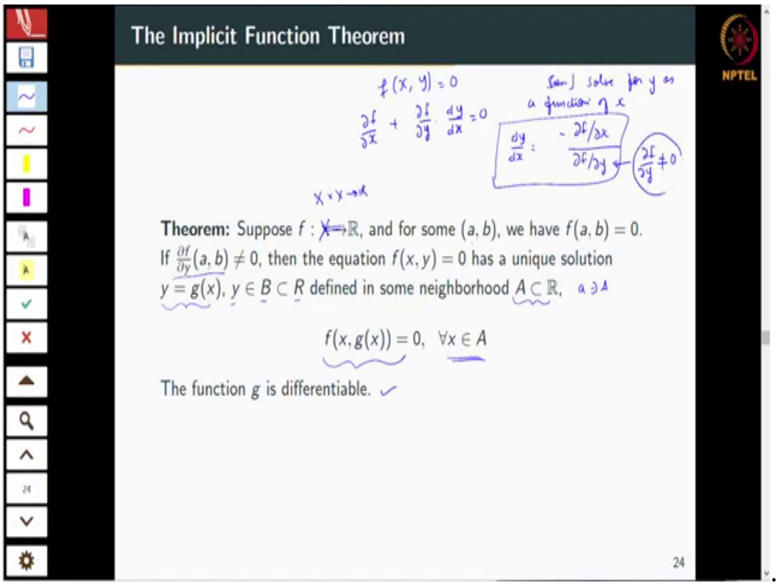
So, first is, if I look at the first statement when I a unique curve of fixed points passed through mu comma x is 0 comma 0 ok. And this curve was such that, this curve was tangent to mu equal to 0 at x equal to 0, this one. So, this curve when I say this curve was tangent. So, this one, precisely is talking of this one this is the tangent right. So, mathematically I can write this in the following way that this means that some d mu by dx equal to at 0 was equal to 0 ok.

And the second condition which said that the curve laid entirely on one side of mu equal to 0. So, the condition 2, this is incorrect this would mean that d 2 mu by dx square is actually not equal to 0 and so far not writing strictly greater than or less than 0 and I will I will tell you shortly why ok. So, let us say now I am looking at a general scenario, general scenario of x dot is f x comma mu again 1 D bifurcation ok.

So, with a fixed point x comma mu equal to 0 0. This is my equilibrium and additionally I can also say this one df by dx. So, suppose this also holds true hold to 0 and so ok. So, what are these conditions mean? This two conditions first. So, whenever I want to evaluate this quantity it should first mean that I should be able to write mu as a function of x only then I can I can differentiate right.

So, this mu should actually be a function of x and then I should be able to evaluate its value at x equal to 0 and this should go to 0. Which means; so now, I have a function of the form f mu comma x equal to 0; where, I should be able to write mu as a function of 0 I go here to the implicit function theorem sorry, where I wanted I had a function f x comma y equal to 0 and I wanted to write y as a function of x such that f x comma g of x equal to 0 and this was possible only when df by dy was not equal to 0.

(Refer Slide Time: 13:05)



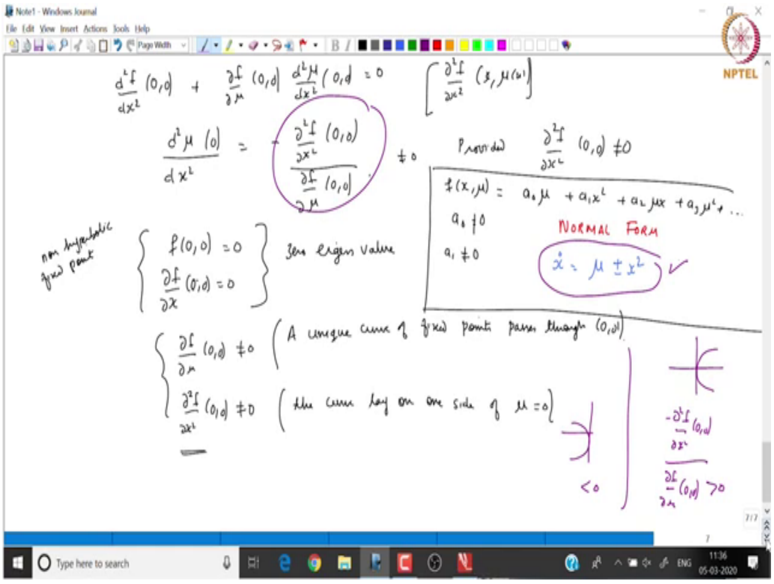
Therefore, equivalently I can write this condition now as df by d mu at 0 comma 0 is not equal to 0. This actually means now that I can actually write mu as a function of x ok. So, if this holds ok, this is a third condition right which seems it seems obvious right. So, if this holds then by the implicit function theorem, there is a unique mu equal to mu of x with mu of 0 equal to 0 such that I can write f x comma mu of x equal to 0 right. So, the reason I am doing this is now, I want to write everything I do not really want to explicitly compute what is this function mu of x. I want to write everything only in terms of f that is given to me right this one ok.

