Nonlinear System Analysis
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Lecture – 24
Stability Notions: Lyapunov and LaSalle's theorem
Part 01

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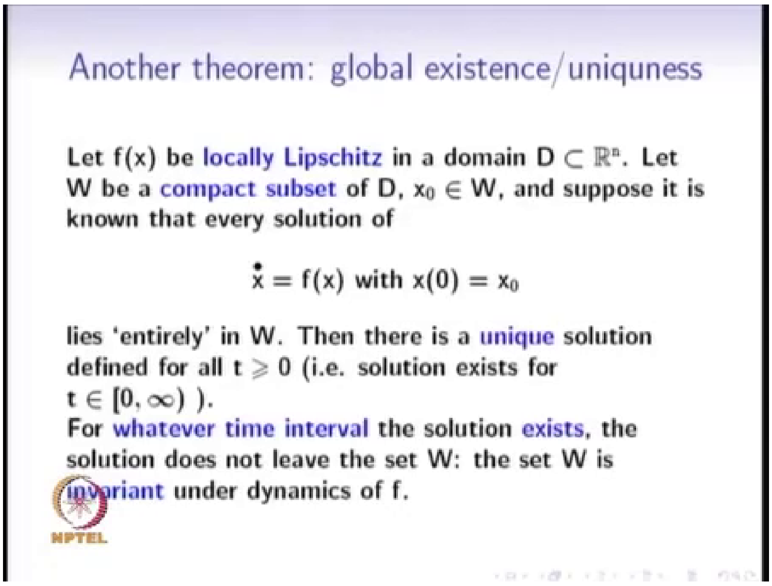

We will now move on to stability to the notion of stability. What do we want to say about stability of an equilibrium point?

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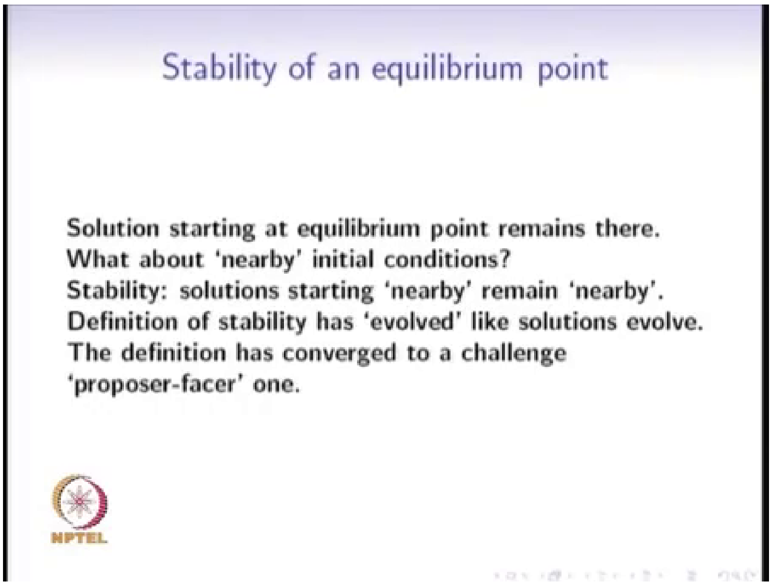

So, we would like to say that a Solution starting at equilibrium point. Of course, we know that solution starting at an equilibrium point remains there but, what about nearby initial conditions? Can we say that solution starting nearby also remain nearby? So, stability is what? Solution starting nearby, near an equilibrium point remain nearby. So, we are going to try to quantify this nearby and this nearby.

So, one should also note that the definition of stability itself has evolved, like solutions evolve for a dynamical system even the notion of definition of stability, even that notion of stability has evolved over the last few decades, and finally, it has converged to what we will see in the next slide. So, this is best understood as this particular definition is best understood as, as a challenge yeah.

So, it is like somebody proposes a challenge and somebody, who is facing the challenge tries to answer, tries to meet that challenge.

