Nonlinear System Analysis
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Lecture – 28
Stability Analysis and types of stability

Hello, welcome to this week 6 lectures on Linear Systems Theory. So, last week we had done lots of analysis about equilibrium points, we did not define properly the notion of stability, but several characteristics of those equilibrium points; why are the phase space, Gave us some information whether the trajectories around the equilibrium point were coming back to the origin or going away from the origin or the equilibrium.

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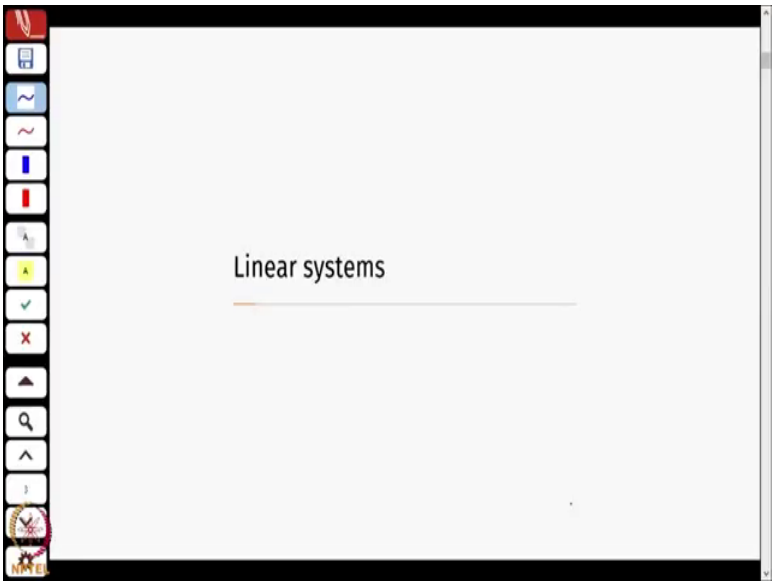

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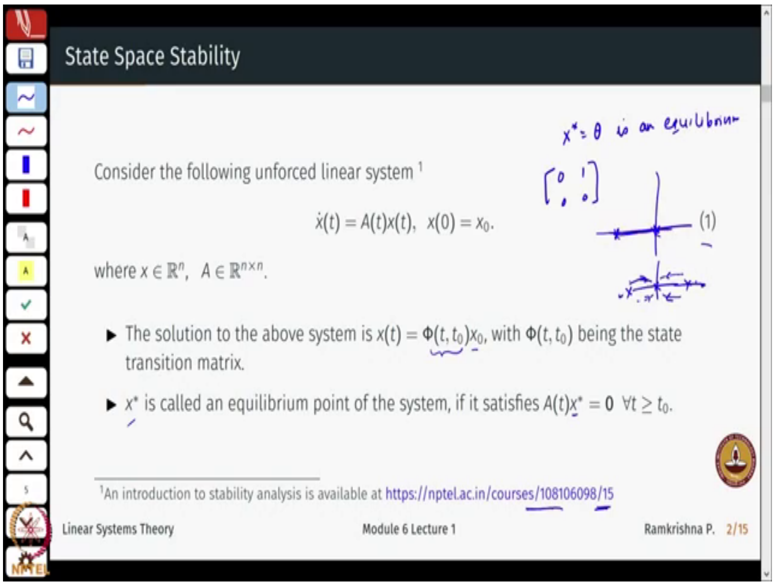

So, today we will give a formal notion of stability. So, of course, we will do mostly in the linear setting and in general, in the time varying setting . I will start with the following unforced system, I mean I will not talk of inputs as yet from x dot equal to A of t x t with a certain initial conditions and x is in general coming an n dimensional vector.

In the previous control course, you would remember stability or one way of verifying stability is to look at the poles of the system and then you had several characterizations of the poles being to the left, to the right on the imaginary axis repeated on the imaginary axis and so on.

All this came from a bit of input-output notion of stability right, and the condition to verify or the way we derive the pole base condition for to verify stability, comes actually from what is called as the absolute integrability of the impulse function, of the impulse response of the transfer function.

We will not revisit those things. If you are interested, you can just go back to one of my earlier lectures that is listed here, just to get a little warm up on the BIBO notion of stability, what was the bounded-input bounded-output notions of stability.

I will not elaborate that but here, what we will essentially focus is on the state-space stability, more related to the kind of solutions we computed for the system of linear equations starting from the concept of a state-transition matrix.

So, for this class of systems what we know is that the solution is well, given by x of t is phi which is the state transition matrix and an initial condition. What we also know is that, x star is an equilibrium point of the system if it satisfies A x star equal to 0. And in most linear cases, the you know the origin x star, the 0 vector is an equilibrium.

It is always; even if the a matrix I mean is not invertible say, if it is of this form still 0 is an equilibrium there could at some point of time we will, multiple equilibrium. Say this is an equilibrium point the origin maybe, somewhere this is also an equilibrium point.

So, the line joining these 2 points is also will this is any point between these 2 points or the straight line joining these 2 points will also be in the equilibrium. So, we will have essentially an equilibrium subspace. That is what we talked about in one of those conditions in equilibrium in previous week’s lectures.

So, fairly we will just in generally, denote the notion of equilibrium as x star and mostly we will deal with the origin. Even if the origin is not the equilibrium; we can shift, change the coordinates or shift the origin. Slightly different than the non-linear case because ok, there we possibly we will not have an equilibrium subspace, but we will have, say isolated equilibrium point say equilibrium point here, equilibrium point here, here and so on.

And each say, if maybe the phase space goes something like this, sorry it could go this way right this and this. So, this is a stable equilibrium these 2 equilibriums could be unstable and so on. So, we have what they also called as call as isolated equilibrium points.

But for us, we will just assume that or we know that origin is the equilibrium and that will be the equilibrium of interest. But just to for notational purposes we just call it generally as x star.