So, this condition, so this condition together with this condition, I will call them say star of 1 and say I am call this 3 and 4. Assuming that the first two conditions which we know are f f and 0 0 is 0 and the partial of f partial of x at 0 0 is also equal to 0 ok. So, what does this mean now in terms of f? So, assuming, I this condition is now satisfied I can say f x comma mu of x equal to 0 and if I differentiate this with x. So, I have df by dx of x mu comma x equal to 0 this actually is the partial of f, the partial of x, x mu x plus partial of f partial of mu x mu x d mu by dx ok.

Now, I want to find out what is at 0 0. Evaluate this at 0 comma 0, because all these are write at 0 and again this is also write at 0 ok. So, let us do some manipulations and I get the following expression d mu x by d of x at x equal to 0, this is negative partial of f by partial of x at 0 comma 0 divided by df by d mu 0 comma 0 ok.

Now this condition; so what should what should we do that the partial of f with partial of a 0 0 the numerator should be 0 and the denominator should not be 0 and this is this essentially is what ensures that.

(Refer Slide Time: 17:47)



Similarly, I compute, I will skip some steps here it should be easy to easy to verify. d 2 f by dx square at 0 0 sorry, plus d f by d mu at 0 0 d 2 mu by d x square differentiating f. So, I start so, this all starts by computing d 2 f by d x square at x mu of x and then trying to evaluate this at 0 comma 0.

So, this will lead to me condition which looks something like this is equal to 0, then d 2 mu 0 comma 0 d x square. So, this will just be 0 ok. So, what is the; what is the condition that should be satisfied that if I go back this should actually not be equal to 0 and this is. So, d, d 2 mu by dx square at 0 is not equal to 0 this is this will happen provided d to f by dx square 0 0 is not equal to 0 to summarize.

Student: (Refer Time: 19:42).

To summarize the conditions for saddle node bifurcation are the following. f at 0 0 is 0, the partial of x partial of x equal to at 0 0 is equal to 0. This will ensure the 0 eigen value sorry and then additional conditions df by d mu at 0 0 is not equal to 0 these conditions comes via this one and the last condition would be d 2 f by dx square 0 comma 0 is not equal to 0.

So this means this condition. So, we are also here interested in relating mu writing mu as a function of x uniquely right so this should be unique because in this bifurcation diagram; there was only one unique curve passing through the origin. And this why I am assessing on this unique will be clear when I do the transcritical bifurcation ok. So, the first condition here ensures that a unique curve of fixed points passes through 0 comma 0. And this ensures that the curve lay weighted lies only on one side of mu equal to 0 ok.

So, if I were to just write down a Taylor series expansion of a vector field; which has which satisfies these two conditions, this also means that it has a non hyperbolic fixed point ok. Fixed point or equilibrium point which is essentially mean it passes through the origin. This Taylor series expansion we look something like this. So, f x comma mu is a 0 mu plus a 1 x square plus a 2 mu times x plus a 3 mu squares and some higher order terms ok.

So, if I just verify these two conditions of df say first say I want to verify d f by d mu equal to 0, this will give me a condition that a naught should not be equal to 0 I am again verifying that at 0 0. Next would be I am verifying this condition d 2 f by dx square is not equal to 0, this will give me a condition that sorry should not be equal to 0 and the second condition would mean that a 1 should not be equal to 0 ok.

