Nonlinear System Analysis
Prof. Ramasheshan
Department of Electrical Engineering
Indian Institute of Technology, Madras

Lecture – 23
Problems on Bifurcation of Theory

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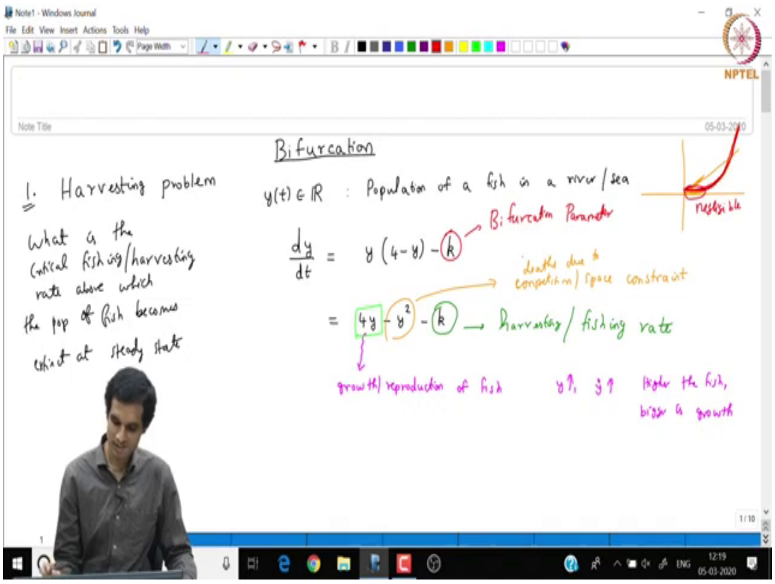

So, good morning everyone. I am Ramasheshan. I am a PhD student at IIT, Madras under Professor Arun Mahendrakar. So, I will be giving a tutorial lecture on Bifurcations. So, I will be solving some 3 problems relating to bifurcation analysis. I have uploaded the problems in the NPTEL portal. So, there are about 10 problems for your practice along with the solutions. So, I will be explaining 3 of them in here now, the rest of them you can see the material uploaded and then if you have any doubts as visual you can ask in the forum. So, we will begin now.

So, the first question is what is called as famously the harvesting problem. So, I will first describe the problem. So, here as usual it is a one-dimensional dynamical system. So, here y of t belonging to R it is the dynamical variable it is nothing, but population. Let us say of a fish in a river or a sea and the dynamics of the population of the fish is given by this equation dy by dt is equal to y into 4 minus y minus k.

So, let us first analyze why this model make sense for the scenario that is presented. So, if we expand this, we will get 4y minus y square minus k. Now, this 4y arises because of the growth or reproduction of the fish. So, more the fish, more they reproduce right; so higher the fish bigger is the growth rate. So, to capture that proportionality we are introducing this term 4 y. So, this 4y captures the growth or reproduction of the fish, but it cannot grow forever, right. As the fish population increases we cannot keep on growing to infinity right, there should be another term that reflects competition of space constraint or some other food competition when the population is very high. So, we have a another d k term which is minus y square that captures the competition or deaths due to competition or space constraint.

And why it is minus y square is because if you see the function y square it grows much much slower than y in the beginning and after sometime only it takes over. So, if we see y square function on contrast it to the linear function y, initially it is much much slow. The initial effect of y square is very negligible. So, at lower populations, we expect the competition to be a negligible effects. So, when the population density is very low, there will typically be a enough space on enough resources for everybody, right, that is why we have a minus y square here. It is only when the population gets really high we get to have problems, and when the population gets really high competition is a more severe issue than growth rate.

So, that is why we want a function which for lower values of y is much much smaller, but takes over very largely at higher values of y and this function y square takes it. And we add a minus y square here because we want we want the d k rate. So, since this captures the death, so it should to be a negative. So, initially the death rate due to competition is really low whereas at a higher population the death rate is really high. So, this term is a competition term. And this term k is called the harvesting rate or a fishing rate. So, this, what does it reflects is, let us say somebody is fishing the river or sea where the fishers are living. So, they keep on depleting the fish at the constant rate k.

So, this is the dynamical system that is given to us. And the problem that is asked is what is the critical fishing or harvesting rate above which a population of fish is becomes extinct at steady state. So, intuitively we realized that when the harvesting rate is really high the fish population should get extinct, right. So, we have to do that mathematically, we have to prove this mathematically.

So, this is a bifurcation study because here if you see it is a dynamical system and k is the parameter. So, k is the bifurcation parameter here. So, as k increases we want to know what happens. So, we want to study the equilibrium points, how they behave. So, let us do the bifurcation study now. So, first let us analyze this problem graphically.