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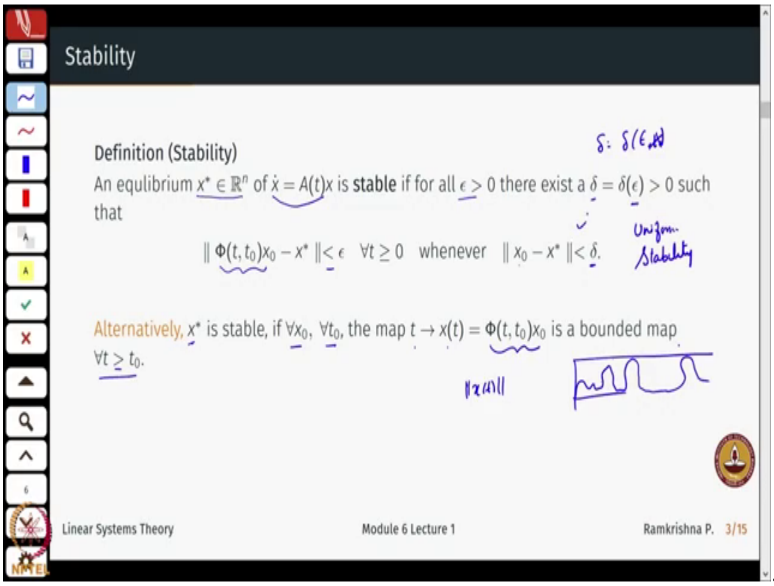

So, the first definition of stability, what does it mean? Again, I am looking at stability is always defined with respect to an equilibrium x star. So, an equilibrium x star of the system x dot is A t x and I just for ease of notation I just drop the time argument for x is stable if for all epsilon greater than 0, there exists a delta which is a function of this epsilon, such that if I take the norm. So this is the solution, right?

How the solution trajectory moves? Phi t 0 x 0 will give me the solution, the norm of this solution minus x star is less than 0 whenever x 0 minus x star is lesser than delta. We will come to this right where does the epsilon come from, where does the delta come from, in some cases, you also have that the delta is depending on the initial condition and in this case it is not depending on the initial conditions, so, it is usually called as the also referred to as the concept of uniform stability ok.

So, we will we will drop this t naught for the moment and we will say well, this we are just dealing with a uniform stability, ok. A slightly milder version of this to give you an intuition of what this what the statement means or a little understanding.

x star is stable if for all initial conditions, for all initial times the map t from x t right, which is given by this solution. This is my solution is a is a bounded map for all times t greater than or maybe even with equal to t 0. Essentially, it means that phi or this solution x of t ok, so it is a bounded map. So, it means the values of x never go unbounded. So, if I were to just draw a little picture this could be solutions, whatever and so on right.

So, these are all bounded. Now, what does what does this actually mean? Ok. So, let us see a pictorial version of it ok.

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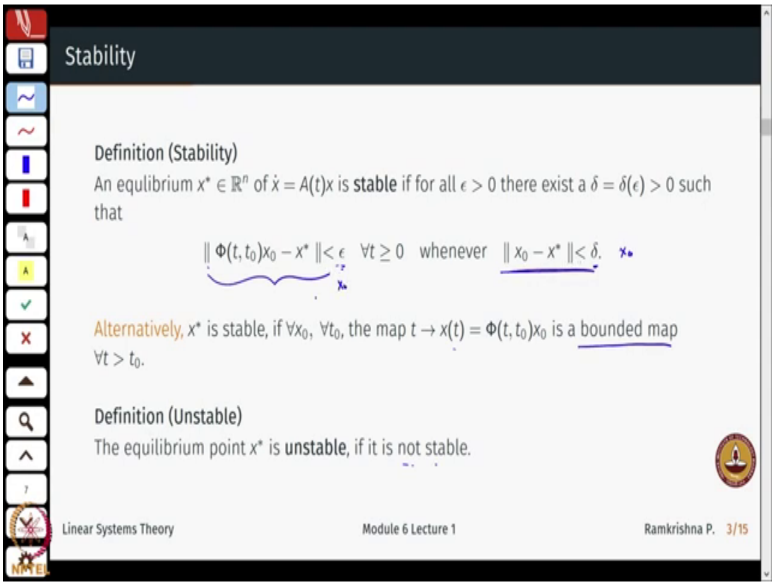

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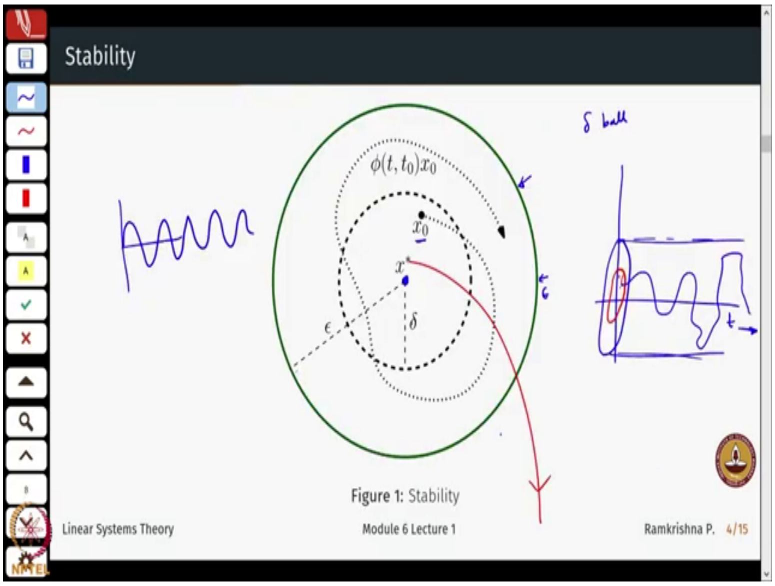

So, I start with the x star as my equilibrium point here. Let this x naught come from a delta ball essentially a delta ball would been a ball centred at the equilibrium and of radius delta and take any initial condition so, I have this delta ball and take any initial condition within this delta ball, this x naught.

So solution will so, if a if I just see this dotted line, the way it goes its always confined to this epsilon ball of radius, no this is epsilon ball is again the ball centered at x star and of radius epsilon, ok.

So, if I just move this picture 90 degrees, it would look something like this. Ok I will try to draw it as an as neatly as possible, say somewhere I will just use this red color for the delta ball. So this is my delta ball and I will I will come back to blue. So, what does this mean of?

Ok,.I start at some initial condition here, and as time goes by I am just within the solution is just within the cylinder it could be go here, go here and so on.

But it will never breach this cylinder right, so it will just move along the cylinder. So, it is here I am actually even adding time axis. So if I tilt it, so these solutions we will just move along the cylinder which is in this in this represented in green, ok. So, essentially it means that the solutions do not go like unbounded.