So, in general I can write a normal form of this bifurcation of the form x dot this mu plus minus x square. We started off saying this x dot is x minus mu square, but I can still do the same when I have x equal to plus mu square right. I am so, I am sorry this actually should be the other way this should be mu minus x square ok, nothing change I think this just a typo there.

So, I can also write the same say if I just instead say I have a vector field mu plus x square I would still have let us just quickly do this right. So, in this case my equilibrium points would be given by x. We use a different color for this x square is minus of mu here for mu greater than 0, there will be no equilibrium for mu less than 0, I will have two equilibriums one of which will be stable one of which would be unstable.

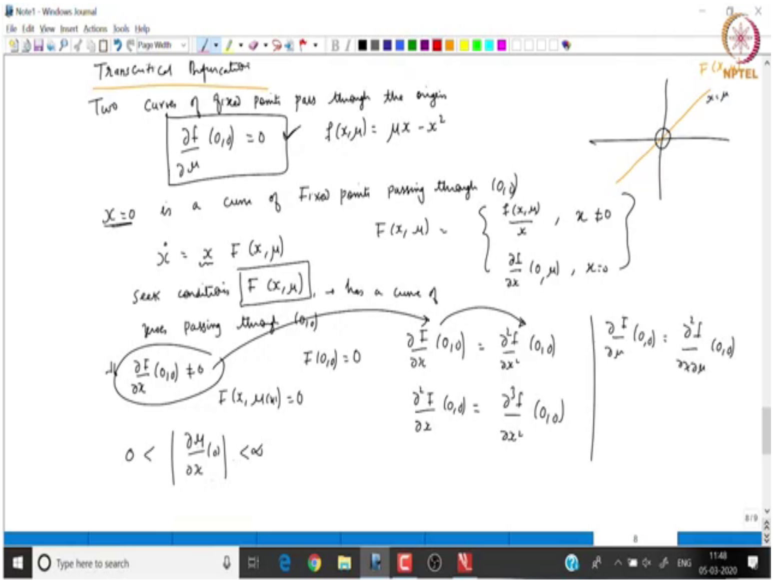
So, if I let me choose another color. So, this one, so this is for mu plus x square. So, I can have that so on for mu less than greater than 0 I will have no equilibrium just for this vector field and when mu is less than 0, I will have some equilibrium points here you can easily find out which one is the stable one and which one is the unstable one this vector field also expresses or also has a saddle node bifurcation same as this vector field here right.

So, the reason why I just said that this should not be equal to 0 is because I started off with a possibility like this, but there could also be a possibility like this and in both cases this condition is true d mu by dx at 0 is equal to 0 right. So, in general these class of bifurcations can be expressed by vector fields of this form ok. So, this is essentially called that the normal form and it is this it is kind of easy to you can check both of these conditions and draw those bifurcation plots and check why they are actually the same. In both cases you will have a saddle node bifurcation.

So, in the first case. So, here if I write it if my bifurcation plot looks like this then this quantity here, this one this minus d 2 f by d x square at 0 0 divided by df by d mu at 0 0 will be greater than 0 and this kind of bifurcation, this will essentially be less than 0 ok.

So, this is in general this is a general normal form of a saddle node bifurcation ok. So, now, that we understand things a little better. We will try to derive quickly the conditions for the other bifurcations, transcritical bifurcation ok.

(Refer Slide Time: 27:16)



So, let us revisit this. In this ok, two curves of the fixed points, they intersected at the origin the curves existed on both sides of the line mu equal to 0 and the stability properties changed as they passed through mu equal to 0 ok.

So, here we had a unique fixed curve, a unique curve of equilibrium points passing through the origin here I have, I do not have the uniqueness is lost here the there is no unique fixed curve which passes through x equal to 0 and mu equal to 0 of the origin. That is the first contrast which we see in which we see with when we compare to the saddle node bifurcation.

