Nonlinear System Analysis

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Lecture – 25
Stability Notions: Lyapunov and LaSalle's theorem
Part 02

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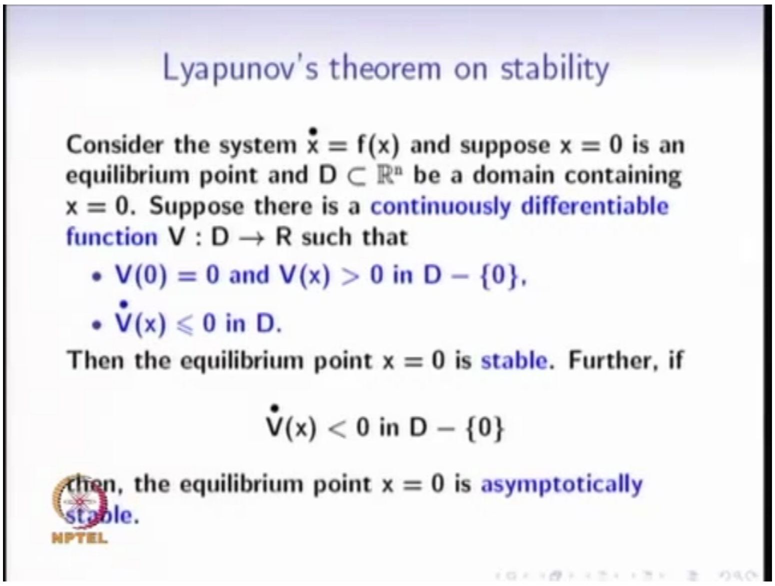

So, in the last time we just saw the definition, the statement of the Lyapunov's theorem on stability and asymptotic stability. We also saw an outline on the proof. Today we will do a very quick review of the theorem statement and we will proceed with the proof for both stability and asymptotic stability.

So, consider the dynamical system x dot is equal to f of x and, suppose x is equals 0 is an equilibrium point. And this is this equilibrium point is in a domain D, a subset of R n. Suppose there is a continuously differentiable function V, from this domain to R such that V is equal to 0 at the equilibrium point 0. And it is positive for all other points and its rate of change with respect to time is less than or equal to 0; it is non positive in this domain.

If such a V exists which satisfies these properties, then the equilibrium point is stable. Further, if the rate of change of V is negative for all points in the domain except the origin. Of course, in that case this equilibrium point is not just stable, but in fact, asymptotically stable.

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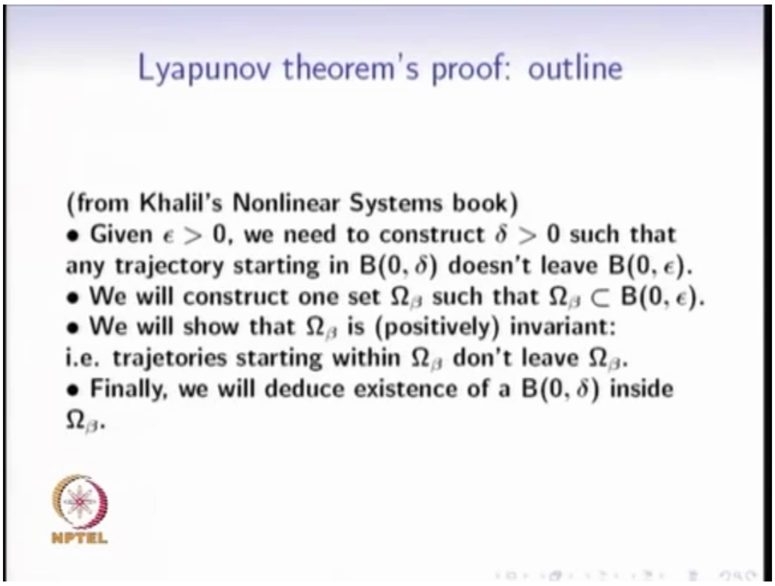

So, the proof that we follow is from the book by Hassan Khalil's on Non-Linear Systems. So, what is required to be done in the proof, what is this to prove stability? Whenever somebody gives us an epsilon greater than 0 we need to construct this delta greater than 0 such that any trajectory starting inside this ball B comma delta does not leave this other ball B 0 comma epsilon.

So, in this notation is that the first 0 is the center of the ball and delta is a radius, similarly here. So, in order to construct this delta, we will construct one set omega beta such that this omega beta is contain inside this ball B epsilon. And we will show that this omega beta set is positively invariant; that is trajectory starting inside the set do not leave the set.

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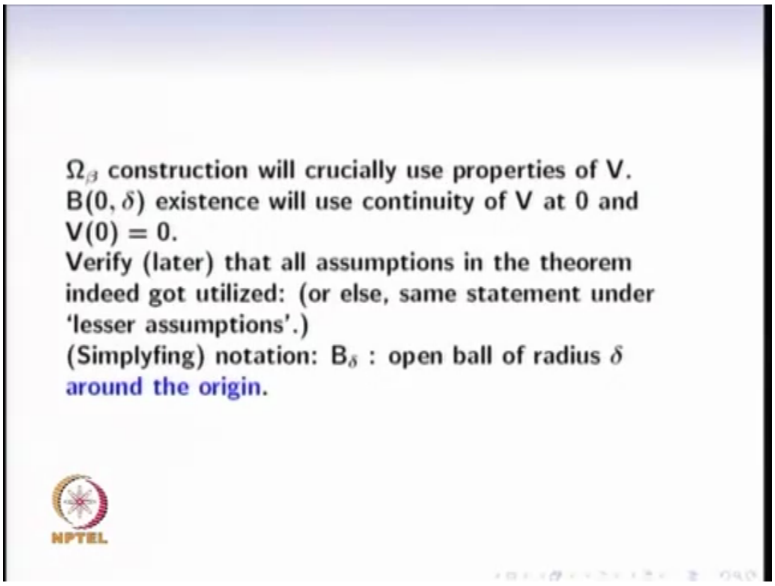

Finally we will also reduce an existence of a ball B delta inside this omega beta. To do all this, we will crucially use the properties of that Lyapunov function V. The existence of this ball B 0 comma delta we will use the continuity of the Lyapunov function at 0 and the fact that V at 0 is equal to 0.

Of course, whenever we prove something at the end of the proof, it is important to verify that all the assumptions in the theory statement indeed got utilized in the proof. Or else one could in principle prove a similar statement the same statement under lesser assumptions. Since we did not utilize some of the assumptions, we could consider relaxing those assumptions and having the same theorem statement.

So, for the rest of this proof, we will since all the open balls we are considering are centered at the origin. We are temporarily we have decided that the origin is equilibrium point, whichever point is equilibrium point we can always shift the coordinate such that that point is the origin. And since 0 is the center we will only denote the radius of the ball hence B delta is an open ball of radius delta. The around the origin part we do not require to mention again, again.

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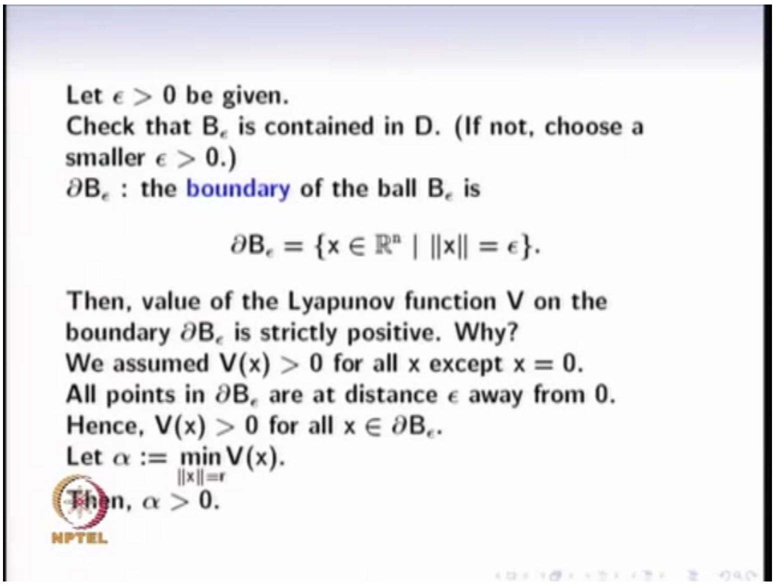

So, let epsilon B given let epsilon greater than 0 be given. So, first check that B epsilon is indeed contained in the domain D; if not one can choose epsilon that is slightly smaller. So, for this ball B epsilon we construct the boundary of the ball. So, what is the boundary of the ball? B epsilon contains all the points which have radius strictly less than epsilon, from the which have distance strictly less than epsilon from the origin; the its boundary is the set of all points whose distance from the origin is equal to epsilon.

This is the boundary of the ball this is the surface of that sphere. On this particular boundary on this ball, we will now look at the Lyapunov function how that behaves. So, notice that this Lyapunov function, the value of the Lyapunov function on all the points on this boundary is strictly positive.

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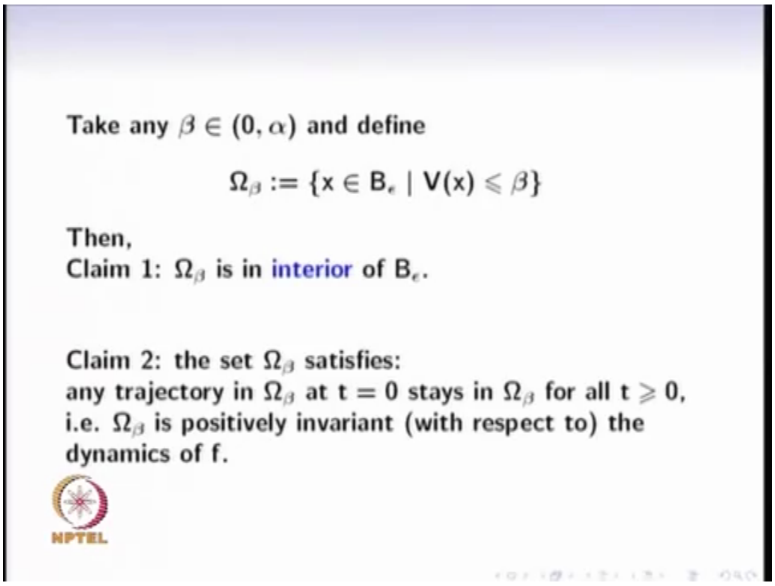

Why? Because it is equal to 0 only at 0, at any other point is positive and all the points on this ball are epsilon away from the origin. And hence the Lyapunov function is strictly positive.