So for example, if just say, I started initial condition here and just keeps going this way, this is an unbounded solution and this is un an unstable behaviour, right. So, this is a very basic very weak notion of stability that this could just be doing whatever the solution could just be doing whatever they want. 1 simple example is the linear oscillator that we did, right. So, it just keeps on oscillating this always stable right.

So, that is what this says, right. So whenever the initial conditions are in a delta ball the solutions will be confined to an epsilon ball or that the solution or this map t to this to the solution space is always a bounded map, right. That is what is this a very loose or the basic definition of stability. What is unstable? There is only 1 definition whatever is not stable is unstable.

So that is the only definition of unstability, you do not really check how the solution goes to infinity are they doing whatever and so on. So, whatever is not stable is unstable ok. So, that is that is the only definition of a system being unstable, ok.

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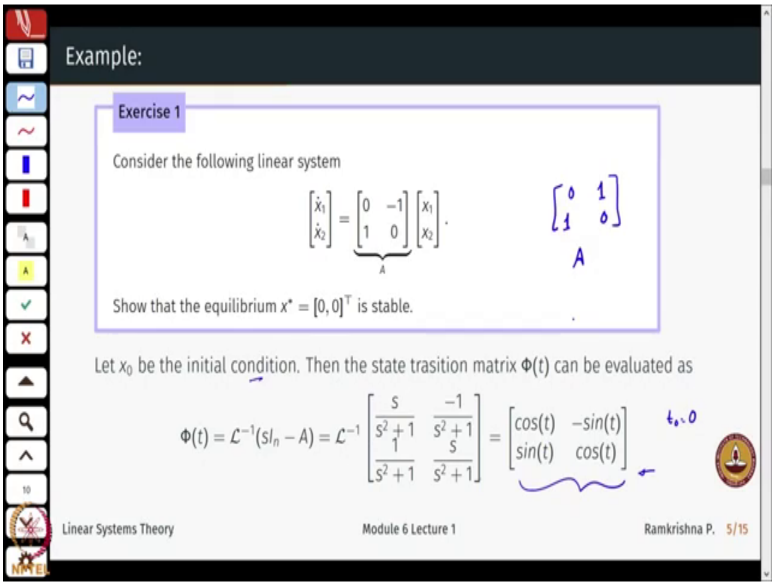

So, just to give you a little very basic seemingly trivial illustration of what is happening in this in this picture.

So let us start with this a second order system x 1 dot is a minus x 2, x 2 dot is a x 1 and so on. show that the equilibrium is stable. So, is easy to verify that 0 comma 0 is an equilibrium. Let us first compute the solution right, I am just let us for simplicity assume that the initial time is 0.

So, let x naught be the initial condition, that the state transition matrix will be know how it is computed this way, I do not, I will not go into the steps of this, but by now, we know how to compute the state transition matrix.

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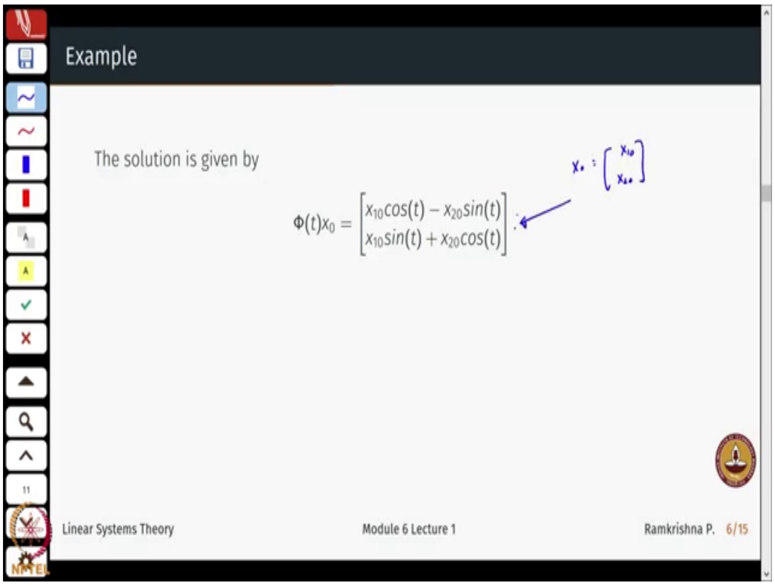

So, given any initial condition, x naught which is two-dimensional vector, x 1 naught or x 2 naught, the solution will look something like this, ok.

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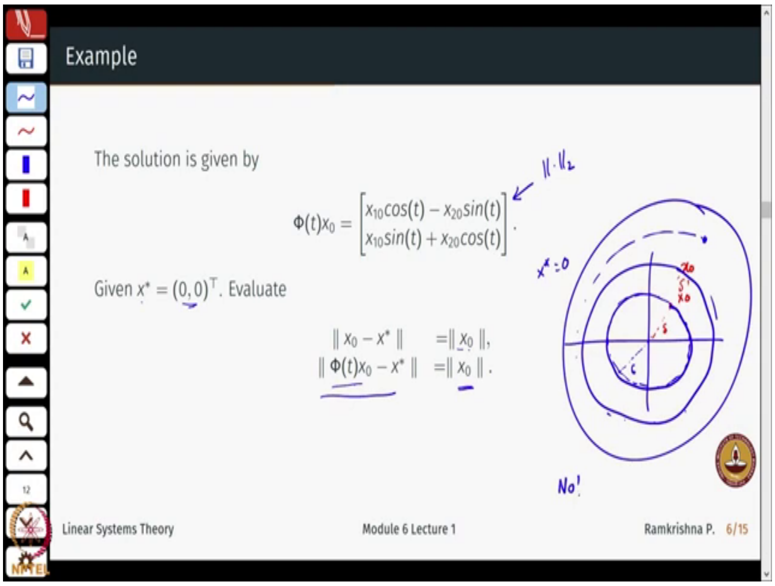

Now, given x star which is just the origin, what I would want to evaluate is check for these 2 inequalities right.

If the initial condition is within a delta ball or the solutions within a an or an epsilon ball or an solution always bounded, ok. So, let me compute this norm, right. So, this could be 2 norm or infinity norm or whatever, because we know that all norms infinite dimensionals are equivalent. So, give me 1 norm, I can always find an isolation with the other or I can always bound it with the other norm.