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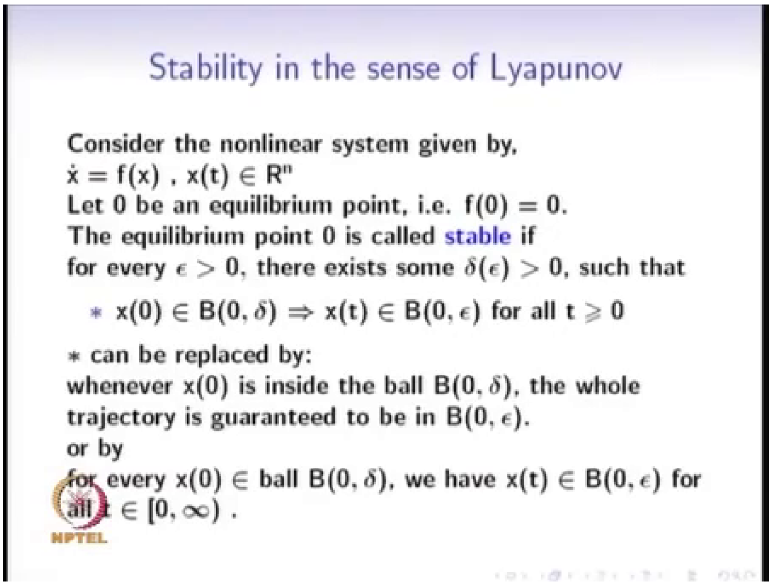

So, what is this challenge proposer facer definition of stability? So, Consider the nonlinear system given by x dot is equal to f of x, in which at any time t, x of t is an element of R n, x has n components. Let 0 be an equilibrium point. So, for convenience we are assuming that the origin itself is a equilibrium point. If it is not origin that is a equilibrium point, but some other equilibrium point we are studying, we can just shift the coordinates there and be studying with new coordinates, in the new coordinates the origin is again the equilibrium point.

So, lets 0 be an equivalent point that is f 0 is equal to 0. Then the equilibrium point 0 is called stable, if for every epsilon greater than 0, there exists some delta greater than 0 such that, for every initial condition x naught inside this ball inside this ball centered at 0, and of radius delta for every initial condition inside this, we have the property that x of t belongs to this other ball again centered at 0 and of radius epsilon, for all t greater than or equal to 0.

So, when do we call the equilibrium point stable? Somebody proposes that for this epsilon. Can you find the delta? The equilibrium point will be called stable, if no matter what epsilon somebody proposes we are able to find the delta greater than 0, such that as long as you start inside this initial as long as your initial condition is inside this delta ball, your whole trajectory lies inside this epsilon ball yeah.

So, this star condition here, it is a very important condition, where the delta, where the epsilon comes here is a very important part of the definition. This star can also be replaced by, whenever 0 is inside the ball, the ball of radius delta centered at the origin, an open ball. The whole trajectory is guaranteed to be inside this other epsilon ball, epsilon is what somebody else proposes to us and delta is what we are able to calculate and find.

We can also replace the star by for every initial condition x naught inside the ball, B 0 comma delta, we have x of t inside this other ball; B 0 comma epsilon for all time 0 to infinity.

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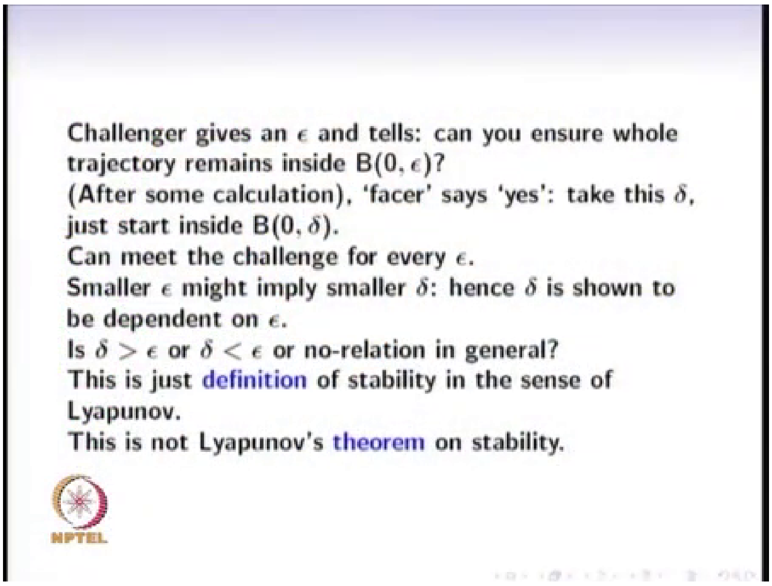

So, what I have said in a previous slide? This challenger is a person who gives an epsilon and tells can you ensure that the whole trajectory remains inside B 0 comma epsilon? After some calculation, the facer of that challenge, the person who meets the challenge says, yes take this delta, just start inside this ball B 0 comma delta, and if you start inside this ball B 0 comma delta we are guaranteed to be inside this other ball B 0 comma epsilon.