So, when there was a unique fixed point, the requirement was that I should write mu as uniquely as a function of x and that was possible when df by d mu was not equal to 0. Here there is no unique fixed point sorry, there is no unique curve that passes through the through the origin right. So, we here we have that two curves of thick fixed points pass through the origin ok. For this two occur, df by d mu should actually be equal to 0. It was supposed to be not equal to 0 for x ensuring uniqueness, but here because there is no uniqueness I would want this guy to actually go to 0.

Because if this were not true there will be only one unique curve that will pass through the origin, but here we have two. So, this we should really violate that condition which we were satisfying in the previous case. So, what we know about the example at hand when we talk of a transcritical bifurcation. In that case we had f x comma mu has mu x minus x square. So, what do I know here is that x equal to 0 is a curve of fixed points or equilibrium points passing through 0 comma 0. This is was this is always true.

And therefore, I can rewrite this vector field small f x u as x times some capital F x comma mu x equal to 0 is always a fixed point. Now where the F x comma mu is defined in such a way that this is equal to f x comma mu divided by x when x is not equal to 0 and it is defined as df by dx, 0 comma mu when x equal to 0 you can derive this should be it should be easy ok.

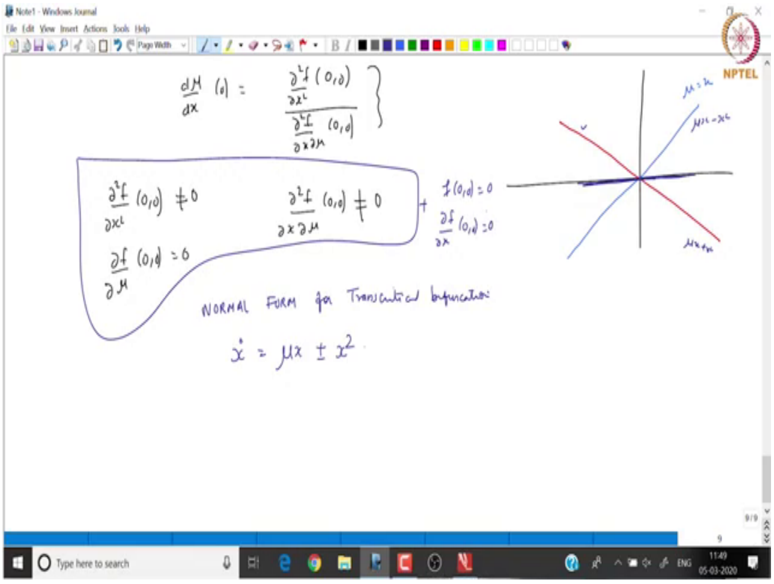
Now, x equal to 0 is already a curve of fix points passing through 0 0 right. Now we want to find additional curve of fixed points that pass through the origin and therefore, we need to now seek conditions not on small f, but now on capital F. Where again F has a curve of equilibrium points passing through 0 0 right. This F has a curve of 0 s or equilibrium points passing through 0 comma 0. So, loosely speaking this F comma 0 0, it will just have a curve like this with for F x mu and this is unique ok.

Now, first thing that we can easily verify is that f of 0 0 is actually equal to 0 ok. df of capital F by dx at 0 0. This is d 2 F by dx square 0 0. So, we will derive conditions on capital F and in turn derived conditions on the small f. That is all we are given right. You are given the small f, this is d 3 f by d x square 0 comma 0 and. So, this way you can verify that. So, and then we have df by capital F of d mu is d 2 f by d x d mu at 0 comma 0 ok.

Now, for this curve f x mu right only this one I omit this one for the moment for this we need a unique curve of fixed points passing through the origin and here x is equal to is related to mu in our example, x was related to mu as x equal to mu right. So, if this is not 0, if dF by d x is not equal to 0 then again by the implicit function theorem I can find in a small neighborhood of the origin a function mu of x such that this is equal to 0 ok.

Now, in the previous case what we had was that if I go back to the bifurcation diagram that the slope should be 0 here d mu by dx equal to 0. In this case here for this green for this line x equal to mu what we want is that the slope should not be 0. It can be whatever here its defined by x equal to mu, but in general we would want to ensure a condition that the slope should not be 0. This can be ensured by the following 0 less than d mu by dx of 0 should not be 0 at the same time it should also not be infinity ok.