So, it does not really matter which norm we choose, but for our purposes we will usually you know look at the 2 norm, ok. So this well, x star is 0, so, this turns out to be just this x naught,. Now the solution phi t x naught minus x star. So, this is computations become easy, right. So, if I evaluate the 2 norm of this solution, it again turns out to be x naught.

So what does it mean, is that well the initial conditions, so the delta. So this, I found a bound to be x naught, but actually I found it actually to be equal to x star. So, this is also x naught, ok. So, let us take a trick to verify or to understand what this means, what we know is the phase space, the phase space of this systems are concentric circles, ok.

So, let me say I take this initial condition x naught. So, this will be my delta ball, ok. Now where when the solutions move? Well, the solutions will move along this circle of radius delta, right. Along all this now this is also equal to epsilon and that is what that is what I mean when I draw these 2 pictures I start with the initial condition at this point. So, this is what was my initial condition.

Then the final conditions will also be or the solutions will just be circling around this again of the same radius . Further, if I say what if my initial condition is here? So, I will have a new delta let me call this delta prime and my solutions starting from here with some initial conditions would just be circling around here, ok.

So, they never go unbounded right, so that is the idea here. So, I can just say take a larger ball like this, start from initial condition here and they will actually the solutions will just be inside this, ok. So, very nice and the trivial example, but it gives us a nice understanding of stability.

It is easy to check that this system will have complex eigen values and so on as a as exercise to check if the system is or an example for instability just check for this matrix and then try to compute the epsilon and delta values and you will find that this will not actually not exist. So, you will not be able to bound this by some number from the above, ok.

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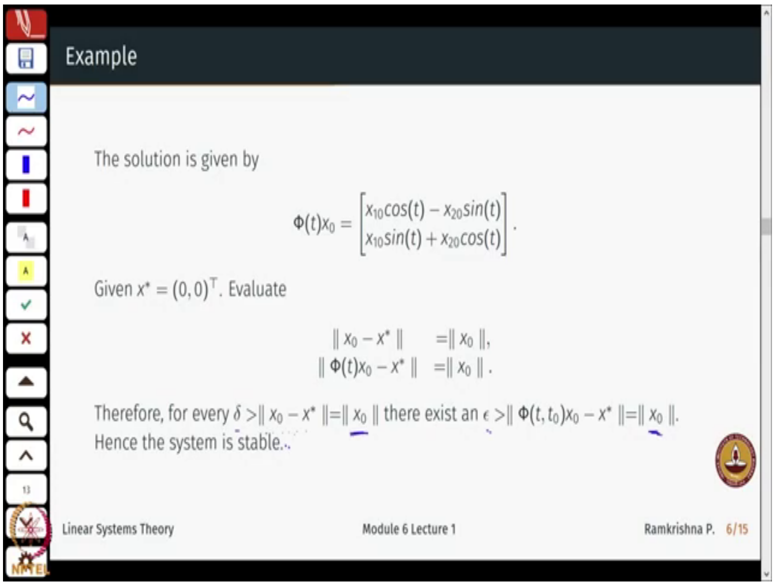

So, I will not really do that verge example of instability because that that it should be easy to check. And this is like what I proved here is I can just check that for every delta greater than blah, which is x naught there exists and an epsilon which is x naught and hence, the system satisfies the definition of stability.

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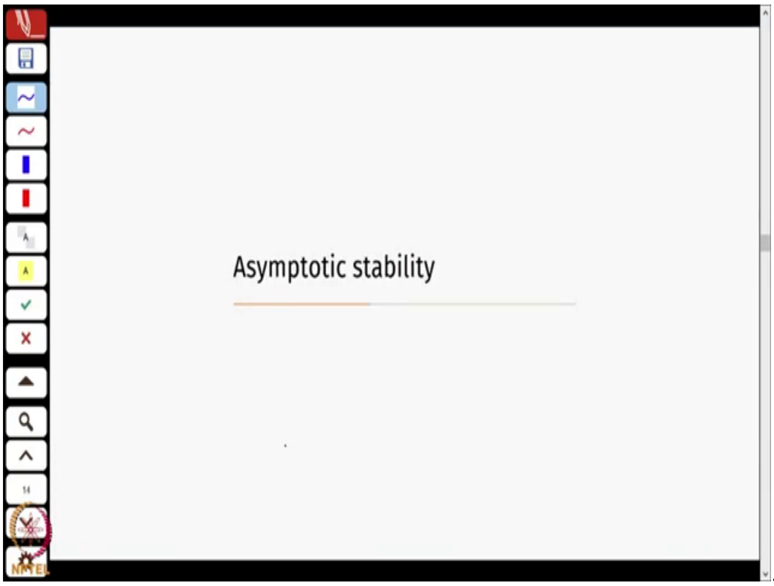

Now what next? I really do not want systems which are I really do not know where they are going, but they are just bounded. I want something stronger, right. So, stability I would rather want my system to come back to its equilibrium or its steady state position when it is subject to initial condition or there is slight perturbation in the system.

Now how do we mathematically define that notion? And that notion is the notion of the asymptotic stability.

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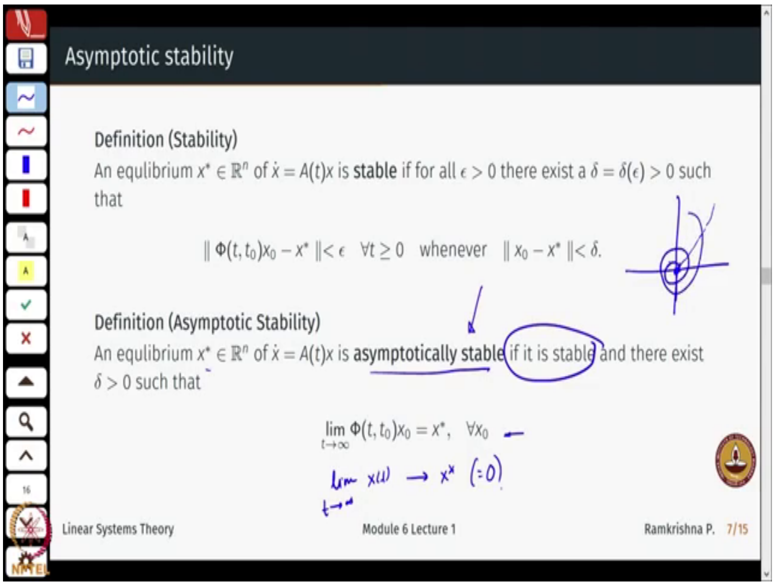

So, the definition states the following. So we will first read the statement and then try to interpret that. the equilibrium again x star of this system is stable if again, so all this epsilon delta exist as they would for. So, the basic definition of stability should be satisfied. Now additionally, on top of that, an equilibrium x star is asymptotically stable.