So, this is a reason. So, if it is positive on all the points on this boundary on this del B epsilon; then we will look for the minimum value that V takes on del B epsilon. On the minimum over all x such that mod x the distance of x is equal to R. The norm of x is equal to R norm of x is nothing but the distance of x from the origin. For all such points we will look at the minimum value that V takes and let alpha B that minimum value.

So, we already know that alpha is strictly positive. Then we take any beta, that is between 0 and alpha; in order to take this beta it is required that alpha is positive and alpha is the minimum that v takes on the boundary of B epsilon ball. So, once we have chosen some beta that is strictly less than alpha, we will define this set called omega beta which is a set of all the points in this ball B epsilon; such that V of x is at most e qual to beta.

So, inside this B epsilon ball, there are various points and at various points the Lyapunov function takes different values. We will pick all those points where the Lyapunov functions value is at most equal to beta. Once you have construct this omega beta set there are two properties important properties of the set that we need. One is that omega beta is in the interior of this ball B epsilon it does not come to the edge, second the set omega beta satisfies.

This important property that any trajectory inside this omega beta at t equal to 0; anything that starts inside omega beta stays inside this set for all t greater than or equal to 0. In other words we use the word that omega beta is positively invariant with respect to the dynamics of f. So, these two properties we will first show.

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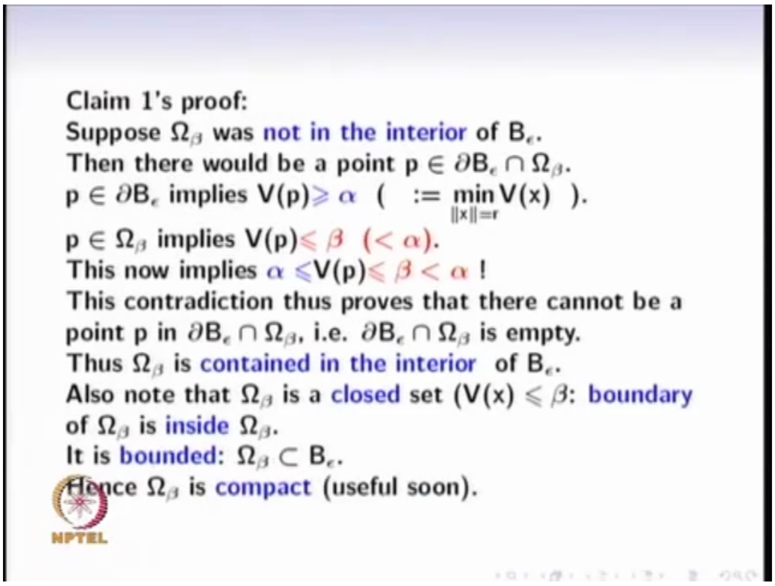

So, the first important property is that omega beta is in the interior of B epsilon; but suppose it was not in the interior then there would be a point that is on the boundary of B epsilon and also inside this omega beta set. So, this part of the proof we will replace and hence i am going fast. So, now, we come to the second part of the proof.

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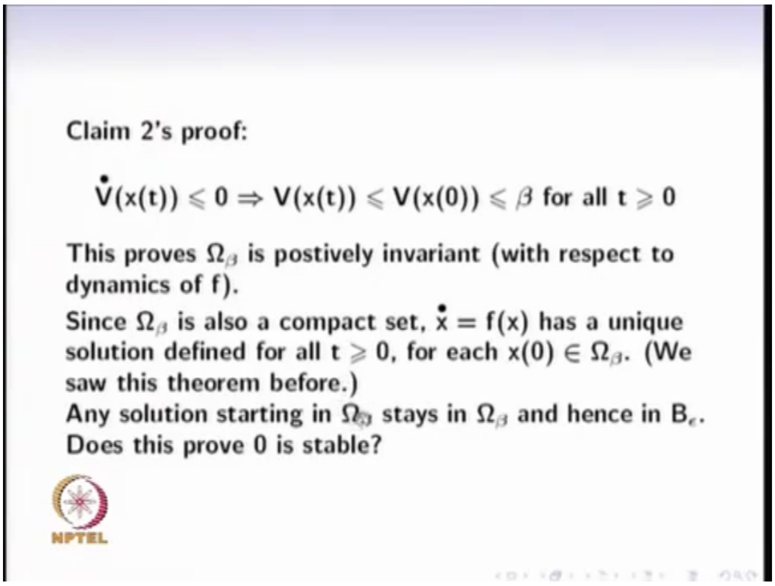

What is the other claim, what is other properties that omega beta set has? We claim that it is also positively invariant, how do we prove this? So, we know that V dot x is less than or equal to 0.

So, what does that mean when we integrate this with respect to time we see that V at x at any time is less than or equal to V of x at 0; which was equal to which was at most equal to beta. So, this particular inequality is satisfied for all t greater than or equal to 0; this inequality can be obtained simply by integrating this quantity. And using the fact that this quantity is less than or equal to 0; yeah we can do it a little more slowly on this piece of paper.

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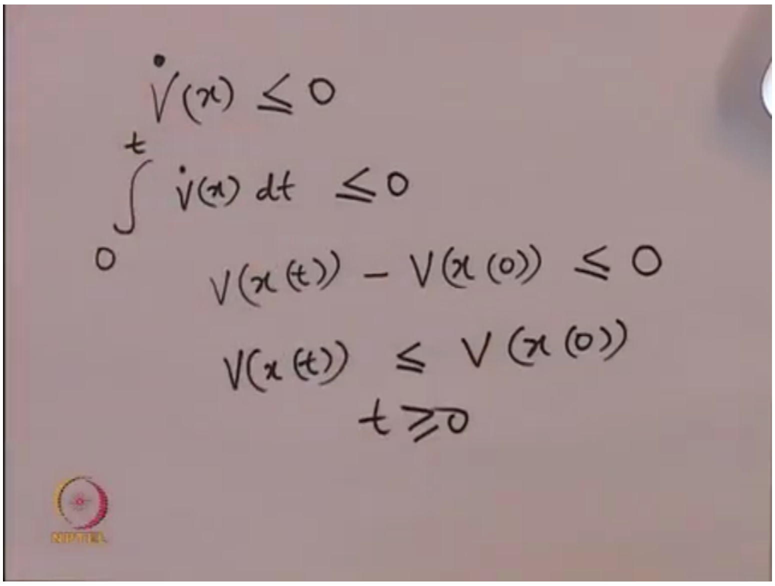

We already have assumed this particular property of the Lyapunov function. When we integrate this from 0 to t of V dot of x this integral is also less than or equal to 0; because the integrating some quantity that is negative that is non positive. So, this integral of the derivative of a function is nothing but V of x of t minus V of x at 0 and the final value minus the initial value. This itself is less than or equal to 0 which says that is less than or equal to V.

Of course this is expected V is a function that is decreasing with respect to time. Hence at any time t that is greater than or equal to 0, this value will be less than or equal to this value. So, that is all that is said in the inside this inequality. And this value itself this value itself was less than or equal to beta, that was the method of construction of the omega beta set; since we are started inside this omega beta set this value is at most beta.

So, this proves that omega beta set is positively invariant, if the value is less than or equal to beta t equal to 0, then for all future time it can only decrease. And hence that trajectory remains inside the set omega beta. Further this omega beta set we also saw just now is a compact set, which means it is a closed and bounded set. Hence x dot is equal to f of x has a unique solution defined for all t greater than or equal to 0. We saw that if you have a compact set which is positively invariant.

And the function is just locally Lipchitz then we are able to. In fact, assure global existence and uniqueness of solution for this differential equation. So, any solution starting inside omega beta stays inside the set and omega beta set inside itself was contained inside the ball B epsilon. Hence this omega beta is also hence the solution also remains inside d epsilon ball.

So, does this prove that the point 0 is stable? No, not yet; we want that the result B delta ball such that solution starting within it remains inside the ball B epsilon. We have only been able to show that any solution starting inside some set omega beta stays inside this omega beta and hence inside the ball B epsilon.

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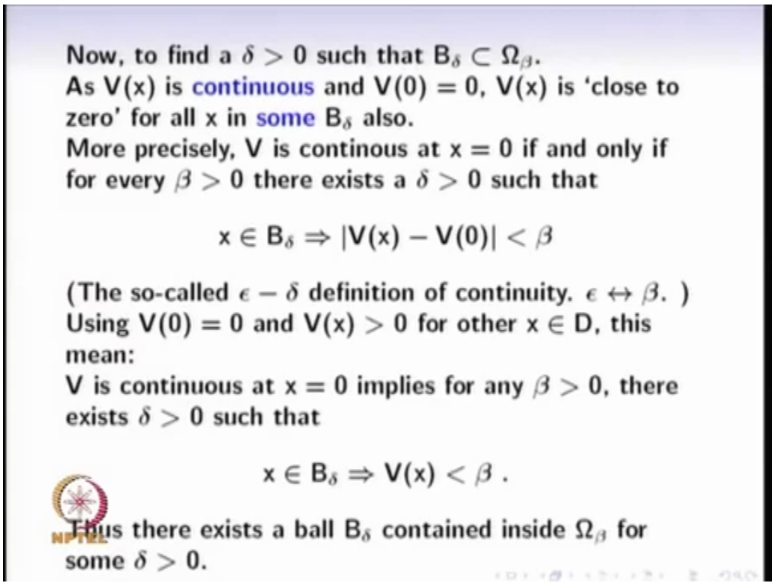

So, how do we find this delta greater than 0, which is contained inside omega beta set? Since V is continuous and V of 0 is equal to 0, V x is close to 0 intuitively speaking since V is continuous at 0 and its value at x equals 0 is equal to 0.