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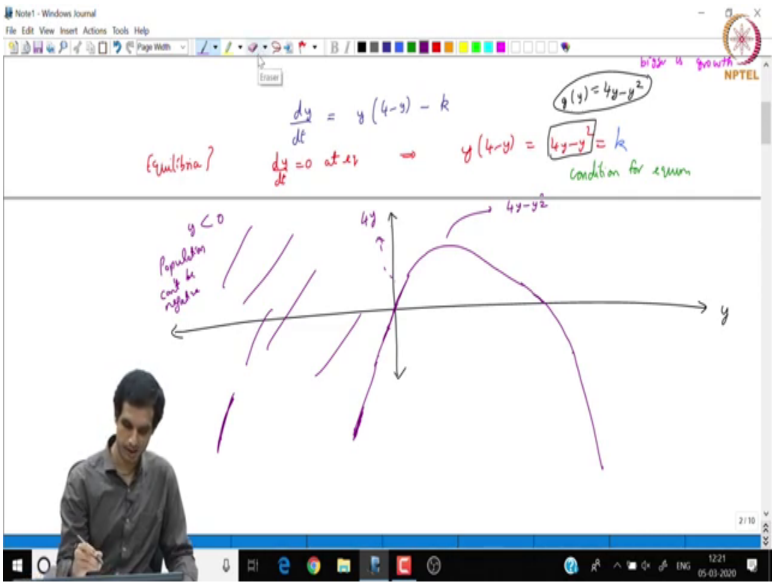

So, we have once again I have dy by dt is equal to y into 4 minus y minus k. Now, the question is what are the equilibria? Where are the equilibria? For equilibria we know that dy by dt is equal to 0 at equilibrium. So, this implies y into 4 minus y which is nothing, but 4y minus y square should be equal to k. So, this is the condition for the equilibrium.

So, the equilibrium parts are nothing, but the solution to this equation. Now, this is simple quadratic equation that we know how to sketch quadratic equations and their solutions. So, here we have, so here is the y and here is this function. So, this function let me call it g of y, g of y is equal to 4y minus y square.

Now, how do this function look like graphically? So, when y is really small initially this looks like the linear function 4y. So, when y is really really small what we have is it looks like this. So, this line looks like 4y. Now, as y becomes really large it biggest to look like minus y square graph because minus y square dominates 4y, where y is really large. How does the minus y square graph look like? It looks like a inverted parabola right, it something like this.

So, what happens is it has to do something in between. So, this is the graph and we are not bothered about y negative because physically population cannot be negative physically. So, we are not bothering about what happens when y is negative. But if you want for your reference it will keep going like this because minus y square it keeps going like this, so the linear and quadratic part kind of reinforce each other. So, this is the graph of the function 4y minus y square, correct.

Now, for equilibrium, so let me erase this unwanted things.

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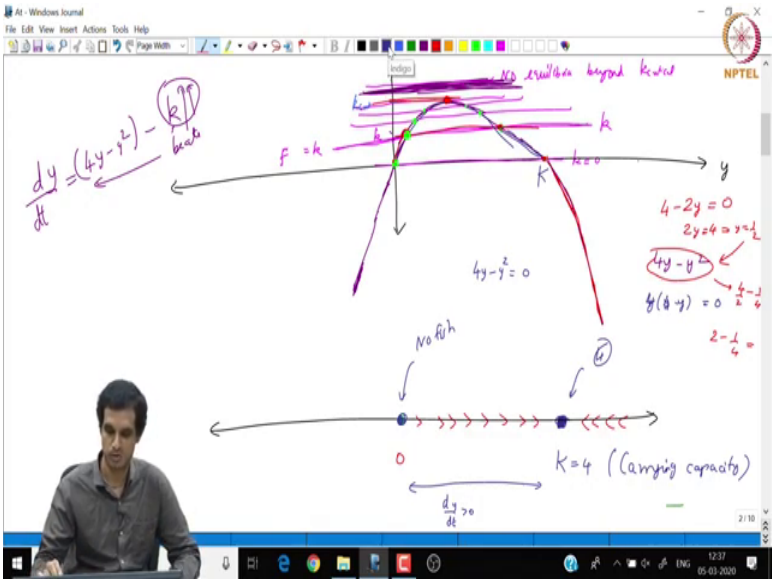

So, this is the graph of the function 4y minus y square. So, for your reference, I am writing the function in. Now, for equilibrium these has to intersect k. So, this is the line y equal to; so, this is the line let us say k. So, this is the place let us say where f of f is equal to k. So, this is the value k.

Now, for equilibrium these two curves have to intersect because these two equations have to hold simultaneously. So, this k should be equal to 4y minus y square. So, typically we say we are get two equilibrium points, but do we always two equilibrium, no. As k increases this line shifts upward we see that the equilibria get closer and closer and after some critical value k critical, so let me call this value corresponding to the vertex of this parabola k critical, we see the two equilibria merge into a single degenerate equilibrium and beyond k critical there are no equilibria at all they just vanish. So, this is related to the critical harvesting rate because that was asked to in the question. So, let us come back.

So, let us analyze the flow of this dynamical system. I will draw the vector field, it is a line, it is very simple. So, let us take the value first k equal to 0. There is no harvesting at all, we are know nobody is fishing in the rivers all the fisher were very happy and happily swimming around. So, now we see that there are two steady states, one is 0 which is I have a one equilibrium, so this is 0, and there is another equilibrium which I call it let us say k. So, we see that when the harvesting rate is 0 there is one equilibrium point at y equal to 0. So, and another equilibrium point which I have called it k, that k can be found by solving the quadratic equation 4y minus y square equal to 0 which here is nothing but. So, if we solve this quadratic equation, it is y times 4 minus y 0. So, the k is equal to 4 here.

And now if you look at the this quadratic function between y is equal to 0 and y is equal to 4 dy by dt is positive. So, dy by dt is positive in this region. Now, when dy by dt is positive means y increases with time right that is what the derivative says. So, if we look at the flow of the vector field it will be like this. So, the population evolves like this. Whereas, when y is beyond 4 dy by dt is negative because dy by dt is this quadratic and this quadratic is negative which means y decreases. So, now, you see there are two equilibria, one corresponds to 0 and we know the population is 0 means there are no fish, another equal to 4.