So, the fact that we are allowed to do some calculation means that delta is allowed to depend on epsilon. So, if we are able to meet this challenge for every epsilon, no matter how small That is when you will call the equilibrium point as stable. Smaller epsilon might mean a smaller delta. Hence delta is shown to be dependent on epsilon in the previous slide, in the definition. So, this is we should be seeing a figure here.

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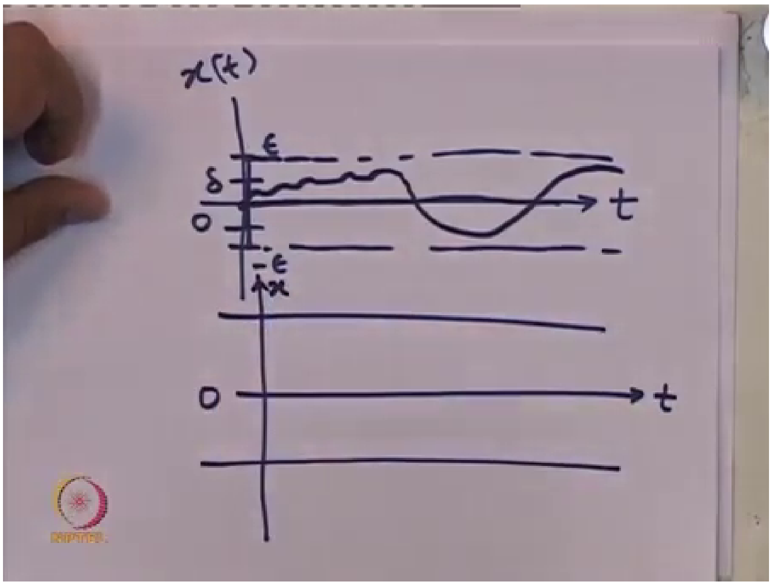

This is the time axis, for the purpose of this figure, x has only one component and 0 is the equilibrium point. So, of course, starting at 0 we have the constant solution. The solution always remains at 0, but somebody proposes as an epsilon ball. So, this minus epsilon to plus epsilon band is here, and by our convention our epsilon ball is an open ball. In other words, this boundary epsilon we should not touch. So, somebody proposes this epsilon ball and tells can you ensure that the trajectory remains inside this epsilon ball.

So, after some calculation we come up with this delta. So, that as long as the initial condition starts inside this, it might leave the delta ball of course, but it will remain inside this epsilon ball yeah. So, for the trajectory to remain inside this epsilon ball, this is a 0 this is another figure with x sorry.

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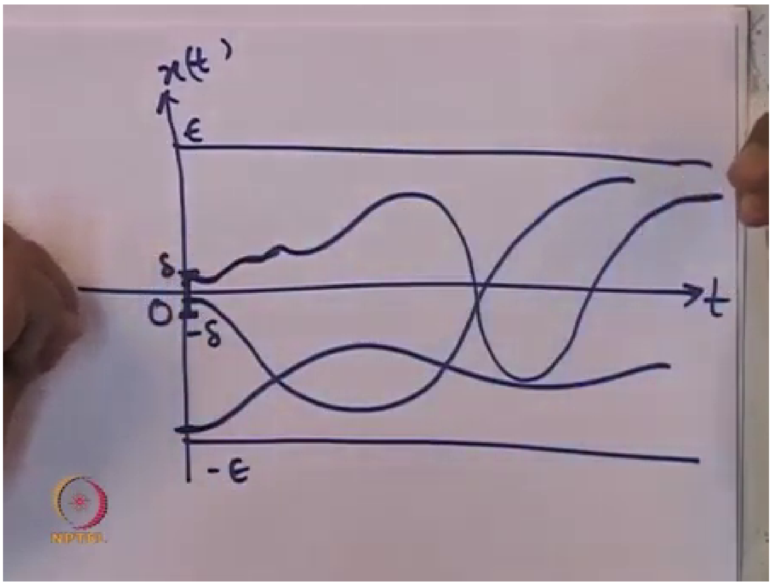

So, this epsilon which is being shown large could also be very small. This is what the challenger decides, how small epsilon should be. This is a 0. So, once this epsilon is given, we might we might know that the trajectory is might have to start within this very small interval. The interval within which it should start, so that it is guaranteed to remain inside this bigger ball yeah, this is the ball. This is what I am showing is the diameter, the radius is this, this is a radius and this is a diameter. For it to remain inside this bigger ball, it is possible that we should ensure that the initial condition lies inside this smaller ball yeah.