It cannot be very large for points that are close to the origin. So, this particular property is what continuity says more precisely V is continuous at x equals 0. If and only if for every beta greater than 0 there exists a delta greater than 0; such that for all points inside this be delta ball V of x can differ from V of 0 by at most amount beta and this is a definition of continuity.

It is a so, called epsilon delta definition of continuity, but since we are requiring epsilon in a different context; we have just replaced epsilon by beta here. Since V of 0 is equal to 0 and V of x itself positive, we can get rid of this modular sign here and V of 0 was anyway equal to 0. So, this particular definition of continuity becomes V is continuous at x equals 0 implies that for any beta greater than 0; there exists a delta. Such that, a for all points x in the ball B delta V of x is strictly less than beta.

This is nothing but to say there exists a ball B delta that is contained inside the set omega beta. For some delta greater than 0, we are able to find some positive delta such that, the ball B delta is contained inside the set omega beta.

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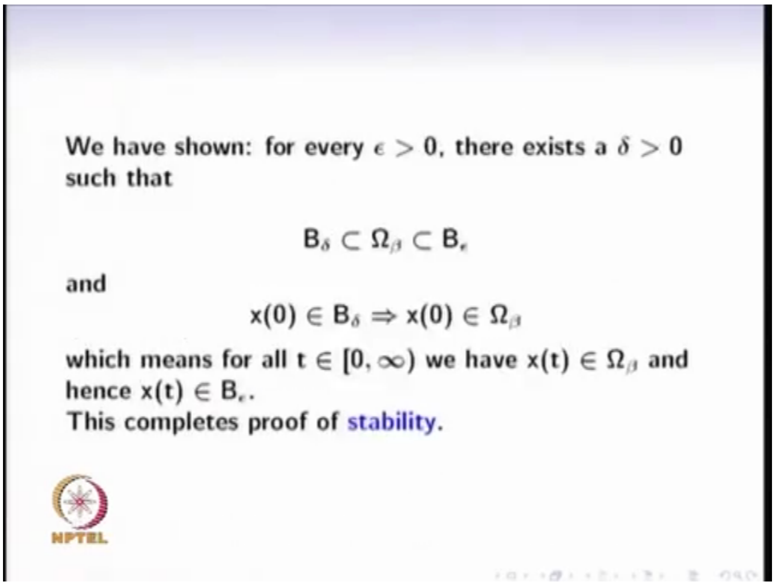

So, what we have shown is no matter what epsilon greater than 0, we start with there exists a delta greater than 0; such that these two inclusions hold B delta is contained inside this omega beta set.

Omega beta is not a ball it is a set only means omega beta set itself is contained inside this larger ball B epsilon. Further, if the initial condition is inside B delta; it means that the initial condition x of 0 is inside the set omega beta also. And hence for all future time x of t is inside omega beta and hence x of t is contained inside the ball B epsilon also; this was precisely what was to be shown to prove that x is equal to 0 is an equilibrium point.

So, this completes the proof of stability for the Lyapunov theorem.

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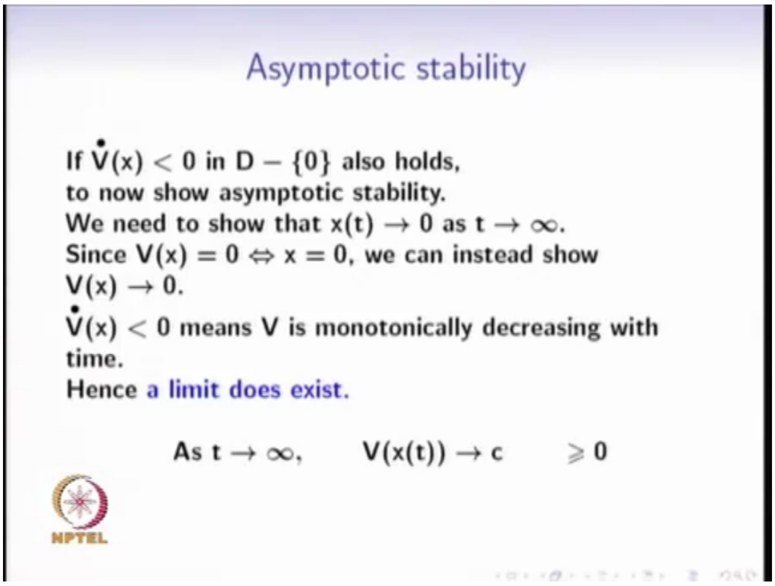

What is to be proved for asymptotic stability? If V dot of x is in addition assumed to be negative for all points except the origin for all points except the equilibrium point, then we are to yet show that the equilibrium point is in fact, asymptotically stable.

So, to show that it is asymptotically stable what is to be shown? We need to show that x of t tends to 0, as t tends to infinity. But in since V of x is equal to 0 only at the origin. We can instead show that V of x tends to 0, why? Because, if V of x tends to 0, that can happen only when x tends to 0 as t tends to infinity.

So, what does this property that V dot of x less than 0 mean? It means that V is monotonically decreasing with time, as a function of time V is only decreasing. And further it is also bounded from below yeah, V cannot arbitrarily decrease for this we need to see a small figure what a monotonically decreasing function can look like. This is V which is a function of x.

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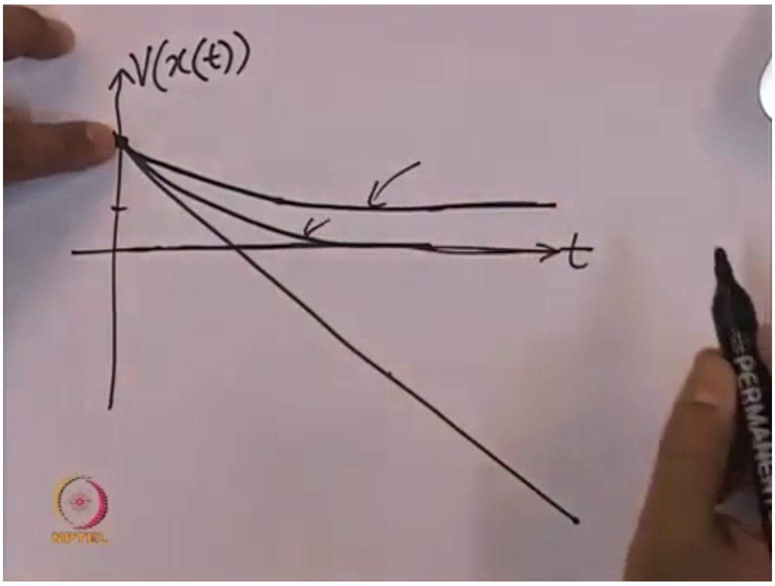

But x itself is a function of time and hence we can draw this. So, here is what value of V at x equal to? Value of V at x 0 is equal to and the function is only decreasing with time. So, there are three possibilities; one it could converge to some nonzero positive value other it could converge to 0 or it could converge and it could it need not converge, it could go on decreasing.

So, at least we know that V is bounded from below hence this is not possible it is bounded from below by 0. The minimum value that x V itself is a positive function it is equal to 0 only at x equals 0 at all other points it is positive. So, it is not possible that at any time instant the trajectory goes, the function V as a function of time cannot become negative; hence it can be only one of these two.

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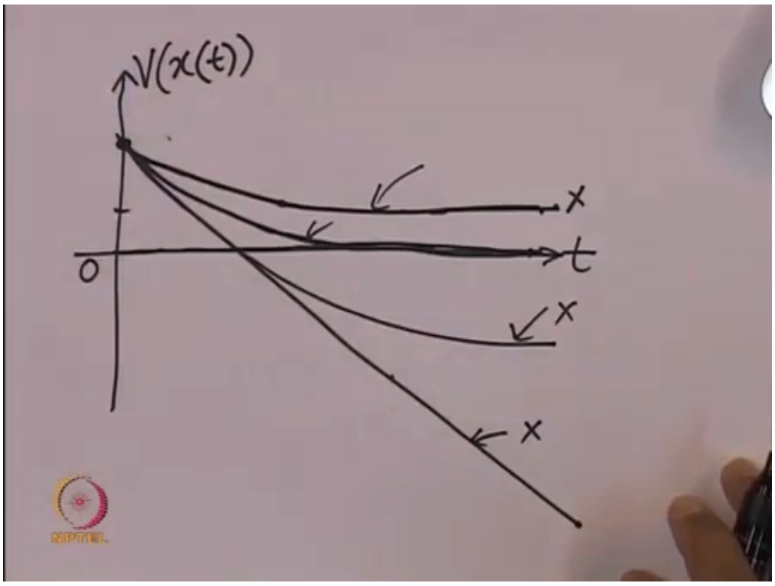

Yeah, also of course, it is not possible that V of x oscillates; it is not possible that why is this not possible? Because here is a region where V dot is positive yeah over this region V dot is positive, but we know that V dot is always less than 0. It is monotonically decreasing, if it is monotonically decreasing such oscillations are already ruled out. Hence we are dealing with one of these three cases is going on decreasing arbitrarily too small to small values or it comes in converges to the 0. Or it converges to some other value or of course, it can converge to some negative value.

This is ruled out this is ruled out both because we know that V is bounded from below by 0 by the 0 function. Hence it can be only one of these and these cases yeah. So, we are going to try and use the various properties of V to rule out this also. So, that we say that the only way we can behave is that it for each initial condition it comes in converges to 0. For V of x tending to 0 as t tends to infinity and this can happen only when x tends to 0.