So, we see that when the harvesting rate is 0 even when we have a slightly nonzero amount of fish they will reproduce, reproduce, reproduce and grow, grow, grow and grow they will keep on growing until they reach y equal to 4. So, there is the population will settle at y equal to 4. And if suppose it gets greater than 4 the competition and all the space constraint they will lead to deaths and the population will level down back to 4. So, this 4 is kind of called carrying capacity of the river in population biology literature.

So, we see that the carrying capacity is 4 when the harvesting rate is 0. So, the population settles at 4. Of course, a physical population of 4 may not make sense, you may treat y as population in 1000s or 10000s to make more sense out of it. Now, what happens when k is slowly increased from 0? So, when k is slowly increase from 0 what happens is the 0 steady state, it shifts somewhere over here.

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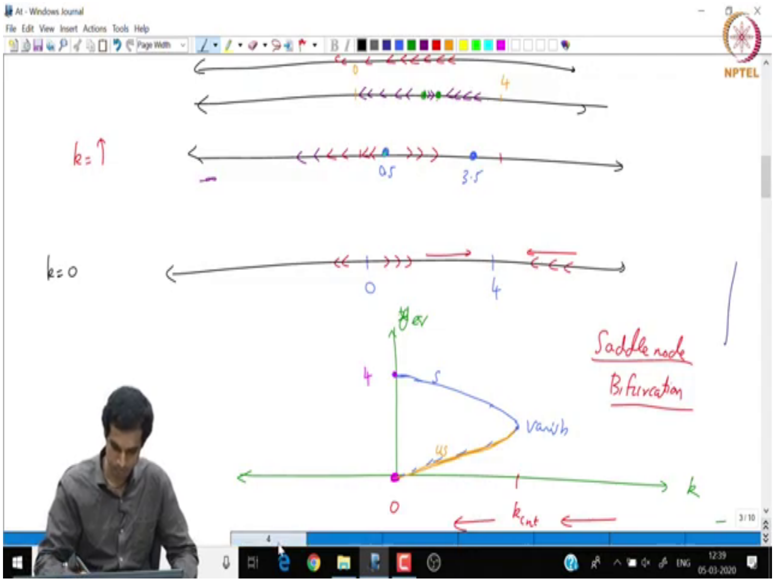

So, let me go to the next page. So, when as k increases, so this is when k equal to 0 we saw there are two steady states 0 and 4 and this is unstable and this is stable. So, the population kind of settles around 4.

Now, as y increases sorry, as k increases what happens is this 0 steady state if you see, so this 0 steady state it kinds of comes here it kinds of comes closer because now, it is an not the intersection of the parabola with 0, but it is the intersection of the parabola with k. So, this comes here and this comes; so, the left steady state moves bit to the right and the right steady state moves a bit to the left. So, what happens here is this these two equilibria they kind of come here, but stability is maintained if you go back the logic; dy by dt is positive in the region in between and it is negative, sorry if this is negative. So, this is negative.

So, now we see a very. So, the population now let us say is 3.5 and this is let us say 0.5 just for some simplicity is may all correspond the actual values. Now, we see a very drastic behavior, now when the population is above 0.5 it settles to 3.5, it did not settle at 4 it still settles are slightly bit of lower value because it is expected right, when you start fishing the steady state it has to be lower. But earlier any nonzero population of the fish tends towards 4 whereas, now here if you see the initial population of the fish has to be beyond some threshold which is 0.5, if it starts anywhere below 0.5 it will decrease. Of course, it cannot go to negative values it will settle at 0.

So, this negative thing is not feasible. The population cannot keep on going to minus infinity. These reflects that our model is not that good. So, of course, I did not expect inaccurate model because I just wanted to explain the overall view. I am not I do not want to give an accurate model. So, we see that. Now, as k increases further these two equilibria they get even more closer which is becomes which becomes worse.

So, they kind of become very very close now. So, this is again 0 and this is 4; so, they kind of sorry. So, now, what we see is this is a kind of come really really close. So, only in this narrow band, you have a nonzero steady state. Anywhere the population starts below this value it will decrease and the fish population become to extinct. But if you are luckily enough to start in this narrow band it will go and reach another steady state.

Now, when k exceeds the critical population level, so which when k exceeds the critical value suddenly there are no equilibria and you observe that dy by dt is always negative because dy by dt is 4y minus y square this quadratic minus k and when k is really high, it will beat this term. So, dy by dt will always be negative now. If you see this always beats this. So, this straight line always beats this curve. So, this beats this. So, the dy by dt will always be negative.

So, what happens tragically? Tragically when k increases beyond the critical value the population just decreases it just the fish all the fish becomes extinct. So, how do you calculate the critical level of harvesting? Again, it is very simple algebra, we have to find this vertex of this parabola and the vertex point the derivative vanishes. So, if you see the derivative of this because for a local maximum derivative vanishes. If you take the derivative, what is the derivative of this? 4 minus 2 y is equal to 0 because we want the derivative to vanish.

So, this gives us 2 y equal to 4y equal to 1 by 2 substituting y equal to 1 by 2 here it gives the value 4 by 2 minus 1 by 2 square; so, this 4 by 2 which is 2 minus 1 by 4, so 2 minus 1 by 4 whatever value this fraction is that is the critical harvesting rate. Beyond this harvesting rate you have doomed, with the population of the fish in the lake or the river or the ocean will go extinct.