2 conditions, 1 if it is stable, and second as t goes to infinity, x of t should tend to x star which could be the equilibrium in this case, for all initial conditions, right. So, let us see what is what is new here, first is so it means that as t. So, suppose I am I am here, so this is my equilibrium, I say I will start from any point, I should actually eventually come back to this point.

I can come back this way, I can come back this way depending on the nature of the equilibrium point. So this is this is what it means. Now why do I superimpose a statement saying that stability is also a necessary condition? Why is it not always obvious that at t is equal to infinity, I am going to the origin? So, why should I additionally on top of that check for stability?

Let us do an example of that. In the linear case it might be obvious, but there might be some classes of systems which would behave very strange ok. So, let us do an example of this.

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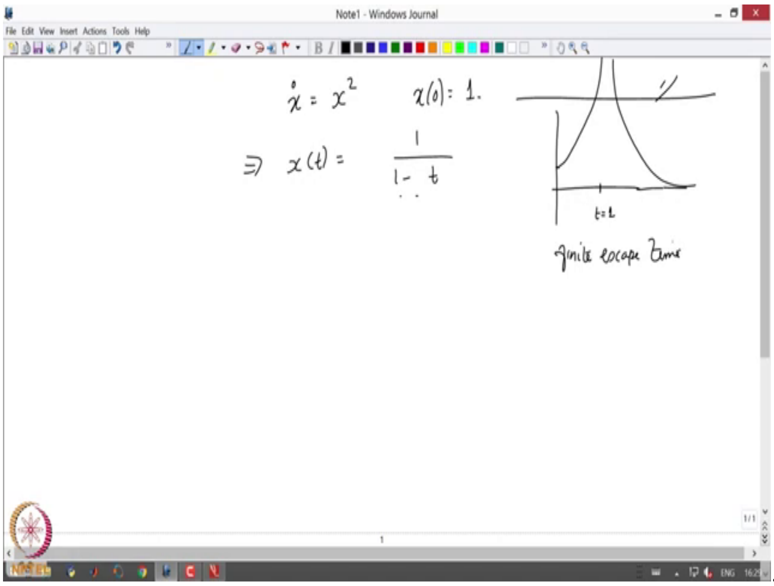

So I have this system x dot equal to x square and let me say that x at 0 is 1, right. Now what is the solution to this? This the solution turns out that x of t is 1 over 1 minus t, ok. So,me strange thing will is likely to happen. So, at t equal to 1, my system will say, start at say x 0 is 1 it will have, it will just blow up right.

So, the value is its not defined for t equal to 1 and you might think that well it will actually come back after a t equal to 1 plus and eventually, if I apply the limit rule what happens as t goes to infinity this might actually come back to 0, right I just draw slightly better, right.

Now is this a stable behaviour? Well, this is not, right. Because the solutions actually go on unbounded. So, what was the definition of stability that whatever the solutions do? They or just they should just be bounded by above from some by some number, ok. This typically could happen in some non-linear systems and this phenomena where at t equal to one the system goes up is a called the finite escape time.

Phenomena not usually or almost never seen or never seen in the case of linear systems and therefore, to have a general notion of stability we may have to impose this extra condition that first I need to check for stability and then I need to check what happens to the solutions as a time progresses, what happens asymptotically to the system?

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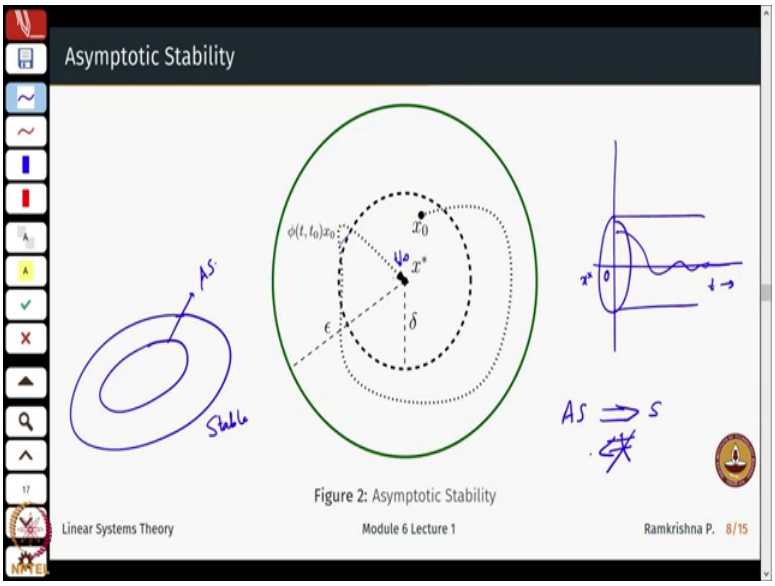

So, I what does that mean? pictorially, I think they should now be like kind of kind of easy to visualize now right. So, if I go back to drawing the cylinder diagram, I start from very initial condition, I think that is nice way to look at it is it just comes back to the equilibrium position.

This is the equilibrium or x star, this is with time ok. So, similarly here, I start with here and somehow I just say that at so this is at t equal to as t goes to infinity, I come back to x star.

So, that is a stronger condition to check right. So for example, is this system asymptotically stable? Well the answer is no. The system is not asymptotically stable. Is it stable? Yes. So, first thing, which is a larger set? the earlier system was stable, but not asymptotically stable.

So, asymptotically stable is a bigger this is a restriction on it, right. So, this will be asymptotically the set of asymptotically stable system, it is a stronger condition. So, not all asymptotically stable systems are stable systems, but not all stable systems are asymptotically stable systems.

So, it is like if I were just to say asymptotic stable and stable all asymptotic stable systems are stable, but all the stable are not asymptotically stable systems. Just a little counterexample to that is this what we saw just now.