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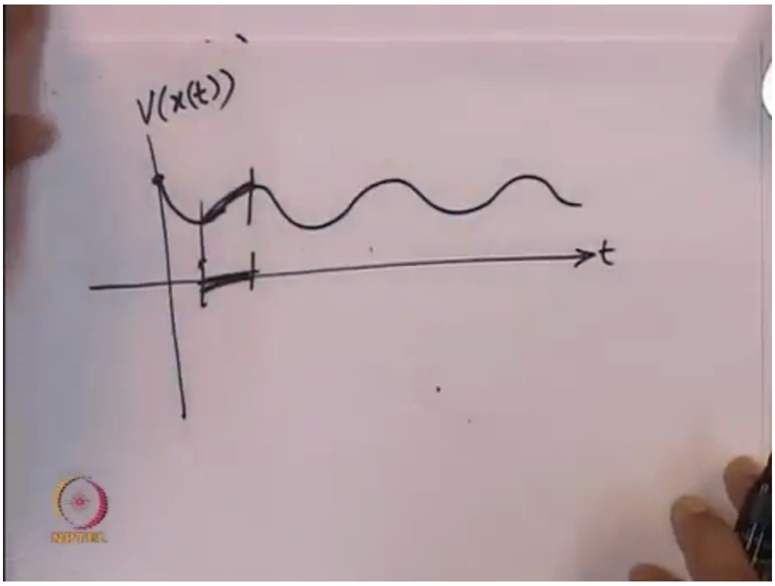

So, this is what is to be shown to prove that it is asymptotically stable. So, the first thing is because V is bounded from below and since V is monotonically decreasing a limit does exist. Suppose this limit is c, as t tends to infinity V of x of t converges to c and we also know that the c can be non negative only it cannot be negative.

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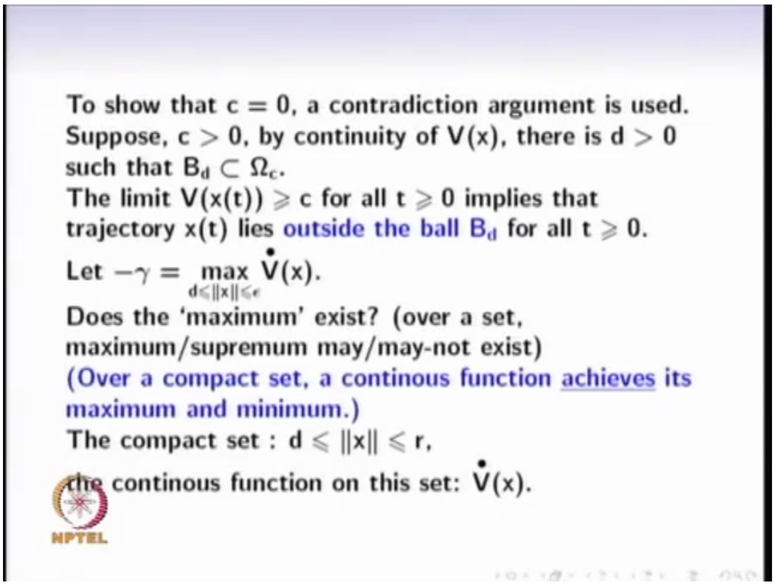

So, to show that c is indeed equal to 0; we will use a contradiction argument what is this contradiction argument? We will assume the con to the contrary, we will assume that c is positive c is greater than 0 and then we will use continuity of V to prove that this cannot happen.

So, suppose c is greater than 0, then by continuity of the function V of x like we had proved in the stability case. Here also if c is greater than 0 then by continuity of the function V there is some d greater than 0; such that the ball B of d is contained inside the set omega c yeah. Instead of beta greater than 0 we have a c greater than 0 hence we construct this omega c set, omega c set is a set of all points where the value of V of x is at most equal to c. And this B d is some ball that is contained inside omega c; we are able to constrain this B d ball again by using the continuity of the function V at x equals 0.

Now the limit, what does it mean that as t tends to infinity V of x of t is greater than or equal to c and the limit is equal to c. It means that, for all t greater than or equal to 0 this V of x of t is greater than or equal to c. And this just means that the trajectory lies outside the ball B d for all t greater than or equal to 0. So, this is a very important argument.

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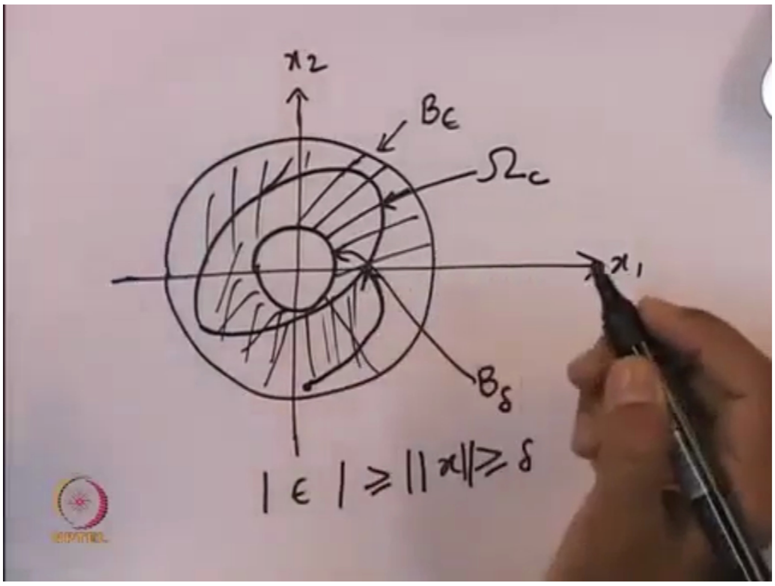

So, we have to see a figure, this was our original ball B epsilon this is our state space x 1 x 2. So, we are assuming for the time being that there are only two state components.

But more generally, this is a ball in R n inside this d epsilon ball we have constructed this omega beta set. Right now, this omega beta we have going to denote as omega c and inside. So, the omega c set as i said need not be a ball it can it might be more general set, but this omega c set does not come too close to the origin. In other words, there is a open ball B delta that is contained inside this omega c set. So, this is this ball which we have called B delta.

So, now look at this region that is outside yeah. So, x greater than or equal to the norm of x greater than or equal to delta, but the norm of x is less than or equal to epsilon. So, sorry say epsilon is epsilon is positive. So, this absolute value sign is not required. So, what does this region signify? It signifies the region outside this ball B delta, but inside this ball B epsilon.

So, B epsilon is open except for the fact it is inside this ball B epsilon, but outside of this ball B delta. So, the fact that V of x is always greater than or equal to c means that the trajectory does not enter this set omega c. If it does not enter inside this omega c it can of course, reach the boundary because the boundaries is where x of t is equal to c; on this boundary it could come, but it does not go inside this set omega c. And this d delta ball is contained strictly inside this omega c set.

And hence we have now concluded that if it starts somewhere here, it can at most come up to this omega c.

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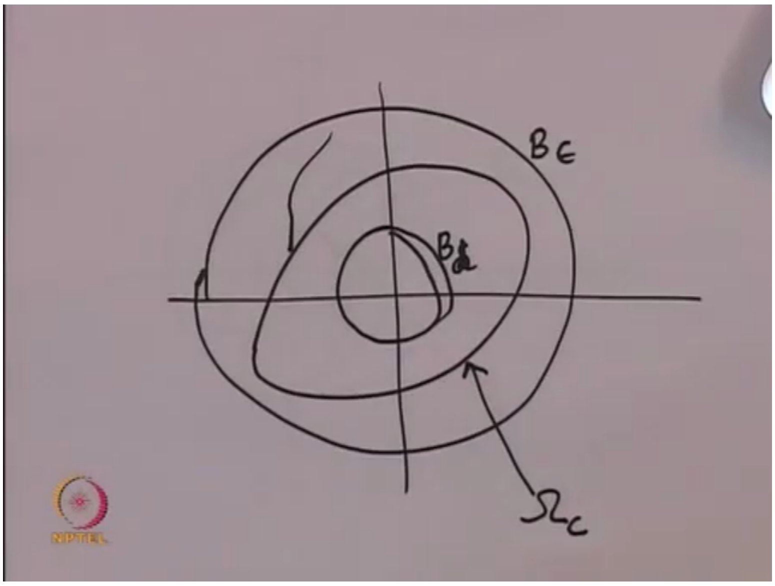

This is our ball B epsilon there is a set which is not a circle its not a ball this is omega c and inside this there is a ball B delta B d sorry, we are now called it B d. Now we have already shown that if a trajectory starts somewhere here it can only come close to omega c it can come to the boundary, but cannot go inside. And hence in particular it cannot go in into this ball B d.

So, what that is precisely the statement that we are saying here that the limit of x of the limit V of x t is greater than or equal to c; it just means that the trajectory x of t lies outside the ball B d for all t greater than or equal to 0. And hence we will now look at optimizing this particular V dot function over the set, over the difference between what is outside B d and inside B epsilon ball.

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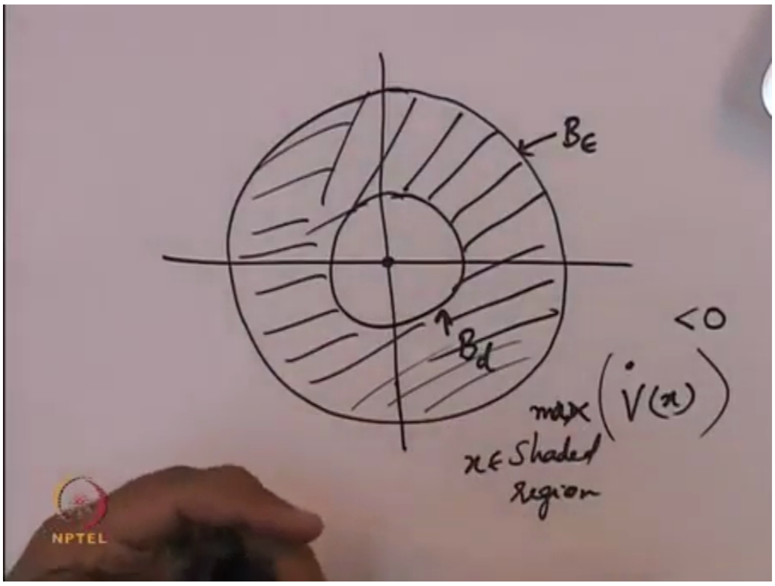

So, over the set this is the ball B d, this is the ball B epsilon the fact that omega c is something sitting in the middle. So, over this particular shaded region, we are going to now look for the minimum value of V dot yeah. So, minimum value of V dot of x at each point V dot is some function it is a rate of change of V; with respect to the trajectory at that particular point minimize this particular quantity over the shaded region. For all x inside the shaded region you want to just minimize this quantity.