So, this is the typical bifurcation example. If you plot the behavior, so if we have a k. So, if you plot the equilibrium y equilibrium versus k, we see that at k equal to 0 we have one steady state at 0 and another steady state at 4. As k decreases what happens is, so this was unstable. So, I denoted by a dot and this is for stables. So, I denoted by the dot. So, this they come closer and closer to each other and beyond this critical harvesting rate, so may be yeah. So, they kind of merge here. After this there are no equilibria at all, they just vanish no equilibria.

So, what kind of bifurcation this reminds you off? So, this blue color corresponds to stable equilibrium whereas, the yellow color corresponds to unstable. So, what does this remind you off? If you look at in the backward direction till a critical value and there are no equilibria and suddenly there are two new equilibria coming out of the blue. So, this is nothing, but a saddle node bifurcation. So, this, so we you do not have any equilibrium and suddenly two new equilibrium are born where one is stable and another is unstable that is the signature of a saddle node bifurcation.

So, the reason for a existence of a critical harvesting rate is because of this saddle node bifurcation. So, with this we can see how simple models can capture the behavior of the system. So, now you move on to problem number 2.

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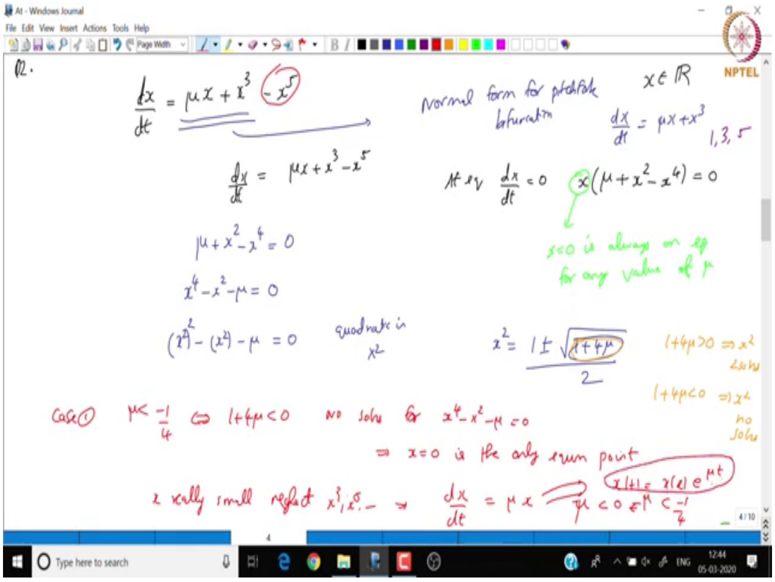

So, there we are asked to analyze the behavior of the dynamical system dx by dt where x is in R, dx by dt mu x plus x cube minus x power 5. So, this is the system.

Now, what are just remind you of? When we have only mu x plus x cube this was nothing but the normal form for a pitchfork bifurcation. So, the normal form for a pitchfork bifurcation was dx by dt is mu x plus x cube right or minus x cube does not matter. So, now, we add a perturbation to this pitchfork. We have an x power 5 perturbation to the pitchfork we are now asked to analyze what is the behavior of this system.

So, I will again write the system dx by dt is equal to mu x plus x cube minus x power 5. Again, we have to find the equilibria. So, the equilibria, at equilibria dx by dt is 0 and hence if you take x common we have mu plus x square minus x power 4 is equal to 0. Now, definitely x equal to 0 is always an equilibrium for any value of mu. So, we are guaranteed the existence of one equilibrium for sure.

So, these are other 4 candidates. So, because these are 4th order polynomial at most we can have 4 solutions. So, when you are lucky enough you will have 4 solutions when we are not lucky we may have we may not have any solution. So, the number of equilibria can be either just 1 which is this along or it can be 1 plus 2 which is 3 or we may have 1 plus 4 which is 5. So, the number of equilibrium points can range anywhere between 1, 3 and 5. So, let us analyze the roots of this equation mu plus x square minus x power 4. It is same as the roots of the equation x power 4 minus x square minus mu equal to 0 right, because there is something is equal to 0 its negative also better be 0.

Now, we can write this as the x square whole square minus x square power 1 minus mu equal to 0. Now, this is a quadratic in x square. So, by the quadratic formula the roots will be x square is 1 plus or minus root of 1 plus 4 mu by 2. Now, as visual the discriminant of this quadratic plays a very very important role determining the number of solutions, right. When the discriminant is positive 1 plus 4 mu is positive x square will have two solutions and when 1 plus 4 mu is negative, x square will have no solutions. So, now, let us now analyze the case. Case 1, when mu is less than minus 1 by 4 which is same as same 1 plus 4 mu is less than 0 because if we rearrange we will get mu less than minus 1 by 4.

Now, an mu less than minus 1 by 4, no solution for this quadratic. So, this implies x equal to 0 it is the only equilibrium point and we want to know whether it is stable or not. So, for stability we want to know how what happens when x is perturbs slightly from 0. When x is really small we can neglect x cube x power 5 and so on and on, this hence we can get dx by dt is equal to mu x and mu here is negative because mu is less than minus 1 by 4 which implies mu is negative.

Now, when mu is negative the linear equation dx by dt equal to mu x is stable, right, because it is an the only solutions, for this differential equation or x of t is equal to x of 0 e power mu t, right. On when mu is negative everything d k is to 0.

