Nonlinear System Analysis
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Lecture - 29
Lypaunov Stability

Hi everyone, my name is Ramakrishna from IIT, Madras and welcome to this lecture series of week 6 on Linear Systems theory. So, last lecture we had defined for ourselves few concepts of stability starting with just the notion of a stable system where the solutions were uniformly bounded, then you had the notion of asymptotic stability which did not tell me about the rate of convergence of my solution to the equilibrium point. And then we had the exponential stability notion which had some information on how fast my solution approaches the origin. And of course, unstable systems the systems which are not stable are unstable systems.

In many of these cases we were talking of solutions converging to the equilibrium point. And in many cases, it may be difficult to compute the solutions explicitly and check whether they are stable or asymptotically stable or not right. Even though we have the beautiful notion of the state transition matrix and computations involving status transition matrix which can give me the expression to the solution explicitly.

Now, are there better ways of verifying this. Loosely speaking can we find an analogous to the stability verification via the location of poles in when I look in terms of a transfer function representation of a system yes.

So, that will be the focus of today's lecture and we will also look at how to not only have conditions to verify stability. But how do we; have how do we even prove some notions of stability.

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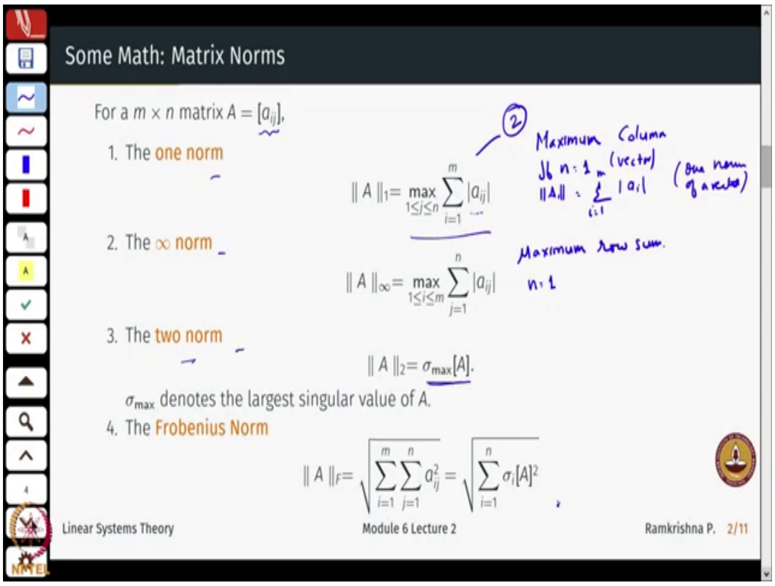

So, to begin with ok. Let us start with a bit of maths preliminaries because we will be dealing a lot with matrices here right. So, I would like to define some notions of on matrices. So, just for any m cross n matrix A. Each element of A has this form a i j the one norm is defined you know in this way. So, I am looking at like in this would could be the summation over the columns and I am looking at the maximum column sum of the matrix.

So, in if n equal to 1; I realized the standard vector case. So, I have a vector now instead of a matrix and the one norm would simply be the summation of all the elements of A say i equal to 1 to m like the standard definition of a one norm of a vector ok. Similarly, with the infinity norm here I am looking at the maximum row sum ok. And similarly I can derive that for n equal to 1 case, I will have a vector and this definition will coincide with the definition of the infinity norm for in case of a vector ok.

Next is the notion of a two norm. So, the two norm is defined in the following way. The two norm of A is just the maximum singular value of this matrix A. And a last notion is that of frobenius norm defined this way ok. We will not use much of this in this course, but just in this nice to know some definitions ok.

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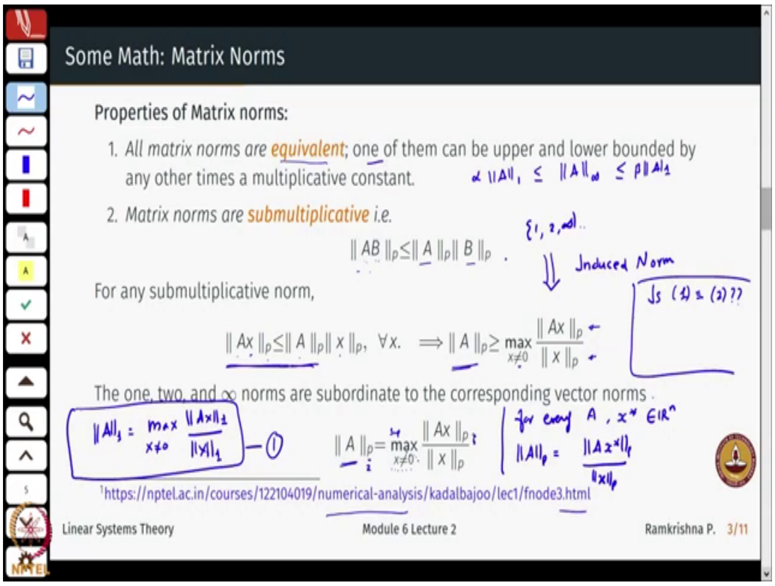

So, what are the properties of norms defined on vector space that will translate to norms on matrix? Well, one thing is obvious that all norms are equivalent like we proved in the vector space case. What does it mean by equivalence at one of them can be upper and lower bounded by any other times a multiplicative constant say, the one norm can be upper bounded and lower bounded by the infinity norm with some numbers alpha and beta here ok.