So, first import property is sorry I am sorry for writing the minimum it is the maximum. We are going to maximize this particular quantity this particular quantity denotes the rate of change. So, its maximum value is the maximum increase, but then this V dot is decreasing as a function of time. So, this V dot at every at every x it is strictly less than 0 why because the point x equal to 0 is ruled out; V dot is equal to 0 only at the center at.

So, hence inside this region V dot is strictly negative and it is some function. If it is some function that is strictly negative we can look for the maximum value of V dot over the shaded region even the maximum value is negative why? Because it is strictly negative for all points here hence the maximum value is also negative.

So, let the maximum value be denoted by minus gamma. So, here is some quantity gamma that is positive why is that strictly positive? Because V dot is strictly negative inside this region and hence it is maximum value, if the maximum exists is also negative. So, the next question that arises is the does the maximum exist? Why because over a set in general the maximum or the supremum may or may not exist.

So, we will see some simple graphs where this can happen, but we will use a very important property that over a compact set a continuous function achieves both its maximum and minimum yeah. And this particular set that shaded region is a compact set ok. So, what is the problem that a function may or may not achieve its maximum and minimum values?

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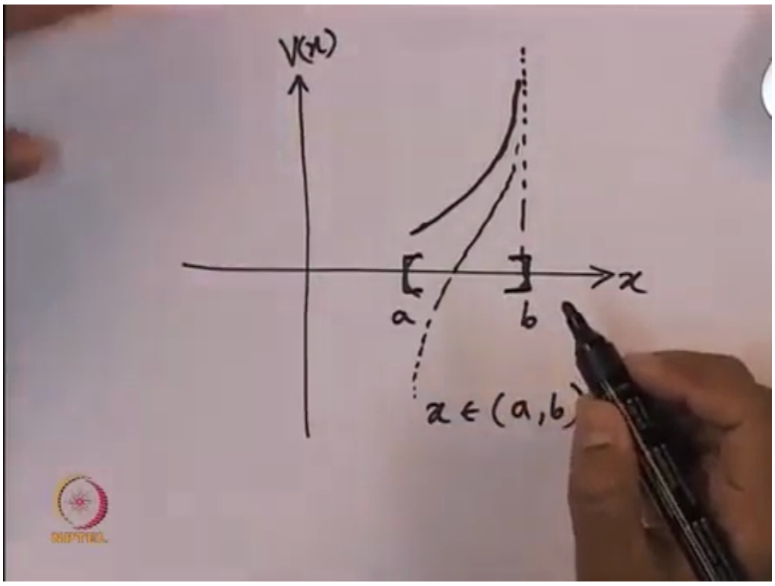

The first important property is we can have this particular set for the purpose of this we require to plot V of x versus x hence x has only one component.

So, what this particular set? It is possible that V of x is going is defined at every point in this open interval in the open interval a to b, for x in the interval a to b it is possible that x goes to infinity it becomes unbounded; as x tends to B it becomes infinity. And hence maximum of course, does not exist even the supremum yeah the value that it can become close to even that value does not exist it is unbounded right. Similarly the minimum value the infimum also may not exist simply because it is unbounded.

So, in our case this cannot happen; because first of all we are dealing with a closed interval um. And moreover this is a bounded interval and another situation where we can have a closed interval a closed set over which the function does not achieve its supremum is if the set is itself unbounded.

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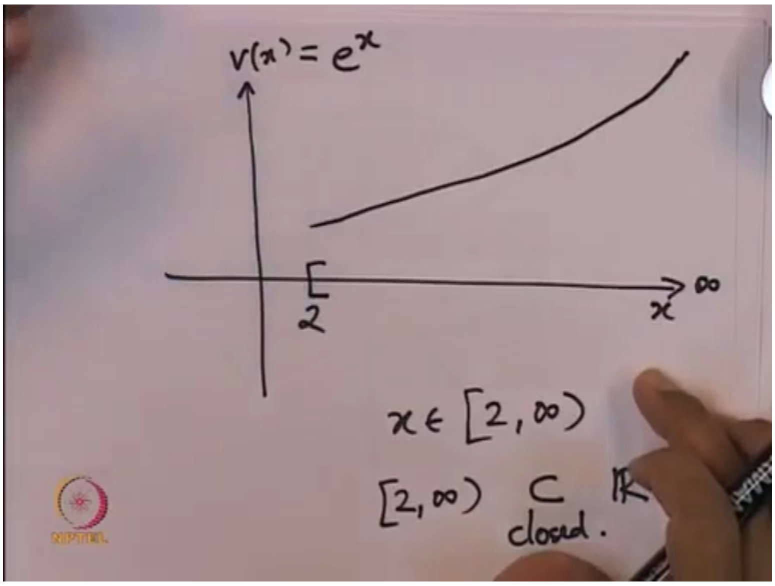

So, consider the interval 2 to infinity. Yeah this is x this is V of x. So, over x in the interval 2 to infinity, this is this set itself is a closed set it is a closed subset of R n of R 2 comma infinity is a closed set of R in.

On this close set just because a set is closed the supremum may not be achieved the maximum may not be achieved the set also requires to be bounded. So, suppose look at the function now, V of x equal to e to the power x. So, this particular function over this closed set does not achieve its maximum, it does not achieve its supremum also. And there is no number to which it becomes arbitrarily close to and is always below that number that number would be called the supremum, such a number does not exist for this function simply.

Because this set even though it is closed it is unbounded; however, in our case we are dealing with a closed set and also a bounded set.

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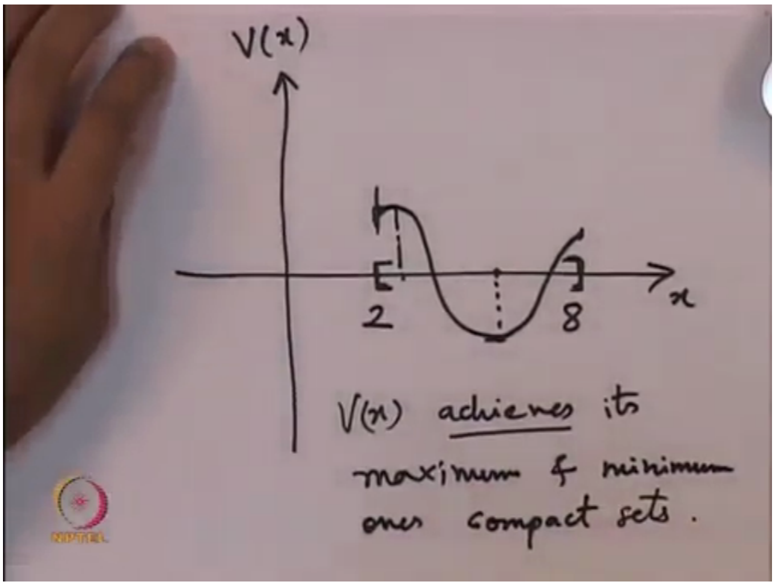

In other words it is a compact set over compact set the function. In fact, achieves the maximum and its minimum. Suppose this is the interval 2 to 8 this is x this is V of x there is this continuous function its continuity is also important; suppose this is the interval.

So, there is indeed a value of x where it achieves its maximum value there is one other value at least where it achieves its minimum value. So, the maximum and minimum are in fact, achieved, that is why we will say that the maximum minimum of that particular function over that set exists why because this interval 2 to 8? It is both a closed and bounded set.

So, V of x achieves meaning of achieve means there is a need of point x where the value of V of x is equal to the maximum value over that set; achieves its maximum and minimum over compact set. Compact we saw already was nothing, but closed and bounded set of course we have assumed that V is continuous, only continuous functions are guaranteed to achieve their maximum and minimum over compact sets.

So, so we are now back to our case where x has many components it is a element in R n. And we are now looking at what is outside the ball B d and contained inside the ball B epsilon inside the closed ball B epsilon but it is written R here. So, this is a small typo. So, the set of all points where the norm of x is greater than or equal to d and less than or equal to R; so, notice that the boundaries are also included and it is bounded from R epsilon R is equal to epsilon.

It is bounded by epsilon and hence it is also bounded set. On this compact set this maximum value is achieved, we really acquire this property that is why the emphasis. And this maximum quantity itself is negative and hence gamma is positive. So, the over this compact set the continuous function on the set that continuous function is V dot of x that depends continuously on x.

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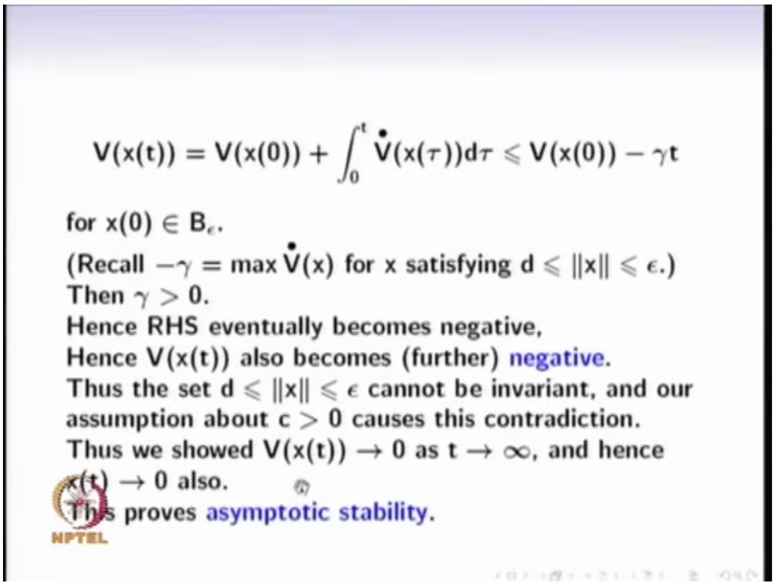

So, since it is we require this property of gamma greater than 0 and that we will use to arrive at the contradiction that we are looking for. So, now, integrating both sides we see that V of x of t is nothing but V of x of 0 plus this integral from 0 to t of V dot of x tau d tau. Now this particular quantity itself is, less than each at each time instant this quantity is less than or equal to gamma. And hence we will integrate this and this becomes V of x of 0 minus gamma t; yeah, recall that gamma minus gamma was equal to the maximum of this.