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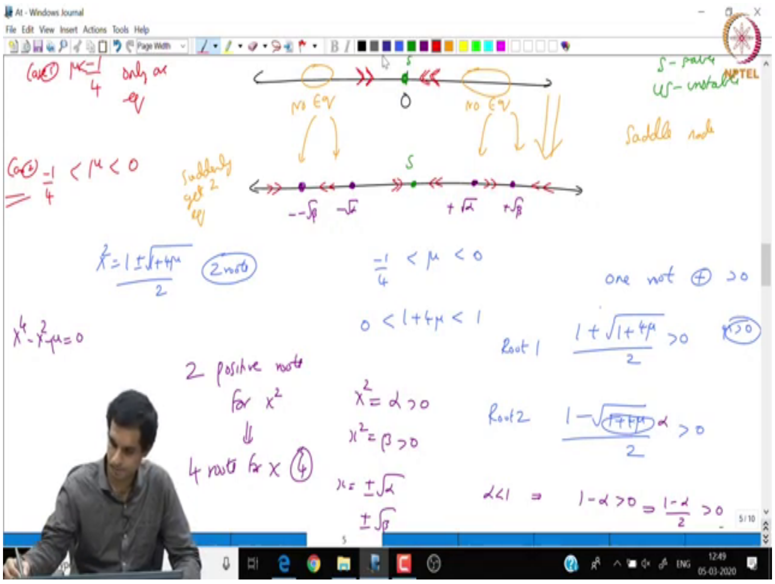

So, in that case what happens just, so in case 1 when mu is negative, we have only one equilibrium which is 0 and that is stable. So, we will have a portrait like this, correct. So, this is the case when mu is sorry, when mu is less than minus 1 by 4. So, when mu is less than minus 1 by 4 only one equilibrium and since mu is negative this will be stable. So, this origin is stable. If I will write s for stable and u s for unstable.

Now, we will come to the next case when mu is bigger than minus 1 by 4 that is still negative. So, in that case what happens is origin is always an equilibrium, no doubt at all indeed and it will still be stable because mu is still negative. By the same argument origin will be stable, but what happens is we witness the birth of two other equilibria because this quadratic if you see x square, what was the quadratic? x power x square is 1 plus or minus root of 1 plus 4 mu by 2. So, this has 2 roots.

And if we see if you look very carefully when 1 plus 4 mu is bigger than, so when this bigger than 0, but it is less than 1. So, when mu is bigger than minus 1 by 4 and less than 0 1 plus 4 mu is essentially greater than 0, but less than 1. So, if it is less than 1, we have both the roots, both the roots, if we compare both the roots 1 root is positive, 1 root plus that is positive because 1 plus root 1 plus 4 mu by 2 is positive because mu is positive and every other thing is positive. And if you see the other root, so this is root 1 and if we look at root 2 which is 1 minus root of. So, when mu is kind of positive you will have a root indeed. But when mu is kind of very small does not dominate, what happens to this? What happens to this root? This root is also positive right because when this is less than 1. So, let us called this quantity alpha if alpha is less than 1, you will have 1 minus alpha is positive. So, this 1 minus alpha by 2 will also be positive. So, this will also be positive.

So, we have two positive roots for x square which means we have 4 roots for x because if x square can take two positive values, alpha and beta, then x can take either plus or minus root alpha and plus or minus root beta. So, we have now 4, we have 4 the maximum possible roots for this 4th order equation x power 4 x power 4 minus x square minus mu equal to 0. So, we now has a maximum number of roots.

And now what happens is, so we can both of these, so these are the positive roots this are plus root alpha and this is plus root beta and this is minus root alpha, and this is minus root beta. We have now two other roots. So, again, this again typically if you see from nowhere we suddenly get two equilibrium. So, we suddenly get two equilibrium from nowhere. So, again it is a signature of a saddle node bifurcation.

So, we did not have any equilibrium and suddenly, we now getting two equilibrium. So, this is the signature of a saddle node. So, if we look at the stability by a continuity of the vector field if this points left this has to point right and this has to point. So, if this point is right, this has to point left and if this points left and this has to point right. So, we see that there are 4 equilibria now, alternative in stability. So, this is the scenario in case 2 when mu is bigger than minus 1 by 4, but still is negative.

Now, what happens in the case next case when mu is positive?

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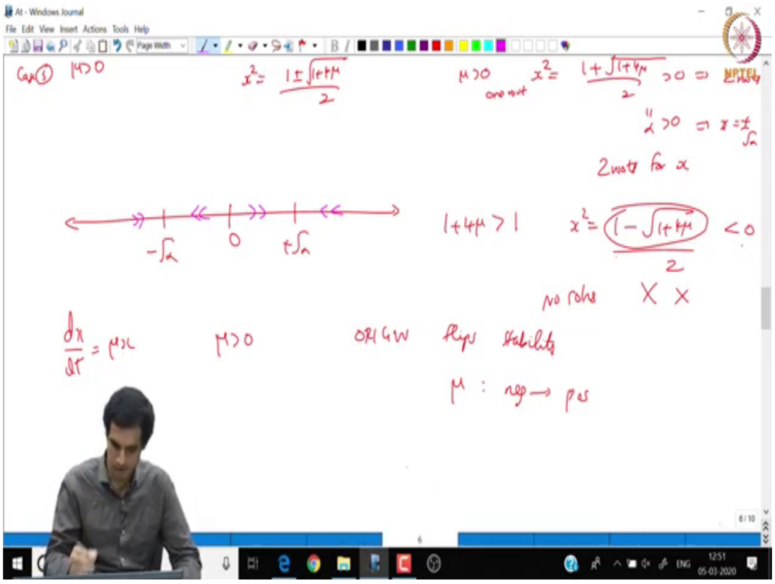

So, now we look at case 3. Mu positive again we have x square is 1 plus or minus root of 1 plus 4 mu by 2 when mu is positive 1 root 1 plus root of 1 plus 4 mu by 2 this is positive no doubt about it. So, we will get two roots.