(Refer Slide Time: 35:57)



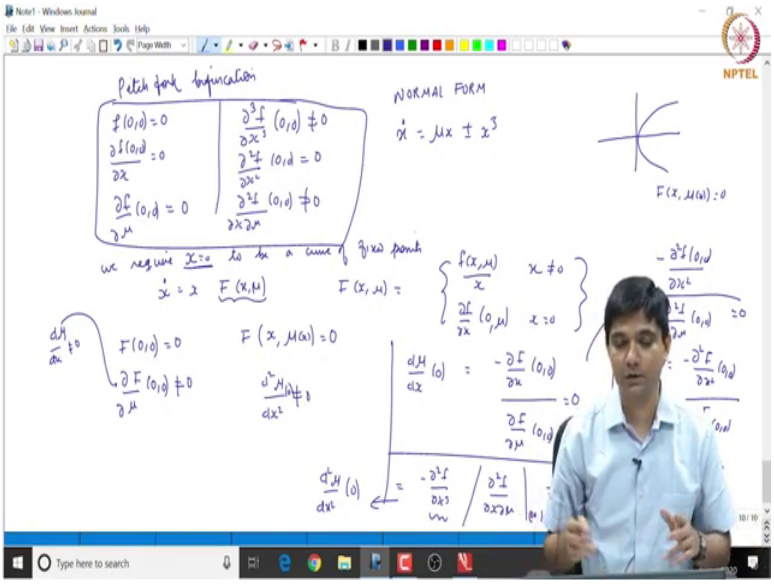
So, this condition is now I we can write in the following way; d mu by dx at 0 this is d 2 f by d x square 0 comma 0, d 2 f by dx d mu 0 comma 0 ok. So, what are now the additional conditions? I need this to be not equal to 0. df by dx not equal to be, so, this will translate to a condition of F in the following way. So, the additional conditions now I need are the following first this d 2 f by dx square, this should not be equal to 0 ok.

Now what we also have was this condition because for the original vector field there were two curves of equilibrium point passing through the origin, this is one then I have df by d mu at 0 comma 0 is equal to 0 ok. And lastly by from this expression here you can say that d 2 f by dx d mu 0 0 should also not be equal to 0 ok. Now in general; so we said that the slope should not be 0 and in our example we had occurred like this mu equal to x ok. So, in general we could also have a curve which is like this it could still have the same bifurcation properties right.

So, when we say this, we do not really say that the slope should be positive. So, that the magnitude I say right d mu by dx will just be bounded and it should not be 0. So, therefore, if I just now look at the normal form, I can write a normal form for this, for this as simply x dot is mu of x plus minus x square ok. So, this will give a possibility of these two. The origin the x equal to 0 will still be the equilibrium point and then you will have you can possibly have two different sets right depending on whether this is mu x minus x square or mu x plus x square ok.

So, this is my normal form for transcritical bifurcation ok. Now similarly, I will derive the bifurcation conditions for which. So, these two conditions plus the standard conditions of the hyperbolic fix point or the fix point passing through the origin or 0 eigenvalue condition ok. This is, these are conditions for transcritical bifurcation ok. Now I will just spend quickly I will do the conditions for pitch for bifurcation ok.

(Refer Slide Time: 40:01)



So, just as a recap what happens in this kind of bifurcation is the problem that again you have two curves of fixed points that pass through the origin ok. So, in this case; two curves of fixed points were passing through the origin ok. One was x equal to 0 and the second given by x square is equal to mu ok. Now x equal to 0 existed on both sides of mu equal to 0 and not only that this x equal to 0 had different stability properties for mu less than 0 where is way to a stable and it was unstable for mu greater than 0 ok.

And not only that after when mu was greater than 0, the other stability so, this guy had the same stability properties right for all plus or minus values of x. So, this is a green line of the while ok. Now, how do we derive conditions for this? I quickly give you an idea and write down the final conditions and the rest should be should be easy to derive ok. So, we will quickly go one by one.