And hence we are integrating something instead of this V dot we are going to replace V dot by the maximum value that V dot can achieve. And hence this quantity to the right hand side will end up becoming larger; why would it become larger? Because instead of V dot we have done some manipulation by replacing something that can be larger; gamma is the minus gamma is the maximum value that V dot can be over that set. And hence we have integrated with respect to time of minus gamma and we get minus gamma t here.

So, what we have computed is, at any time V of x of t is less than or equal to V of x of 0 minus gamma t where recall that gamma was equal to the maximum, minus gamma was equal to the maximum of V dot over that set and we also concluded that gamma is positive. So, now, notice that this right hand side, how does it behave as a function of time? This itself is a line.

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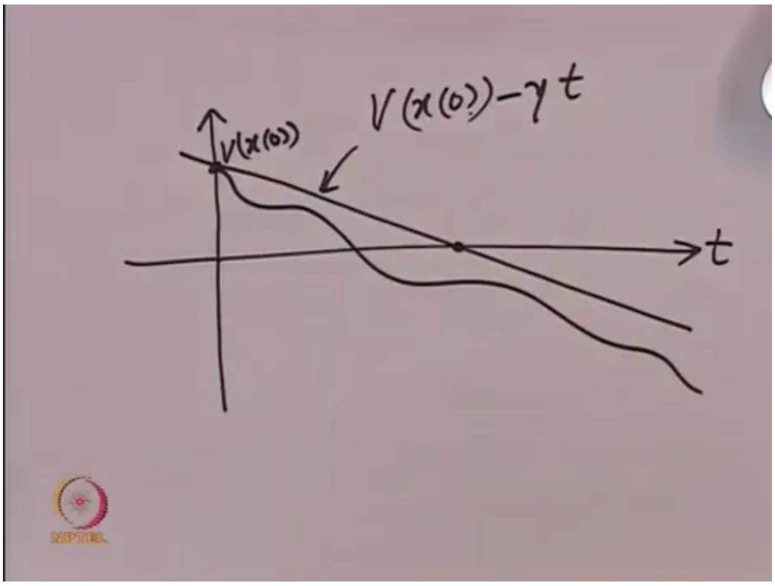

With t as independent variable this quantity, gamma is some positive number and hence this is some line with negative slope with slope minus gamma. Hence it is decreasing like this is V of x of 0 at t equal to 0.

We get this value and hence this is where it starts. And our conclusion is that V of x of t itself is always below this line, this line itself becomes negative for some time this has value of time depends on the value of gamma, but it is guaranteed to become negative. And hence this V of x of t which is what we have concluded is below this line, will also end up becoming negative yeah.

So, coming back to this particular slide V of x of t is less than or equal to this particular quantity this line and this line itself becomes negative. No matter what V of x of 0 is, no matter what gamma value is, because gamma is positive this is some line that is sloped downwards. And hence there is some times in t form with this quantity becomes negative and this quantity which is further less from this will also eventually become negative fine. Hence V of x of t also becomes further negative for that time onwards does this set.

So, what have we concluded? V of x which was guaranteed to be positive has ended up becoming negative. This is a contradiction that we have finally, obtained does this set d less than or equal to norm of x less than or equal to epsilon the set of one such x cannot be invariant that is a property that we used. And our assumption that c greater than 0 has ended up causing such a contradiction what contradiction; V of x of t eventually becomes negative for some time for some finite time t onwards.

So, since c greater than 0 has been ruled out, we have now concluded that V of x of t converges to 0 as c tends to infinity. And hence x of t converges to 0 also this proves asymptotic stability ok.

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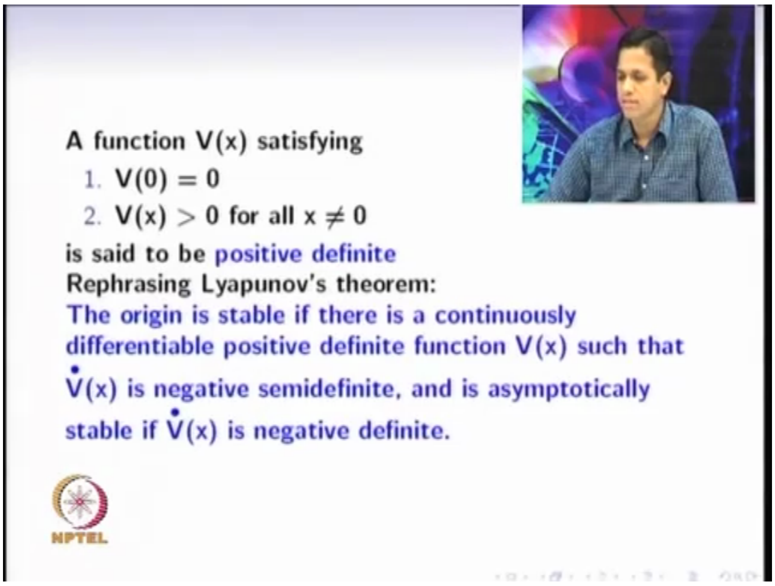

So, this completes the proof of the Lyapunov's theorem on stability and asymptotic stability just some more notation. So, function V that satisfies V of 0 is equal to 0 and V of x is positive for all nonzero points for all points x naught equal to 0; such a function V also called positive definite.

So, in this new words we can replace Lyapunov's theorem that the origin is stable if there if there is a continuously differentiable positive definite function V of x. Such that V dot is negative semi definite and if V dot is negative definite, then that origin is in addition to stable also asymptotically stable.

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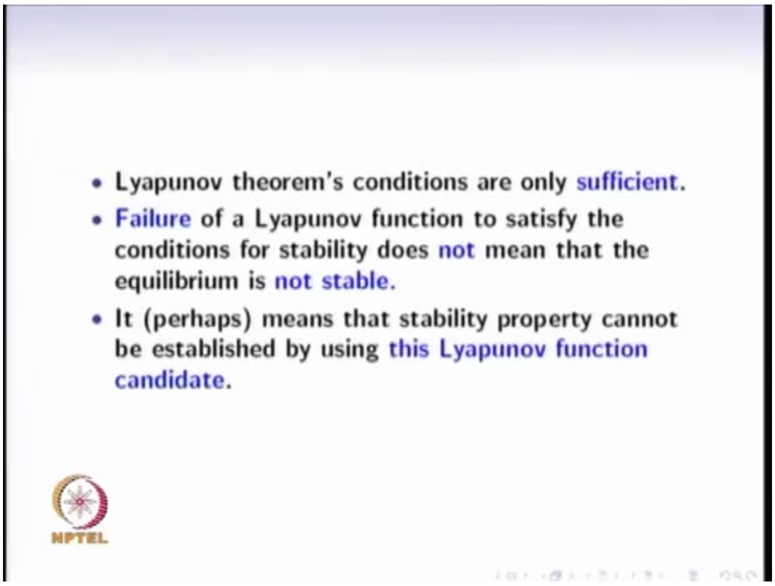

So, its important to note that the Lyapunov's theorem are only sufficient. If there is such a continuously differentiable function satisfying these properties, then we can go ahead and conclude that the origin stable yeah. Let us see this failure of a Lyapunov function to satisfy the conditions for stability in the Lyapunov theorem. If some Lyapunov function fails to satisfy those conditions it does not mean that the equilibrium is not stable. It perhaps it means, that the stability property cannot be established by using that particular Lyapunov's function candidate.

So, we will call it a Lyapunov function only if it satisfies the properties, if you have a candidate that fails to satisfy those properties, we cannot go ahead and conclude that the origin is not stable because it does not mean that. It perhaps means that that candidate is to blame that candidate may not be the correct function. And we might we should perhaps with looking for other candidates which will satisfy the conditions that are stated in the Lyapunov's theorem for stability.

So, before we go to some more results about Lyapunov's theorem not just for locally stable, but globally stable we should draw a figure of how this Lyapunov's function is helping.

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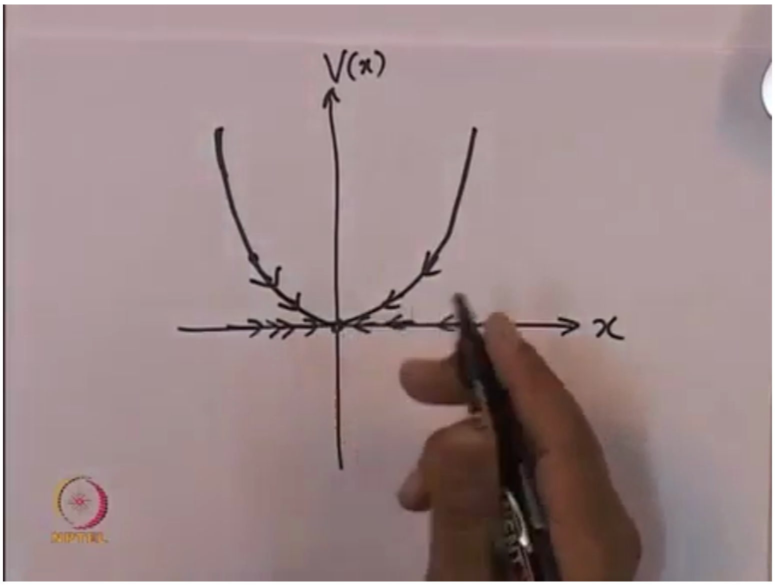

This is our x this is V of x. So, we want to now conclude that all points are all the arrows are directed towards the origin, we are able to do this by the existence of some function V. So, this function V is positive that is why it has to always lie about the graph about the x axis. And here because the directions are always directed towards the origin it turns out that. This function V itself when differentiated with respect to time is always decreasing yeah.

In other words, if you are able to find a function V that is decreasing as a function of time if it is decreasing strictly and the function itself is positive. Then the only way it can happen is that that particular equilibrium point is asymptotically stable yeah.