So, let us say we call it as alpha which is positive we will get two roots x is plus or minus root alpha, we will get roots for x. But when mu is positive 1 plus 4 mu is bigger than 1. So, the other root 1 minus root of 1 plus 4 mu by 2 this becomes negative because this becomes bigger than 1, square root of a number bigger than 1 is still bigger than 1; 1 minus a number bigger than 1 is negative. So, x square equal to some negative number does not make sense. So, we have no solution for this.

So, this has only two roots now apart from 0 there are only have plus root alpha and minus root alpha and if you do linear stability analysis dx by dt is mu x when mu is positive it is unstable. So, origin flips stability, it changes from stable to unstable when mu changes from negative to positive. So, we will just have something like this.

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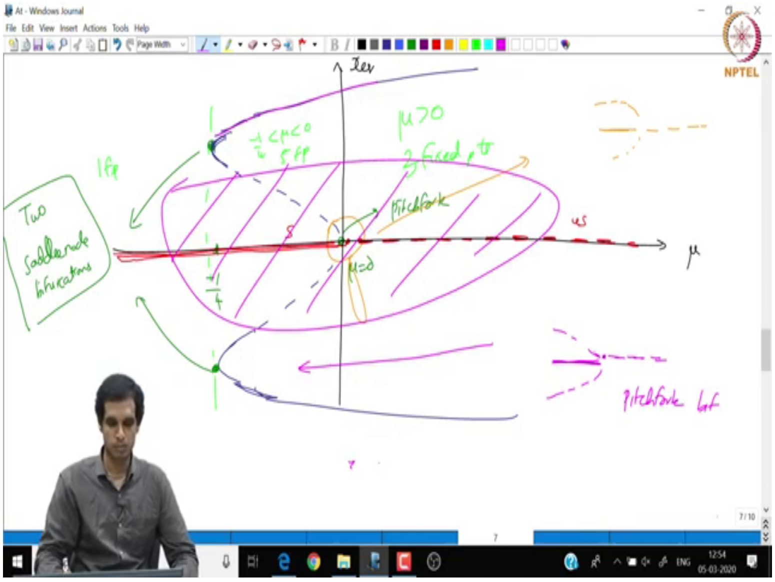

Hence if we plot diagram of how the equilibria vary with the parameter what we will have is, so this is mu and these are the equilibrium values. So, origin is always one equilibrium we saw, so e 0, x equal to 0 is always an equilibrium, but where mu is positive it is kind of stable whereas, when mu is negative it becomes the unstable.

Now, when mu is minus 1 by 4 something funny happens, out of the blue we get 2 mu fixed points. But we see when mu is positive only these two survive, these two they disappear, right. So, they kind of merge they kind of merge and then they disappear there are only two fixed points. So, when mu is positive, 2 plus 1, 3 fixed points including the origin and in this range when mu is bigger than minus 1 by 4 and negative we have 5 fixed points and here as usual we have only one fixed point.

So, if we look at the reverse, what is happening? If we look at the reverse as mu changes from positive to negative, there is an unstable equilibrium that is becoming stable and giving raise to two other unstable equilibrium. So, this must be unstable strictly speaking. So, let me, so one of them is stable the other one is unstable. So, unstable equilibrium is becoming stable and giving raise to two other unstable equilibrium. So, this is nothing, but a pitchfork bifurcation. So, what is happening here when mu equal to 0. So, when mu equal to 0 we witness a pitchfork, so that means, if you locally zoom this diagram if you zoom this one look it will be a pitchfork bifurcation.

Now, when mu is minus 1 by 4 we have two saddle node bifurcations happening simultaneously. So, these are two saddle node bifurcations. If we put the mathematical conditions on check vertex mu this is you are indeed find these are two simultaneous saddle node bifurcations. So, this is the bifurcation behavior of the dynamical system. So, if you add x power 5 if you did not add x power 5 then we add only this part of the diagram. So, this zoomed part. So, this happened when you do not have the x power 5 ordinary pitchfork addition of x power 5 in the dynamical term gives raise to these other behavior this other branches. So, we now have 5 branches now.

So, with this we complete this problem. So, we will now move on to the next problem.