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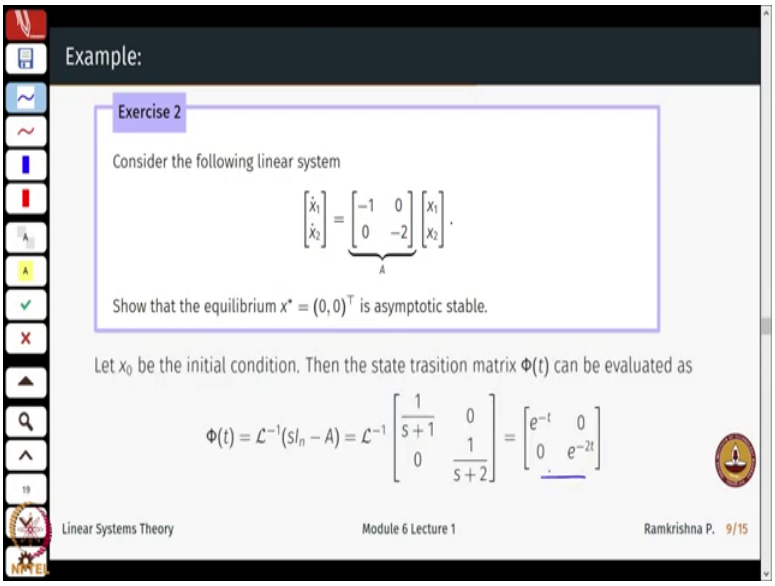

So, let us do it again seemingly simple looking example, if I just want to ask you to compute the state-transition matrix, it is a solution it will just be like by-heart right, because it s in the diagonal form and then the e power e t is easy to compute.

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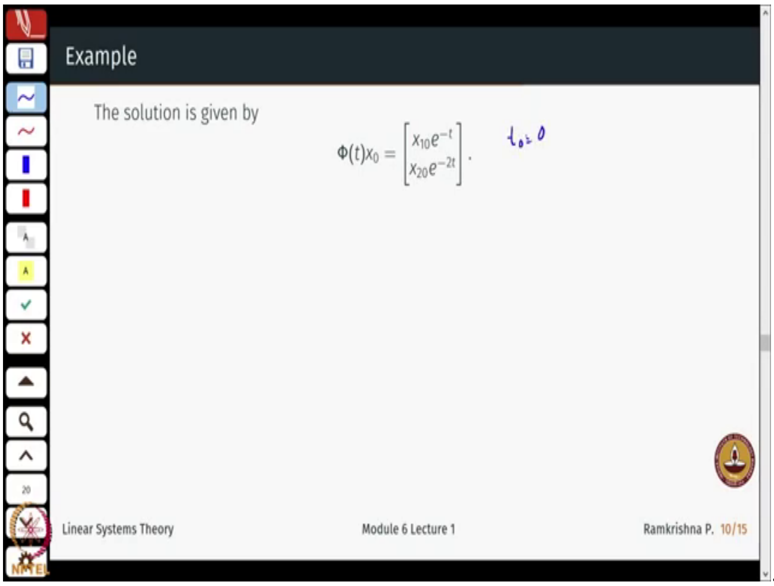

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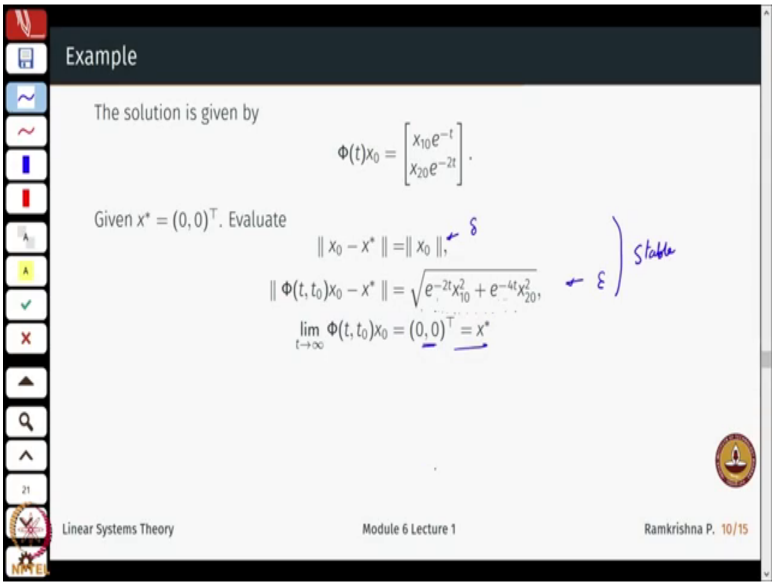

Let us say, I start with certain initial condition x 1 naught x 2 naught. Again, for simplicity I assumed t naught is 0. First, can I check for stability? That will be easy to check, right. So, this is my the delta ball this will be how the epsilon ball would look like. So, the first 2 conditions will tell me that the system is stable.

Now how do the, so I can actually find the upper limit of this right, so I will just leave that as an exercise, but what look at this expression here, what happens it as 2 time, 2 terms which are you know exponentially decaying, right and therefore, yeah if I put the limit as t e goes to infinity, I will just end up here right.

So, limit t tends to infinity, phi t naught x naught is the origin which is the equilibrium of interest.

Again, simple example, but good enough to verify what is what is happening here.

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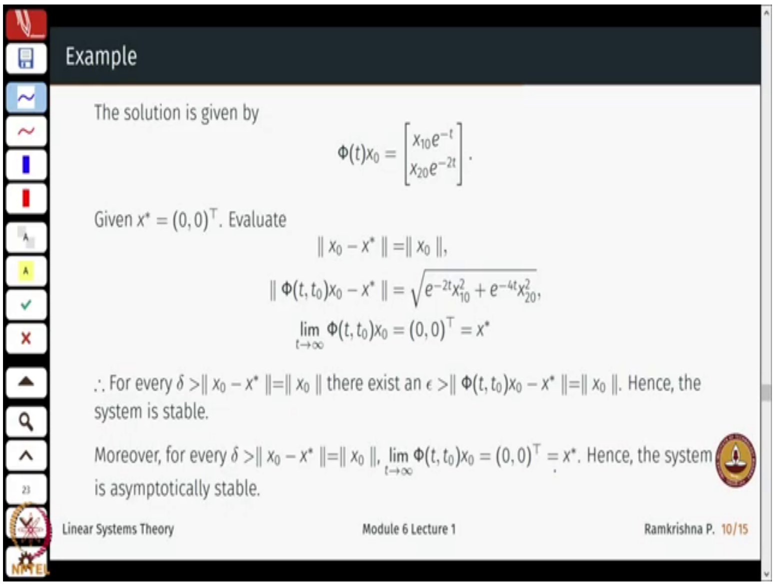

Is that are we happy when we say this? This is just the statement which concludes that the system is first that the system is stable, and second that this system is also asymptotically stable, right. So, just repeating that and writing down there firmly.