So, it is not possible that there is such a function which is decreasing which is decreasing strictly along the trajectories and is itself positive; is equal to 0 only at that point that function V. If such a function exists it is not possible that this particular equilibrium point is unstable it is guaranteed to be asymptotically stable, this is what the Lyapunov function says. We can take one very simple example for this purpose.

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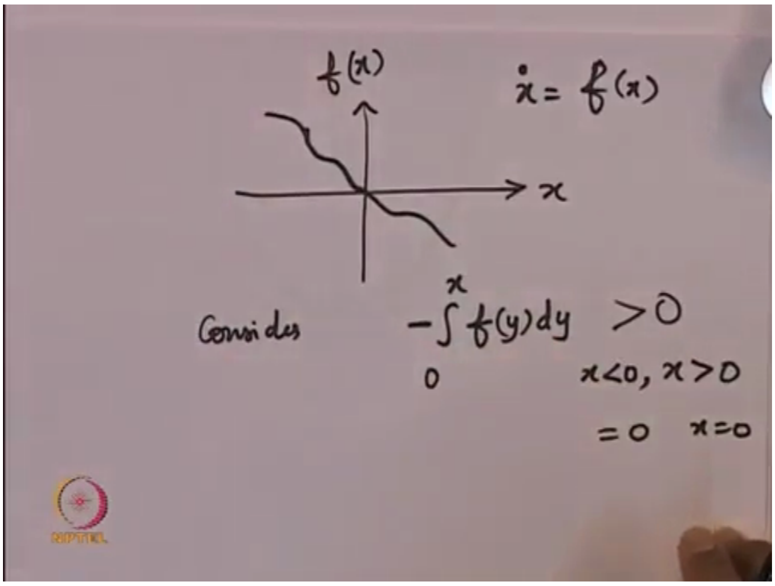

So, consider suppose this was our dynamics x dot is equal to f of x in which graph of f was equal to like this yeah. So, here is some function. So, we know that if V if f satisfies this. So, it is important to note that in this example we are plotting f versus x while we are actually going to construct the Lyapunov function which is also positive except at the point x equals 0 and rate of change V with respect to time V dot is required to be negative.

So, this dot all our by convention all the dot signifies rate of change with respect to time, f is itself a function of x. So, consider integral from 0 to x of f of f of y d y it is customary to use different variable here and here yeah. So, this particular function can we say that this is positive for this particular graph? What does this particular function signify? It signifies the area from here to here yeah.

So, it seems like this graph for x positive this graph is lying below the x axis. And hence this greater than 0 is not satisfied. The negative of this function seems to satisfy this property what property? It is for x greater than 0, this area is has the sign that the areas negative, area under this function f. For x greater than 0, this because this function lies below the x axis, the area is negative. And after putting this negative sign this quantity is positive at least for x greater than 0. For x equal to 0, it is the integral of some function for the width 0 and hence this equal to 0 for x equal to 0.

What about for x negative? For x negative we will instead consider the integral from x to 0; from x to 0 this particular quantity has positive area. And hence from 0 to x it has negative area but so, after this negative sign has been put this again this one is satisfied. So in fact, this greater than 0 satisfied for both x less than 0 and for x greater than 0, only for x equal to 0 it is equal to 0.

So, perhaps this one will serve as our Lyapunov function candidate, it is some function which is positive for all nonzero x and is equal to 0 for x is equal to 0. So, we will now show that this particular Lyapunov candidate indeed serves the satisfies the properties of Lyapunov function.

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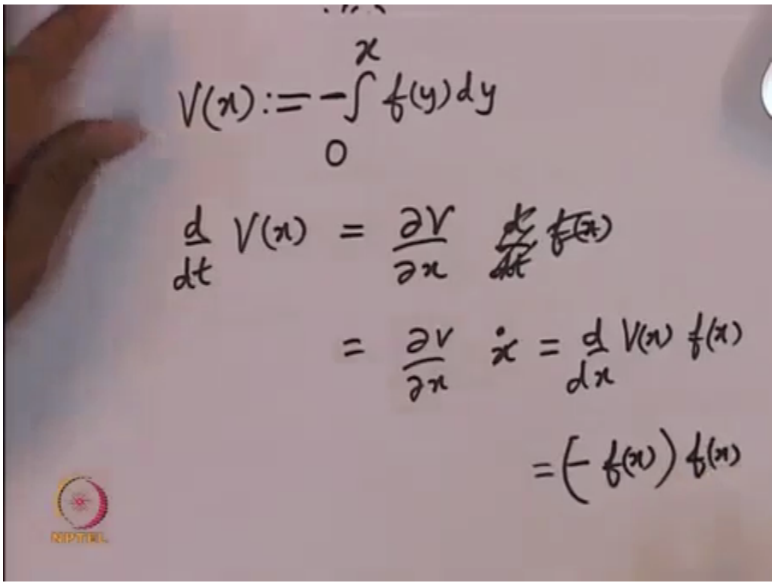

So, V of x is now defined as integral from 0 to x of f of y d y. Now what is the next thing? d by dt of V of x is now equal to derivative of this with respect to x and then that so, so, this is nothing, but del V by del x times d by dt of f of x; del V by del x times sorry this is equal to del V by del x times x dot.

Because this is a composite function, V is a function x which is itself a function of time to differentiate this quantity with respect to time, we will first differentiate V with respect to x and then differentiate x with respect to time. And of course, V is a function of only one component x in this case. So, this partial derivative could also be replaced by ordinary derivative. So, this is d by d x of V of x times x dot is nothing, but f of x.

So, what does it mean to differentiate this particular function with respect to x? Here is a function, which is growing with respect to x growing at what rate? Precisely this f y when you differentiate such a function with respect to the endpoints. Then we get precisely the quantity that is inside sorry we had a negative sign in our definition of Lyapunov function, that negative sign is really crucial. So, this is nothing but minus f of x this is the rate of change of V of x times f of x. So, we have finally, concluded that, V dot of x is equal to minus f of x whole square.

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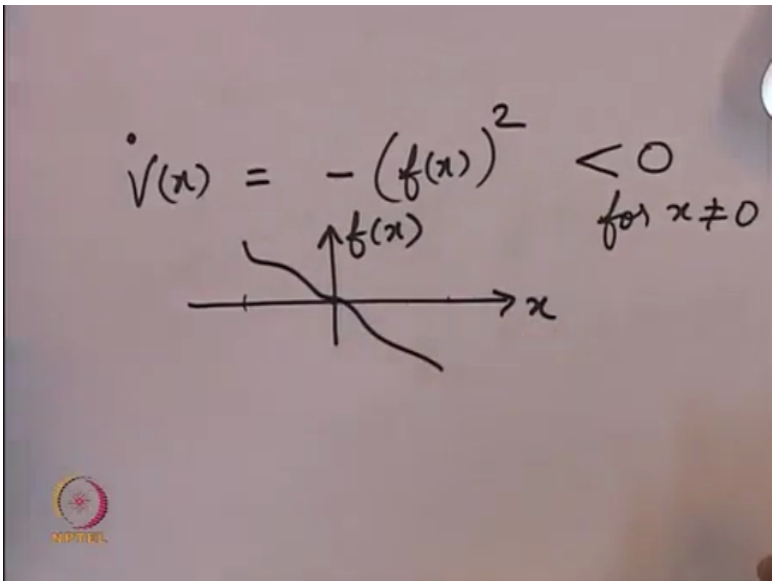

And recall that f of x graph that was drawn was like this. The graph indicated that f was equal to 0 only at the equilibrium point x equal to 0. What are all the equilibrium points? x equal 0 only. Only at that point f is equal to 0; which means that that is the only i that is the equilibrium point for this interval that we have drawn the graph is an isolated equilibrium point. And hence this particular quantity is less than 0 for x nonzero yeah. Since f of x is nonzero for all nonzero x this particular quantity is less than 0; is strictly negative.

So, here is a function V which is obtained as the integral of another function. Hence it is automatically continuous continuously differentiable; also and it is positive we already saw its rate of change is negative. And this proves that this particular equilibrium point is asymptotically stable. So, this is one way in which without knowing the precise formula for f by just using the property that f was positive for x less than 0. f was negative for f for x greater than 0 by using just the continuity property of f, itself we have been able to show that this equilibrium point is in fact, asymptotically stable equilibrium point.

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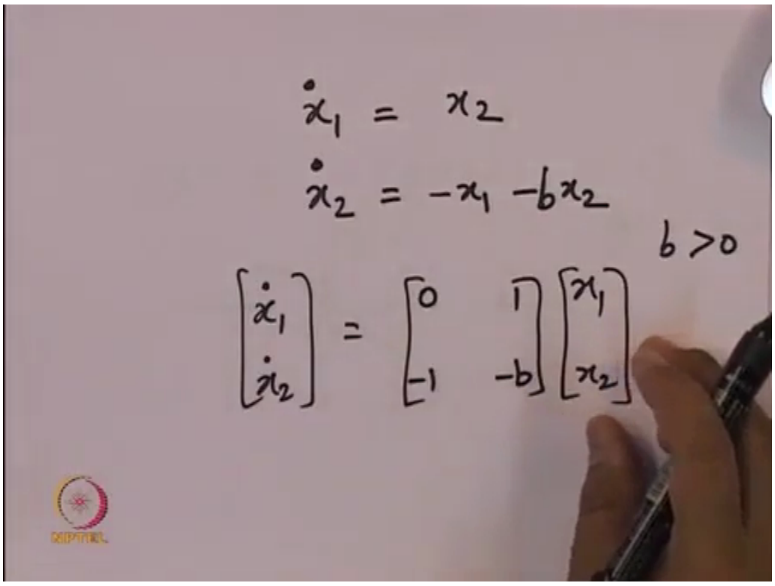

We could take another example with two dimensions consider x 1 dot is equal to x 2 and x 2 dot is equal to minus x 1 minus x 2 yeah. So, rate of change of x 1 is just equal to x 2 rate of change of x 2 is equal to minus x 1 and also some d times x 2. Where b we will assume is positive. So, this is an example of a pendulum with friction. So, we will see this example in more detail, when we consider relaxing the conditions in the Lyapunov's theorem.