As long as it begins from here, from the solutions to the differential equation we know that it remains inside this epsilon ball. Another solution here also remains inside this epsilon ball. Of course, this solution might also remain inside this epsilon ball, but it is possible that every solution inside with this much distance, does not remain inside this epsilon ball. To ensure that it remains inside this epsilon ball we are forced to make this delta very small may be.

So, but the fact that this delta is greater than 0 is what defines this particular equilibrium point as stable that no matter what epsilon somebody gives us we are able to do some calculation and propose this delta. So, that if the initial condition starts inside this delta ball, the whole trajectory remains inside this epsilon ball. So, one is this once this epsilon is specified, is this delta unique yeah? If suppose after lot of calculation we have found this delta, and somebody else does a similar calculation, but obtains a different delta.

Can we say that one of the two deltas is wrong because the delta should be unique, or is it possible that for the same epsilon there are many deltas yeah? So, this is the question that we can answer without too much effort.

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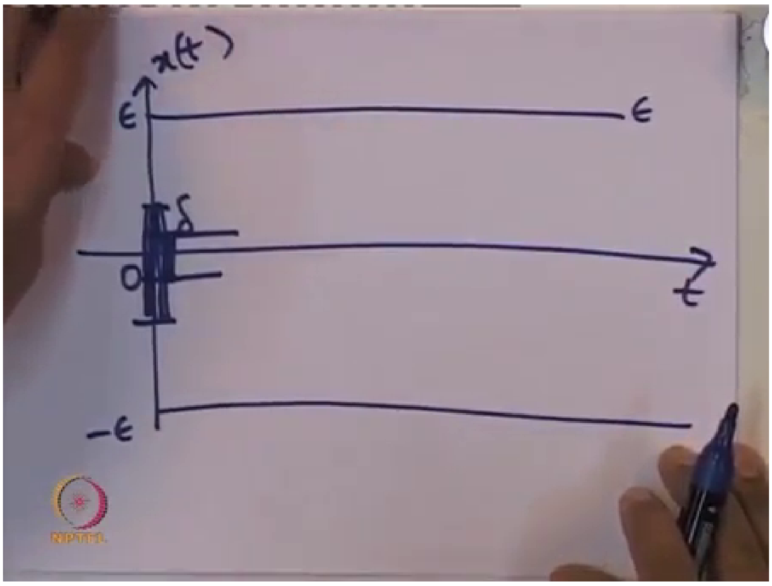

So, this is only a guarantee. Suppose this is our epsilon ball, and after lot of calculation one person finds this delta ball yeah. If we start inside this interval, then the trajectory supposed to remain will is guaranteed to remain inside this epsilon ball, but this same guarantee will automatically be satisfied for this smaller band also. If the initial condition starts inside this smaller ball then also it is guaranteed to remain inside this epsilon ball.

Why? Because once we are sure that starting anywhere inside this initial condition band, assures that the solution lies inside this epsilon ball then we know that inside a smaller band also if we have begun, we are guaranteed to remain inside this epsilon ball.

Of course, this smaller one, the smaller initial condition ball is a more conservative one but this only tells that the delta is not unique. Once we have found the delta is greater than 0, we can take another delta that is strictly smaller and positive and still that delta which comes in the definition of stability that delta that condition is satisfied for this smaller and positive delta also.

There is only to note that delta is not unique. One could ask the question can we make this delta larger and larger and for each epsilon there might be a unique largest possible delta. In that sense it might be unique ok. So, it is also clear that when epsilon is made smaller then we might have to make the delta smaller. One could ask the question in general is delta what is the relation between delta and epsilon.

Our figure appears to show that, the delta is smaller than epsilon. But in general, should such a relation be satisfied, the delta is smaller than epsilon or delta greater than epsilon or should such a relation need not does such a relation need not exist. So, this, please note that this is called the definition of stability in the sense of Lyapunov. Yeah this is just a definition. This is not Lyapunov's theorem on stability.