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So, are we happy with this? Well one may ask how fast are we coming to the origin? Or how fast are we converging to the equilibrium point?

So whenever I say origin, it is also means it also means the equilibrium point. So, these things will be used interchangeably origin the equilibrium and so on. So, I would also want my system say for example, it takes forever right, if it is a very heavily dammed system, maybe the response is very slow. And therefore, I would also like to see what is the rate of convergence to the equilibrium point?

And that therefore, I need to define further a stronger notion or do I can I quantify the rate of convergence? And that is where it comes in the notion of exponential stability. Ok what does it mean?

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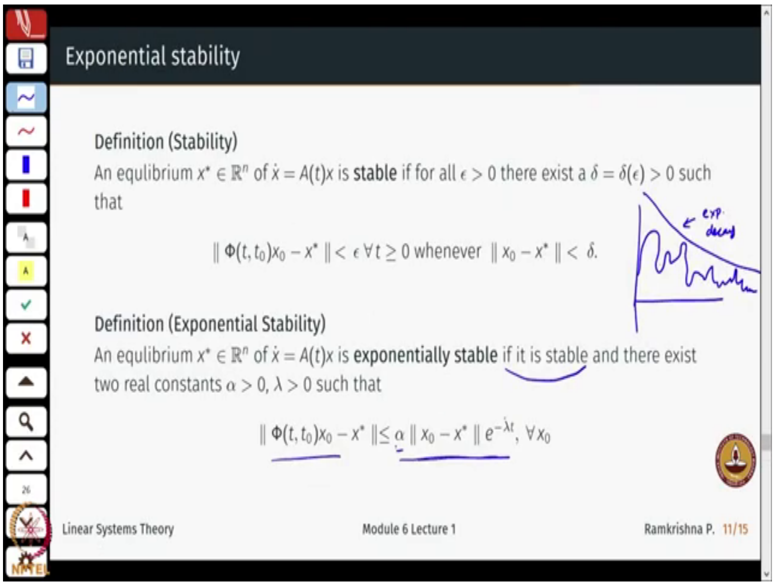

Again, the basic thing is to look at the definition of a stability; ok, when is the system additionally exponentially stable? Of course, first I should check if it is stable or not and then there exists 2 constants alpha and lambda such that something like this holds.

Ok what does this means? Say suppose, I have a solution which starts from initial condition, it goes whatever like this, like this and maybe eventually goes to the origin. Now, exponential stability would mean can I actually bound this from above by an exponentially decaying curve? Right so this is this is a exponential decay, right.

So, if my solutions are always under an exponential curve then this is called an exponentially stable system. Mathematically, well this is an my solutions. So, this is my solution is bounded by an exponential function. Right, this is exactly what is written here, right. So, this is this will be called the decay rate and so on right, ok

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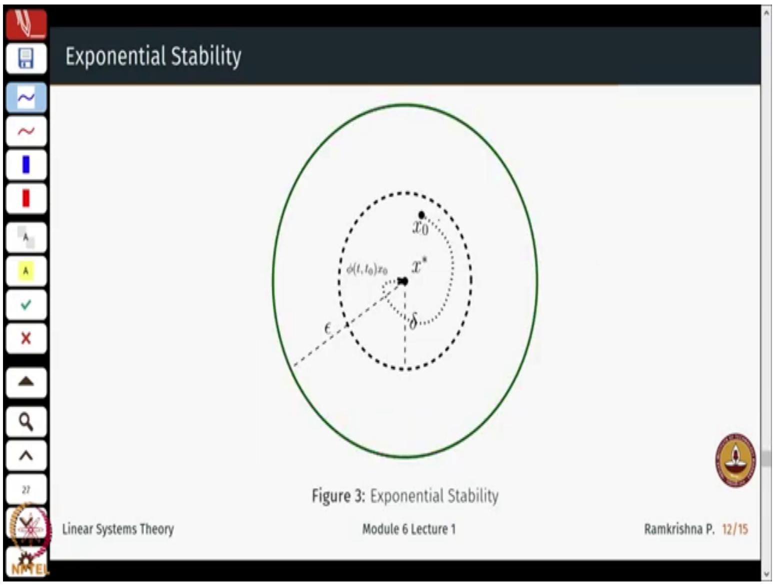

So, I can just write down a picture which looks like this, but I think it is easier starting from what we know earlier, it is now easy to interpret this picture.

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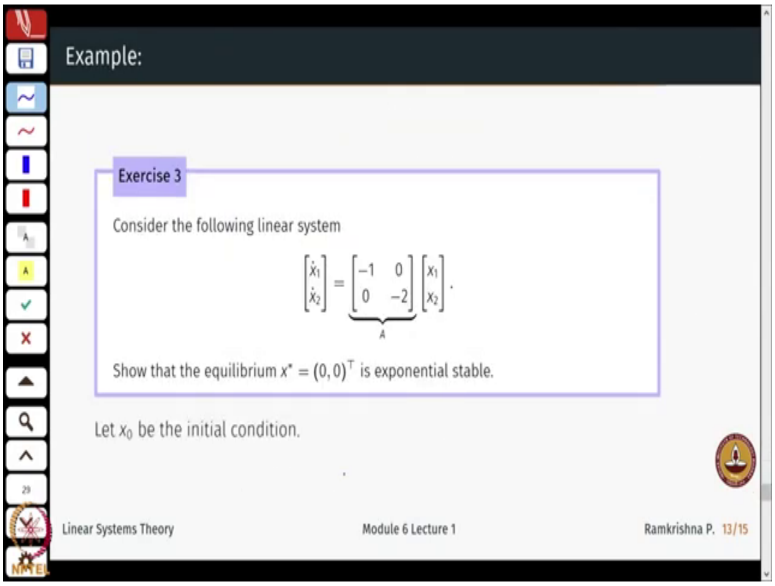

Again I come back to this to this system.