Or at least this particular example we will use and check whether this is whether the equilibrium point is stable and asymptotically stable. Of course, this is a linear example, we can write x dot is equal to a times x yeah. And one could in principle find the eigenvalues of this matrix and already conclude that all the eigenvalues are in the left half plane, but that is just one way of doing it. We will now use the Lyapunov function argument for proving that the equilibrium point is stable.

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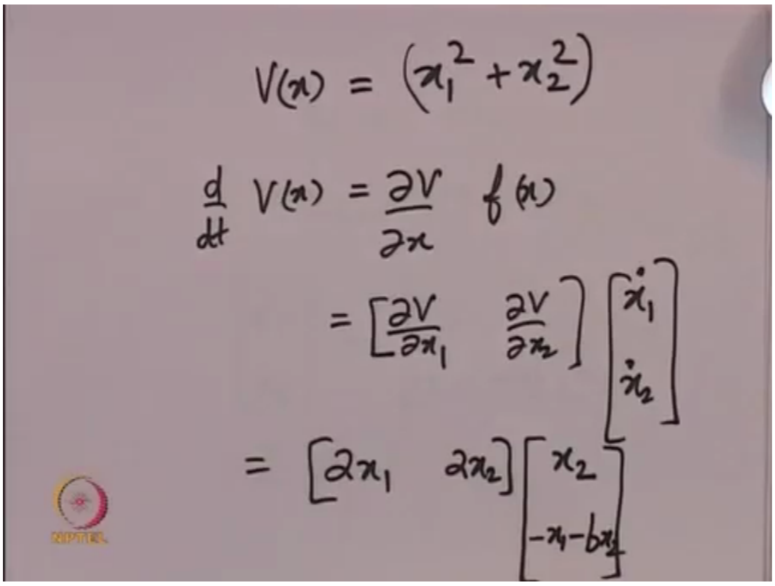

So, take on the Lyapunov function, it requires some effort to guess to guess a candidate. From physical exam for physical systems it is possible to take the actual energies and sum them up. But for more general systems, it requires an effort which can come with experience of how to guess a Lyapunov function candidate. And then try to show that it is stable why because, if one candidate does not serve the purpose it might require some effort to guess another candidate. And still succeed in showing that the equilibrium point is stable or asymptotically stable.

So, here we take this particular candidate d by dt of V of x is nothing but delta V by delta of x times f of x f of x is nothing, but x dot. So, partial derivative of V with respect to the vector x is nothing, but del V by del x 1 del V by del x 2 times x 1 dot x 2 dot. So, the partial derivative of this function with respect to x 1 is nothing but 2 x 1 partial derivative of this particular function with respect to x 2 is nothing but 2 x 2. x 1 dot was equal to x 2 yeah let us see this previous here. x 1 dot was equal to x 2 and x 2. We will write as minus x 1 minus b x 2 minus x 1 minus b x 2.

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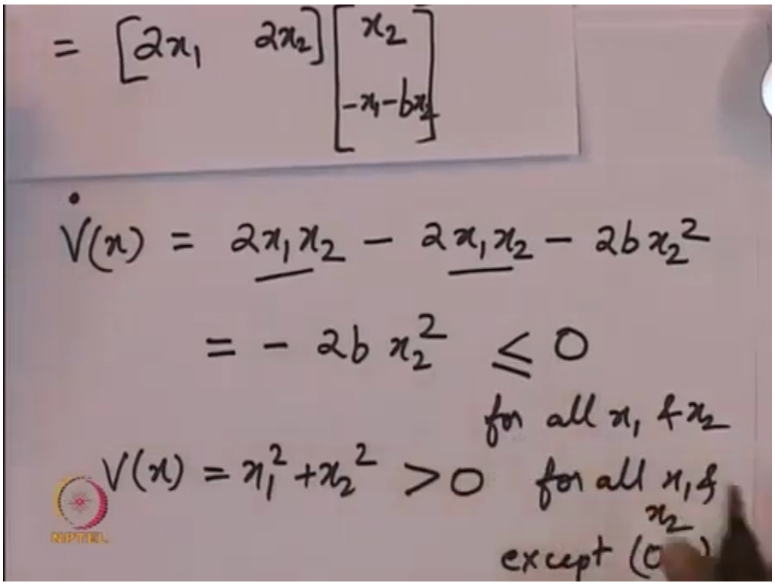

Upon evaluating this particular we get 2 x 1 x 2 minus 2 x 1 x 2 minus 2 b x 2 square; this is what we get as equal to b dot x. So, we can cancel this and this and this is nothing but minus 2 b x 2 square by a. Since we are assume that d is positive this particular quantity is less than less than or equal to 0 for all x 1 and x 2. It turns out that x 1 does not appear at all, in the definition of this in V dot of x.

But in any case for all x 1 x 2 this is less than or equal to 0. And also we forgot to verify that V of x itself which is equal to x 1 square plus x 2 square is greater than 0 for all x; for all x 1 and x 2 except of course, except 0 comma 0. Except when both components are equal to 0, except for that that case this Lyapunov this particular function is positive. So, in other words it is a positive definite function that is why it is a candidate. And we have in fact, checked that its rate of change is also non positive it is less than or equal to 0 for all x 1 x 2.

And hence the equilibrium point for that particular dynamical system is a stable equilibrium point. That is all we have been able to show what have we shown, that this particular function.

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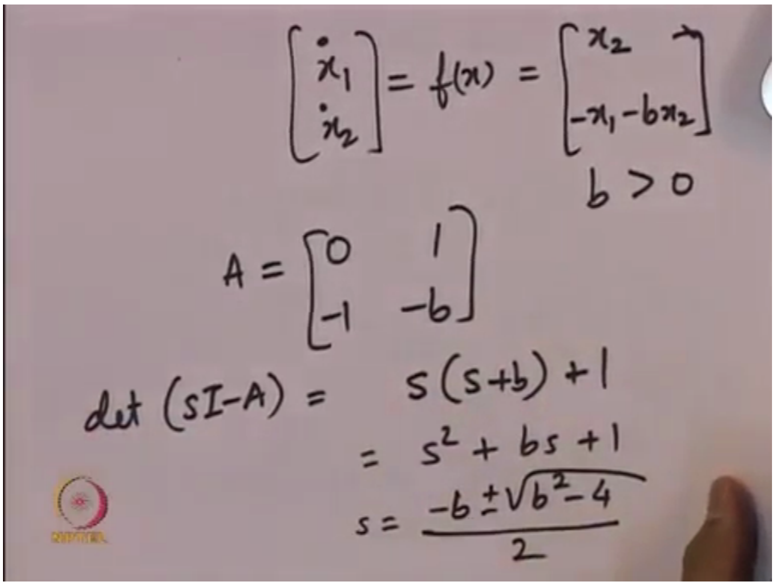

x 1 dot x 2 dot is equal to f of x given by x 2 minus x 1 minus b x 2 for some b greater than 0. This particular systems equilibrium point is the point x 1 and x 2 equal to 0 both equals 0 the origin that equilibrium point we have shown is stable. Because we have demonstrated that there is one Lyapunov function, whose rate of change is always non positive.

So, does that mean that this is not asymptotically stable? Let us check that matrix A we had got was equal to 0, 1 minus 1 minus b. And now when we do s I minus A determinant we get s times s plus b plus 1 is equal to s square plus b s plus 1. So, here what are the roots of this particular equation? These are some two points whose product is equal to plus 1.

In other words they both have the same sign, moreover their sum is equal to minus b yeah. So, the roots are nothing but s equal to minus b plus minus b square minus 4 ac b is nothing but b square minus 4 over 2. So, the sum of the two roots is equal to minus b yeah and the product is equal to 1. So, these are some two points on the unit circle.

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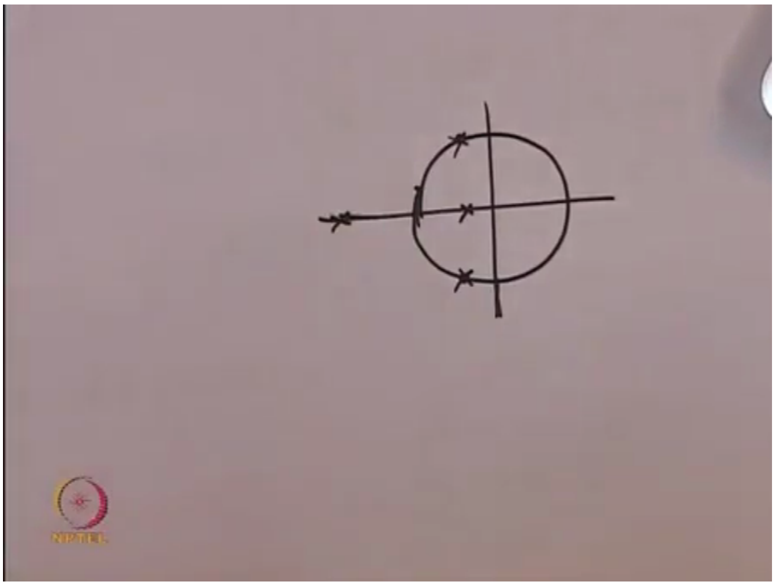

How do the two routes look? It is a two points on the unit circle. So, they are to be conjugate pairs, the product is equal to one and their sum is equal to minus b. Because the sum is negative they have to be on the left half of the complex plane, they cannot be on the imaginary axis also because then the sum would be equal to 0.

So, we know that the matrix has eigenvalues in the left half complex plane; as a result they are either here and here or they are here and here such that the product is still equal to 1 depending on the value of b. So, here are two points which are both in the left hand complex plane. And hence we know that the origin is asymptotically stable, but our Lyapunov function candidate only helped us to prove that the origin is stable. We were not able to use that particular candidate to show that the origin is asymptotically stable, we will resolve these issues in the next few lectures.

But it is important to note that the Lyapunov theorem is only a sufficient condition, if we were able to find a V that satisfies those properties; then we can go ahead and conclude something about stability or asymptotic stability. If V does not satisfy those conditions, perhaps we should spend some effort on finding another V or there is a possibilities are the equilibrium point is indeed unstable for that also we will see some conditions on V ok. These are the things that we will cover in the following lecture.

Thank you.