So, the reason that we have emphasized we have spent lot of discussion on stability on the definition of stability is, because it is a difficult concept and to understand the definition properly is very important to understand the theorems on stability. So, before we proceed to the Lyapunov's theorem on stability we note that what we have seen so far is the definition of stability, in the sense of Lyapunov. After having seen stability what do what do we mean by asymptotic stability?

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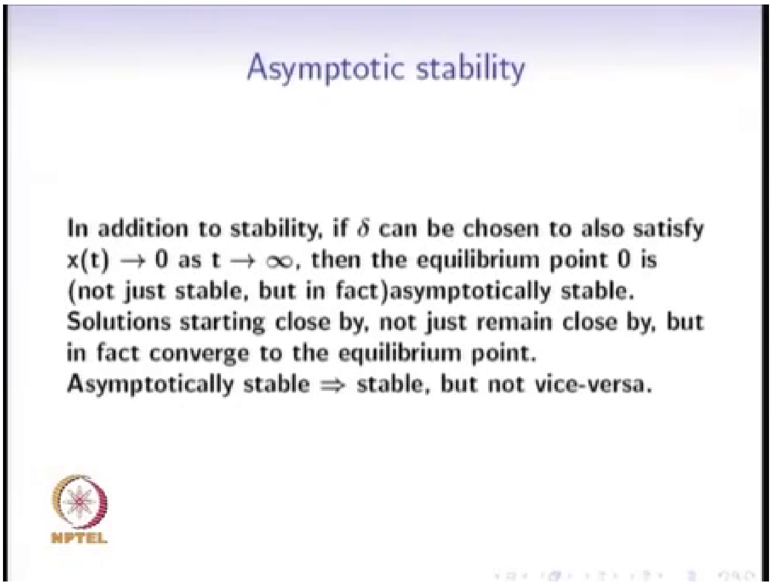

So, in the definition of stability once you are given an epsilon, you were required to find a delta and that meets a certain condition. In addition to that condition required in the definition of stability, if delta can also be chosen to satisfy this additional condition, that x of t goes to 0 as t goes to infinity. Yeah, so what was 0? The equilibrium point; x of t converges to the equilibrium point as t goes to infinity, then the equilibrium point 0 is said to be not just stable but in fact, asymptotically stable.

So, we will call the equilibrium point asymptotically stable if it is stable for it to be stable we already know that for every epsilon. We have to be able to find a delta such that all initial condition starting inside the delta ball are guaranteed to have the entire solution inside the epsilon ball. This delta which was chosen to satisfy this condition, in addition to that if it can also be chosen to satisfy this additional condition that the solution converges to 0, the equilibrium point as t tends to infinity then that equilibrium point is not just stable, but also asymptotically stable.

It was already stable because delta satisfied the condition that the definition of stability required. In addition to that condition it has delta satisfies this additional condition and hence that equilibrium point is asymptotically stable.

For every initial condition inside the delta ball, we also have x of t goes to 0 as t tends to infinity. Solution starting close by not just remain close by remain close by is what x of t, is contained inside the epsilon ball meant. So, they not just remain close by, but in fact, converge to the equilibrium point. We had assumed that 0 is the equilibrium point. So, solution should also converge to the equilibrium point, for every initial condition starting inside the delta ball.

If this is satisfied then we will say that the equilibrium point is asymptotically stable. Asymptotically stable naturally means that the equilibrium point is also stable but not vice versa. For just stability we do not require that the solutions converge to 0, we only require the solutions to remain inside an epsilon neighbourhood.

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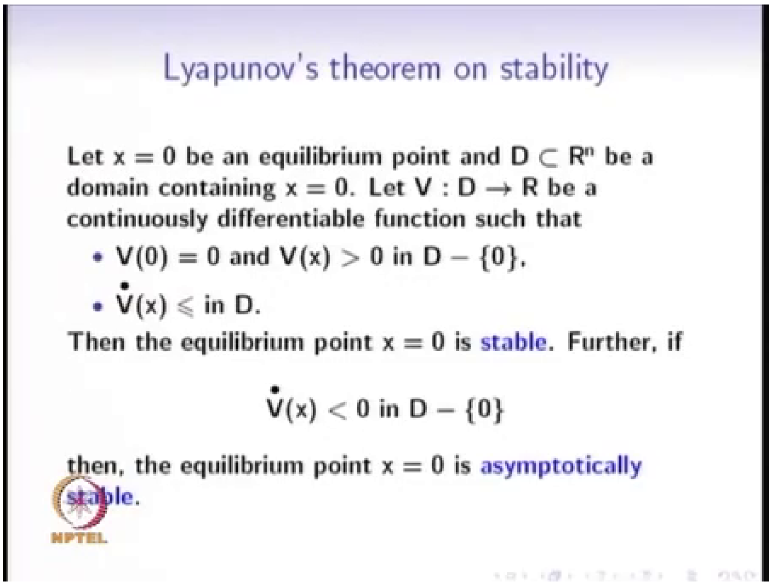