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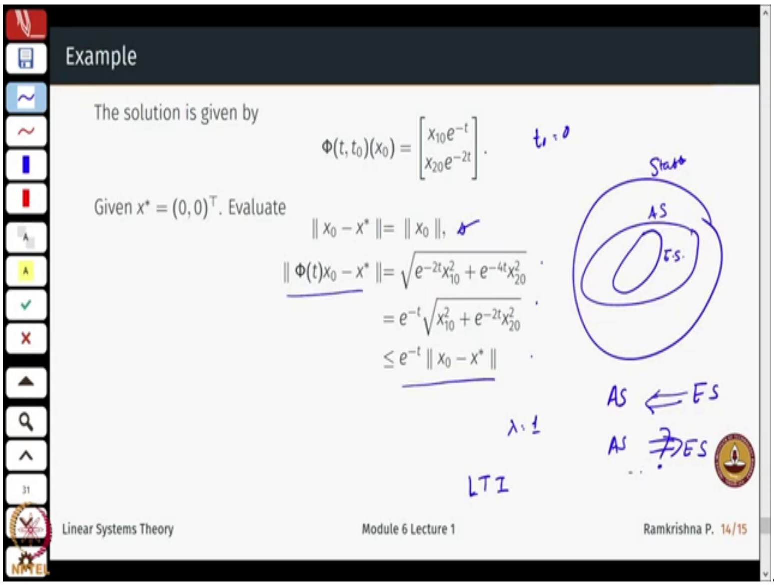

Now, I want to verify if this exponentially stable and then if it is exponentially stable, what could be the values of this alpha and lambda? Ok I will I will again skip the initial condition skip the you know, the little computations. Again, t naught could just be the t naught equal to 0, ok.

So, again this is this is from previous so what does the solution satisfy? Phi t x naught minus x star is just bounded by e power minus t. So, it decays at a rate of lambda equal to 1. So, this 3 steps can be can be easily verified. I am just computing the 2 norms here. So, that should be easy. Now, so far, well we know that this is stable.

Something inside could be asymptotically stable, it is a smaller subset and exponential stable is even a harder condition to check right. So therefore, this is the strongest condition to check would be that of a exponential stability. So, stable is just is a bigger set, asymptotically stable systems are stable, exponentially stable systems are asymptotically.

So, now relation between asymptotically stable and exponentially stable. All exponential stable systems are asymptotically stable; but are all asymptotically system asymptotically stable systems exponentially stable? Well, it turns out in the LTI case, which means the Linear time-invariant case, exponential stability and asymptotic stability would actually make like coincide, right. And we will do a proofs of this.

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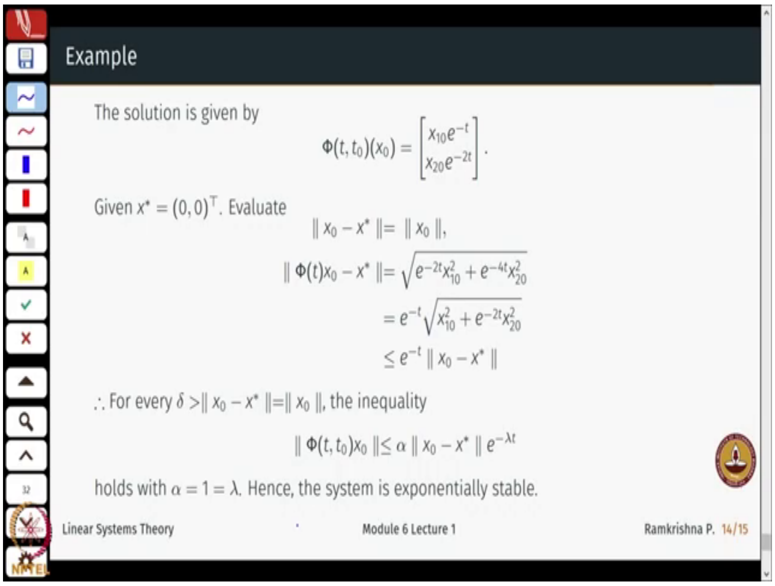

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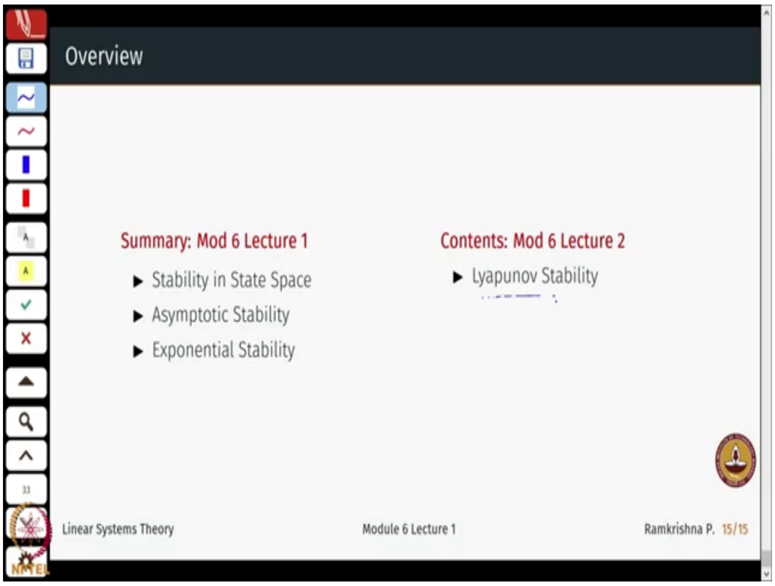

So I just conclude today by just introducing to you the 3 notions of stability. The basic definition of stability, asymptotic stability is a stronger version and something which also tells us the rate of convergence is the is exponential stability. now given a system, now how would I verify stability? Say I am given system in R 4.

Now, can I all the time compute the solutions? Compute the constants? Alpha, lambda and so on or compute various norms that may not always be easy. Now are there effective computation tools which will help us verify stability of systems?

Similarly, in the transfer function case we do not really compute the check if the impulse response is absolutely integrable or not, right. We just translate that to an easier way of verifying which is with respect to the poles.

Similarly, in the state space methods, we will next look at the notion of Lyapunov stability and identify tools which will help us verify stability certificates for LTI systems. Again, much of the tools that we will use will come from week 2 and week 3 lectures of linear algebra and of course, a part of week 4 and week 5. So, that will be coming up in the in the next lecture.

Thanks for watching.