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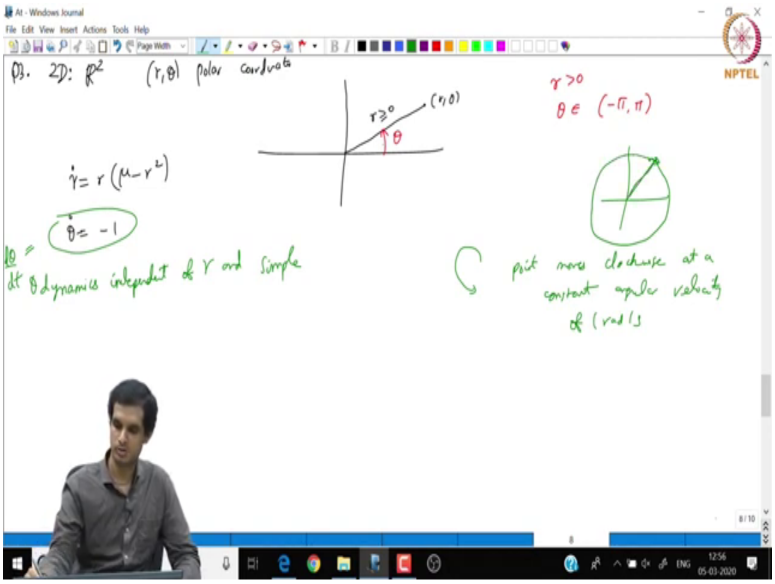

Now, here I present to you dynamical system in two-dimension is R 2, but you see that it is not a really two-dimensional example because one of the variables the dynamics is very independent and trivial. So, I use polar coordinates now in R 2. To recapitulate polar coordinates in R 2 a point is characterized by r and theta where r is the distance from the origin which is always positive on and then theta is this angle. So, we will take r to be positive and theta to be belonging to minus pi to pi.

So, this dynamical system that I give you here is kind of a dynamical system in polar coordinates, but in the polar coordinates we have a very simple behavior, where we have r dot is equal to r into mu minus r square whereas theta dot is equal to minus 1. So, we see this dynamics of theta is independent of r and it is quite simple you just keeps on, it just keeps on increasing at a constant decreasing at the constant rate of minus 1.

So, this clearly means that, so Theesen's theta is this angle this essentially means that the point here moves clockwise because theta is anti-clockwise. So, theta dot is minus 1 implies the point moved clockwise at a constant angular velocity of 1 radial per second. Because theta is an angle, so the derivative d theta by dt will be the angular velocity, right. So, this is just says that wherever the point in R 2 is as for as the theta behavior is concerned it will keep on moving around the circle in minus. If you look at the projection of this dynamics on circle it will just move out the constant speed of one clockwise that is what this equation is saying.

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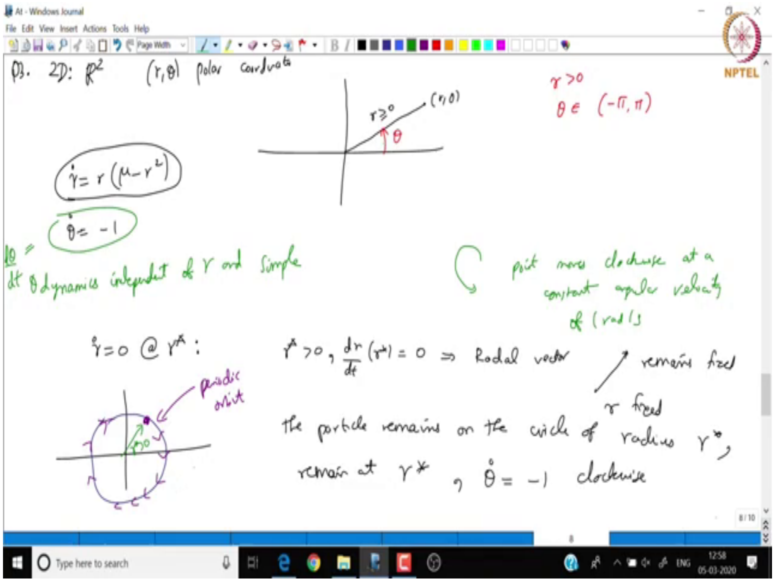

It is the dynamics in the r that that is of real importance to us. So, what is happening let us see. So, the dynamics is r dot is r into mu minus r square. So, when r dot is equal to 0, what happens? So, let r star be, let r dot equal to 0 at some radius, let us say r star. What does this really tell about? So, when let us say r star is positive and we have a dr by dt at r star is equal to 0.

So, this just says that the radial vector joining the point to the origin this remains fixed, so r remains fixed. So, this really says that the particle remains on the circle. So, if the particle starts on the circle of radius r star it will remain at r star ok. On hence, but theta dot is minus 1, so it has to keep moving clockwise. So, when we have a nonzero r star that is becoming 0, what will happen is the particle when it starts anywhere on the circle it will just keep on moving with the constant angular speed of 1 clockwise. So, this corresponds to a periodic orbit when r star is positive. Again this r star has to be positive.

Now, when r star is 0 what happens is r equal to 0 it is not a circle it is a essentially a point, right.

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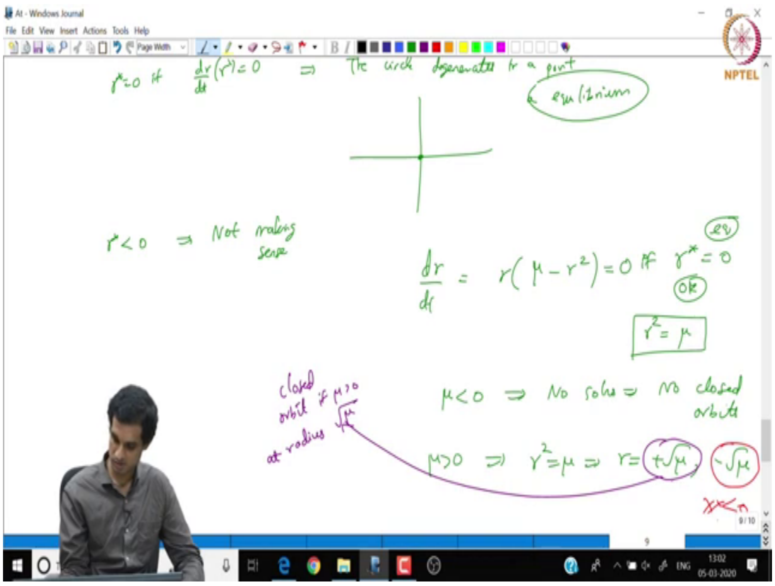