So, we now come to Lyapunov's theorem on stability. After having see the definitions of stability and asymptotic stability in the sense of Lyapunov we are now going to see Lyapunov's theorem on stability. Let x of 0 be an equilibrium point and let D be a domain that contains this equilibrium point. Let V be a function from D to R yeah. So, the domain D is a subset of R n, V takes its values from this D, and is scalar valued yeah. V does not take vectors as its values, it takes only a scalar. Hence R at any point x V of x has only one component.

Let V be a continuously differentiable function. So, it is V itself is continuous and its derivative is also continuous. That is a meaning of continuously differentiable function. Such that V satisfy some conditions, V of 0, the 0 the equilibrium point at 0 V is equal to 0 and, inside that domain at every other point V is positive, V is allowed to be 0 only at the equilibrium point at other points it is positive.

Secondly, V dot is less than or equal to 0. So, we are missed a 0 here. V dot is less than or equal to 0 in the domain D. So, V was a function of x, but this dot here means it is a derivative with respect to time, this I will clarify very soon. So, the rate of change of V with respect to time at every point is less than or equal to 0. If we are able to find such a V which is continuously differentiable which is positive everywhere except at 0 where it is allowed to be equal to 0, and V dot is non positive.

It is less than or equal to 0 in D, if V if there is some V that satisfies these three conditions, then the equilibrium point 0 is stable is a stable equilibrium point. So, what is important to clarify is, this V was not a function of time it was a function of x and x took its values in R n; but how do we go ahead and differentiate V with respect to time, this is one important point that requires the clarification.

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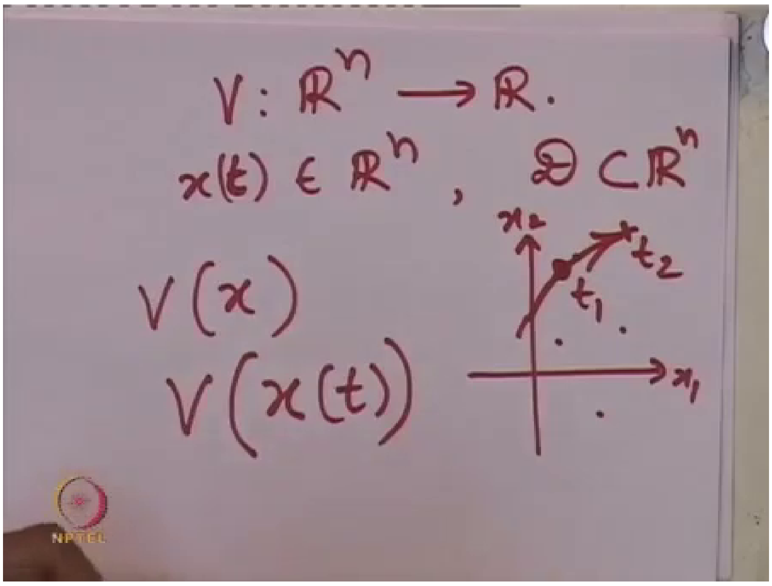

So, V was a map from R n to R yeah. Why because x at any time instant was in R n. Our domain D was a subset of R n, for the time being we assume that D is equal to R n. Hence our function V was a map from R n to R. If a function V was not a function of time how do we go ahead and differentiate V with respect to time? This is something we will quickly see.

So, this V we are going to evaluate at different points x, but through each point we have a trajectory that evolves with respect to time. So, V actually depends on x which itself depends on time yeah. So, because x is changing with respect to time, as x moves to another point value of V will also change. In this sense V depends on time also.

Suppose this is our phase space and this is x 1 this is x 2 and this is some point and this is a trajectory that is evolving with respect to time and at this particular time, at some time t 1 it was here, at another time t 2 it has moved to this point. Because x itself is changing. Along this trajectory we can see how the function V is changing. V has some value at this point, some value at this point, some value at this point. Similarly V has some value at this point immediately after this, immediately further along this trajectory V has a different value.