So, this should be true all the time, this also should be true all the time. So, additionally here what we see is that is there a unique curve of fixed points that goes through the origin, the answer is no. Because, you will have two of them as just discussed. So, the third condition should be df by d mu 0 comma 0, this should be equal to 0. So, this will ensure that there is no unique fixed curves set of fix curve that pass through the origin seem very similar to what we did in the transcritical bifurcation ok.

Now, additionally we also require x equal to 0 to be the curve of fixed points ok. Which means; I should be able to write my vector field as F x, x dot equal to x some capital F of mu, then I can F of mu which is defined exactly in the similar way ok, x equal to 0 ok. So, in order to have an additional curve apart from x equal to 0 passing through mu x equal to 0; what we must have is F 0 0 equal to 0 ok. Not only that now, because for this capital [FL], there is a unique set of unique curve of fixed points at passes through 0 0, we should have this should be not equal to 0 ok. There is only one additional set of fixed curves that passes through the fixed through 0 0 for this capital F norm ok.

And therefore, I can write now mu the same as or I write it previous x, which is just try to look at similarities between all the three and the distinctions of all the three cases that we are trying to derive ok. And not only that this mu right which is derived for the capital function F. So, that unique fixed curve ok, so here I am just looking at F x mu x equal to 0 and this mu of x is such that it lies entirely to one side of mu equal to 0; that means, d 2 mu by dx square is not equal to 0 at 0.

Earlier, we had derived same condition here right. d 2 mu by dx square should not be equal to 0. This was for small f not capital F. The small f had a unique set of fixed points for the saddle node bifurcation whereas, here the big F has a unique set of unique equilibrium curve passing through the origin and it is such that it lies to the entirely on one side of it.

In the previous case for f it was lying on both sides in the transcritical bifurcation. So, we are kind of using both conditions that we derived previously ok. So, I will just quickly write down the steps now, of what these would mean. So, d mu by dx at 0 is minus df by dx 0 comma 0 df by d mu 0 comma 0 equal to 0 similarly, d 2 mu by dx square.

So, I am and deriving all conditions now from mu, conditions on mu sorry. So, here mu should have be in a function of x such that d mu by dx it was not equal to 0 I am deriving from mu conditions on capital F and eventually to conditions on small f as we did in the previous example at the 0, this is negative of d 2 f by dx square 0 comma 0. This and then you have df by d mu 0 comma 0, this should not be equal to 0 ok.

And now I know how to write this as functions of small f. This will become ok, I will just leave this for you to derive by d x cube by d 2 f by d x d mu everything at 0 0 right and this should not be equal to 0 ok. So, one additional condition that I can write now here is the following that from this. If this is not equal to 0, the numerator should not be equal to 0. So, the additional condition is d cube f by d f x cube at 0 0 is not equal to 0 ok.

And the next condition which essentially comes from here. This would look in the following way oh I am so sorry, this was for d mu by dx sorry d 2 mu by dx square at 0 turns out to be this one and this guy will turn out to be the following d mu by dx is minus d 2 f by dx.

I am sorry by dx square over d 2 f by dx d mu 0 comma 0, this should be 0. So, this should be 0, this for this to be equal to 0; we need to have two conditions. The first one is that the numerator should be 0 which means; d 2 f by dx square 0 comma 0 should be equal to 0 and the denominator should not be equal to 0 d 2 f by dx d mu at 0 0, this guy should not be equal to 0.

So, this will in general give you conditions to establish pitch for bifurcation and I can derive an equivalent normal form now, which looks like this, so you have x dot is mu x plus minus x cube. You can also derive it from the conditions on the Taylor series expansion. I will just skip those steps of what will be the coefficients a 0, a 1, a 2 and so on ok.

So, just to just to recap so, from here ok. So, what we have done now is just based on observations with some examples and making use of the implicit function theorem and some basic calculus results derived conditions for a vector field to satisfy to check whether or not there exists certain kind of bifurcations. Transcritical bifurcation saddle node or pitch for bifurcations will not do the how bifurcation that is a little more complex because you are looking at 2 D bifurcations, but for now I we feel its this much is like enough to give you a general idea on the theory of bifurcations.

In the next lecture, so one of our TS will help you with solve with some problems you will also post some solve problems online which will help you to solve the assignments. And for more you can always post questions and we can take up those in those questions.

Thanks for listening.