So, when r star is equal to 0, so at r star equal to 0 if dr by dt of r star is equal to 0 then the circle collapses or degenerates to a point. So, it is essentially an equilibrium. It is not it is not periodic orbit only in equilibrium because the particle its starts here, its states here, it states at r equal to 0 which means it is sustain equilibrium, right. So, this is the thing that we need to be constant about. So, the equilibrium points of dr by dt dynamics if they are positive then they are closed orbits, but if it is 0 then they are it is an equilibrium point. When r star is negative does not make sense, they are not acceptable values because the distance from the origin is always positive.

So, we have to discard any negative solutions we get for this equations. So, let us analyze this equation dr by dt is equal to r into mu minus r square. So, one solution is r equal to 0. So, this become 0 if either r star is 0 which means is an equilibrium at origin or r square equal to mu. So, this again of a quadratic equation.

Two cases, when mu is negative there are no solutions, so which implies there are no closed orbits. When mu is positive we have r square is mu which implies r is plus root mu r minus root mu. But as I said negative values of radius does not make sense, r cannot be negative. So, we can have only positive values of r. So, this corresponds to a closed orbit when mu is positive at radius root mu.

Now, what we are do is we have to analyze the stability of the equilibria on the limit cycle.

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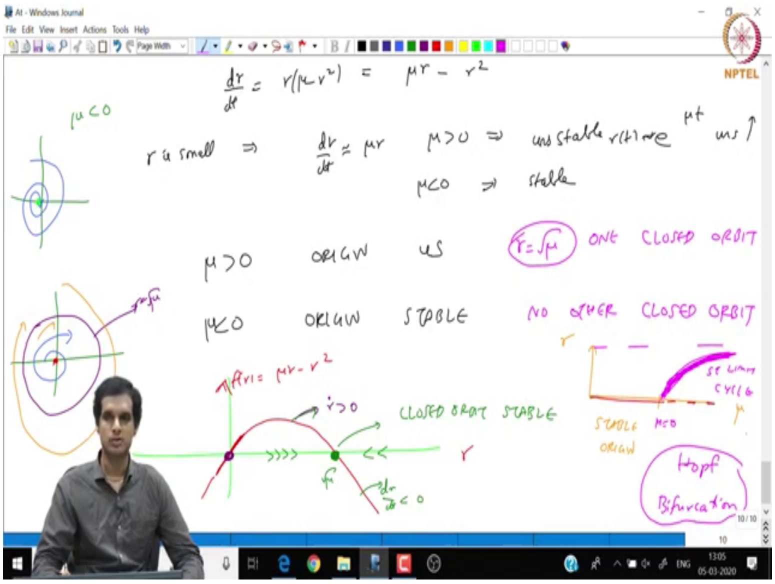

So, go to the last page dr by dt is equal to r into mu minus r square which is nothing, but mu r minus r square. So, when r is small we have dr by dt is approximately mu r because we are neglecting this when r is small. So, when mu is positive we see it is unstable because it is it has a solution r of t grows a v power mu t. So, it is unstable, grows. Whereas, in mu is negative this is stable.

But when mu is positive, so the origin is unstable; when mu is negative origin is stable. But when mu is positive we have a closed orbit, right. So, we have a closed orbit at plus root mu. So, there exist one closed orbit and when mu is negative that are exist no other closed orbit.

And if you look at the linearization about r equal to root mu if you plot this function mu r minus r square so, let this be r and let this be f of r which is mu r minus r square, r is positive that is a value we are bother about. When initially, when mu is positive and very small r it look like mu r graph other it will the minus r square term will take over. So, if you see there are two equilibrium between here to here this r dot is positive, so the radius always increases. When r is bigger than root mu it decreases because this is a negative dr by dt is negative here, dr by dt is positive here. So, we see this closed orbit is actually stable.

So, what is happening here is when mu is positive, when mu is negative there is only one stable equilibrium point. So, origin is stable equilibrium. And any trajectory kind of set is to the origin. Whereas, when mu is when mu is negative origin becomes an unstable; and we calculate it becomes a focus indeed because theta is theta dot is minus 1, so it has to keep rotating clockwise.

So, if it keeps rotating clockwise is spirals out, but that is another closed orbit at r equal to plus root mu, and this is stable any trajectory starting above it will come below and settle, anything starting below will come above and settle. So, when mu versus r, so initially we have only a stable origin, so this is stable origin and after mu becomes positive the stable origin becomes unstable and gives raise to your limit cycle or a closed orbit at r equal to mu. So, this is r equal to root mu because square root function will look like this, right. So, this is when mu equal to after mu is positive it becomes; so, we have a stable limit cycle. So, this is nothing, but the signature of a Hopf bifurcation in two-dimensional systems.

So, we analyze this Hopf bifurcation even though it is a 2D bifurcation we analyzed it in 1D, because we could make use of polar coordinates. Why we could make use of polar coordinates? Because all the limit cycles are circles here.

So, with this we complete the recording. So, thank you. So, if you have any clarification you can ask in the forum for further details.