Similarly, as the x evolves along this trajectory, the value of V or the function V is also changing. In that sense, V is a function of time also because we are evaluating V along the trajectory x. So, we will see what it means to differentiate V with respect to time now.

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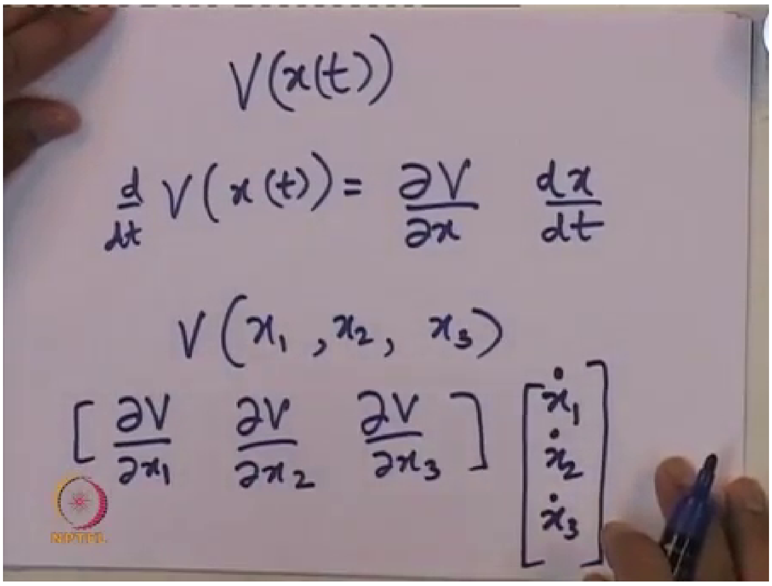

So, V of x and x itself dependent on time t. So, d by d t this is what we call a composite function, V depends on x while x itself is a function of time. So, d by d t of V of x of t is equal to partial derivative of V with respect to x. Partial derivative of V with respect to x because, V itself depends on many variables x 1 up to x n and hence this derivative is not an ordinary derivative, but a partial derivative.

And then x d x by d t, x itself depends on only one variable time one independent variable time and hence this is d x by d t. So, if V depends on x 1 x 2 x 3 then this is nothing but del V by del x 1, del V by del x 2 del V by del x 3 times x 1 dot x 2 dot x 3 dot. So, throughout this course the dot we will reserve for derivative with respect to time. If it is derivative with respect to x or some other variable, then we will just write d by d x del by del x of V.

So, V dot which we saw in our slide is supposed to be is to be understood like this. V was a function of x, V in fact depend on x 1 x 2 and x 3; x 1 itself was the function of time, x 2 was a function of time and x 3 was a function of time. Hence differentiating V with respect to time is nothing but del V by del x 1 times x 1 dot plus del V by del x 2 time x 2 dot plus del V by del 3 del V by del x 3 time x 3 dot. This is the meaning V dot.

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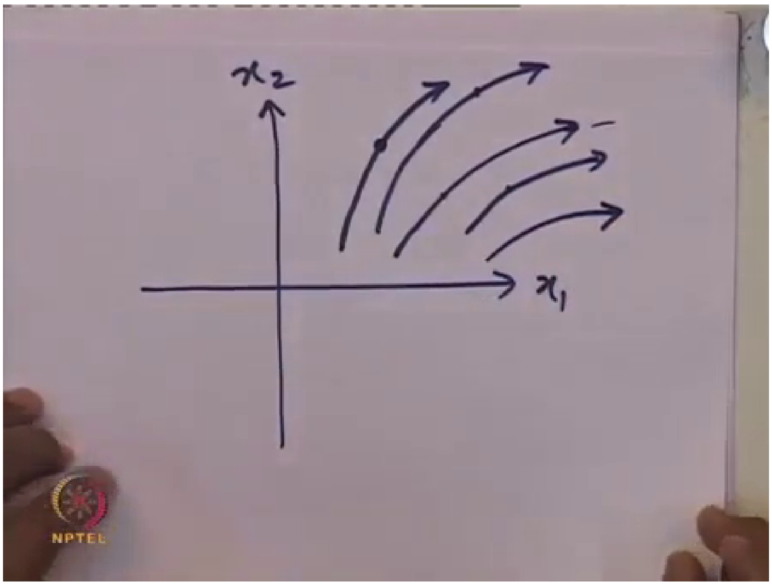

Thus, at each point in the phase space for this figure we are, our x has only two components. At each point V has some value. This is like V is a it is a scalar function, at each point x, x is a vector but the value of V at each point is a scalar. And this is like temperature of a room at each point, the temperature itself has only one component and as a trajectory moves through each point there is some trajectory, this is the direction in which the x trajectory moves at, given that these trajectories are all well-defined at each point.

We can associate the rate of change of V at each point yeah. At each point there is also not just V defined, but also rate of change of V with respect to time defined for each point that rate of change is it is possible to define that because we have all these trajectories, defined at each point and the trajectories themselves have some rate of change defined for them. So, this for those who are more interested in this topic this brings in a use of Lie derivative techniques into dynamical systems.

It is not required for this course as well as we are concerned we want to understand the meaning of V dot, even though V was a function of x and not time. So, here at each point x not just V but V dot is also defined because, we have a differential equation x dot is equal to f of x. So, that that one also remains to be written in this slide, x is equal to 0 is an equilibrium point of the dynamical system x dot is equal to f of x. So, with respect to that dynamical system V dot x is defined. What is V dot of x?

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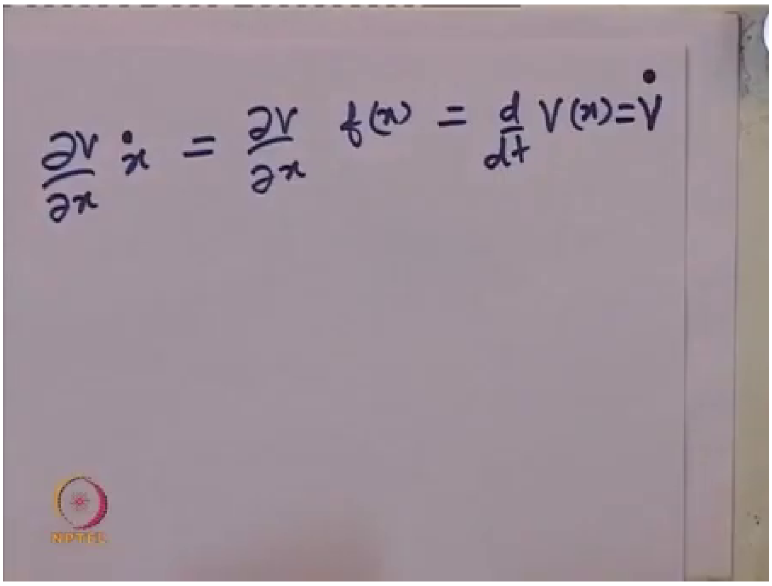

In general, it is del V by del x times f of x. Why because this is nothing but del V by del x times x dot. And this is what? d by d t of V of x is which we have denoted by V dot. This dot here you will reserve only for rate of change with respect of time.

So, at every point x there is a V dot defined and if that is less than or equal to 0, the 0 is also missing here; if that is less than or equal to 0, then the equilibrium point 0 is stable. Further yeah in addition, in addition to less than or equal to 0 if this V dot if this V function V is such that, it is continuously differentiable, it is 0 it is equal to 0 only at the equilibrium point 0 and it is positive at every other point.

And if it is strictly less than 0 over the domain D except 0 in the domain, at all the points except the point 0, if it is strictly negative then the equilibrium point is not just stable, but it is the asymptotically stable. So, this is Lyapunov’s theorem on stability. So, please note that this is only a sufficient condition for stability. When do we call the equilibrium point 0 of the dynamical system x dot is equal to f of x stable? We have a definition of stability. One of the ways to prove that it is stable is if you can find some function V that is continuously differentiable, whose rate of change is less than or equal to 0 and which is equal to 0 at the point and it is positive at every other point.

If it satisfies these three conditions then that V we will call a Lyapunov function and that Lyapunov function helps us to prove that this equilibrium point is stable. If such a function V, we pick if we pick a function V and it does not satisfy these three conditions then we are not able to conclude that the equilibrium point is not stable. It just means that perhaps this function V should have been chosen more properly more judiciously there might exist another function V that satisfies these three conditions and helps prove and helps to prove that the equilibrium point is stable.

This is only one sufficient condition to prove that the equilibrium point is stable. We will see these things in more detail in the following lectures; we also saw a sufficient condition for proving asymptotic stability, this particular function V is called the Lyapunov function. And we will see these functions in more detail and some examples in the following lecture.

Thank you.