Another important property is the sub multiplicative property right. So, if I have two matrices A and B, the sub multiplicative property says that the norm of A times B would be less than or equal to the norm of A and times the norm of p. p could be whatever; p could be the 1norm, the 2 norm, the infinity norm and so.

So, from the definition of the sub multiplicative norm or the property of sub multiplicativity; we can define ok. What is also in literature if usually referred to as the induce norm ok. So, I will quickly run you through what this could be in. So, for any sub multiplicative norm. So, here A is a matrix, x is a vector.

So, A times x, some p norm could be 1, 2 or infinity is less than or equal to the norm of A and norm of x, for all x. Now this would imply that I can write it rewrite it this way that A of p is greater than or equal to the maximum with x naught being equal to 0, the norm of A x divided by the norm of x.

Slight little definition here would be or property would be that this 1, 2 and infinity norms are subordinate to the corresponding vector norms. What does this mean that, we have A of p is actually equal to the maximum or also referred to this as supremum sum over all the x. So, I just check for all the x and find what the supremum of this ratio right. A x p over p little typo here ok.

So, where does this come? So, this equality arises from the fact that this sub from the fact that these gnomes have the property that for every matrix A and I write it properly for every matrix A, there exists a vector x star in R n for which A of p is A x star p, x of p.

In p could be one two and the infinity norms. So, if I just say, just take the case of a 1 norm and this would be the maximum of over x not equal to 0 A x 1 over x of 1 right. So, this is what I would get when I just look at the 1 norm starting from the definition of an induced norm. Now is this norm which I call it 1 label it as 1 equal to the 1 norm that I defined here right. So, if I go back. So, the question is I decide it over here is 1 equal to 2 ok.

The answer is yes. It is a, little proof, not very complicated. So, I will skip the proof, but you can just follow this link of another NPTEL lecture on numerical analysis. So, this nice notes will give you a very nice exposition to the proof of this. Similarly, you can replace a 1 by 2 here you know in this call this p to be 2 and you can prove that this will actually give you the 2 norm and so on with the infinity norm also right.

So, there is a good equivalence between the definitions of the one the infinity and the 2 norm and what I derive you're why are these sub multiplicative property ok. So, of all these properties this is what we will be using in this lecture.

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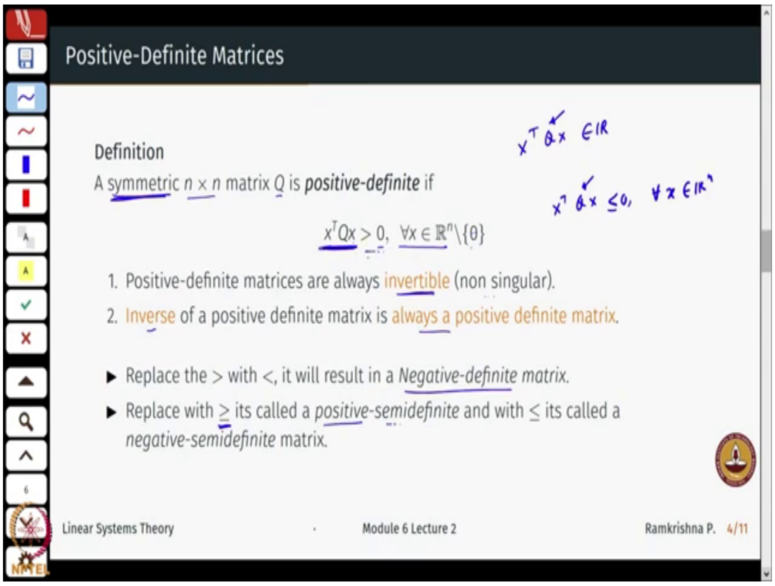

Now, a little to do with definitions or some properties of positive for definite matrices ok. What is the definition I just take an n cross n matrix Q is positive definite if x transpose Q x is greater than 0 for all x except the origin or except this is the 0 vector right ok. Now again, how do I test if the matrix is positive definite or not. Well, if I go by definitions I may have to check all possible vectors which are in R n and verify this condition that x transpose Q x should be greater than 0 ok.

So, this x transpose Q x will be in R, I that can be checked or observed quite easily right. But I would not want to do this for all x and take forever to check if a little even a 2 cross 2 matrix is positive definite or not ok. So, before we look at what are the ways to define, what other ways to check if a matrix is positive definite or not I just list some properties that such matrices are always invertible and this is also a good property that also means that they are non singular.

Another good property is that the inverse of a positive definite matrix is always a positive definite matrix. Again the proofs might be pretty simple so, I just skip ah. So, if I instead write x transpose Q x less than 0 for all x in R n, then this will define for me what I call as a negative definite matrix. And if we instead of this strict you know greater than sign if I just replace it by greater than or equal to it will be called a positive definite so sorry, positive semi definite matrix and with a less than or equal to so in this case, so this will be a negative semi definite matrix.

So, I am just talking the properties of this Q matrix positive definite negative definite positive semi definite and negative semi definite.

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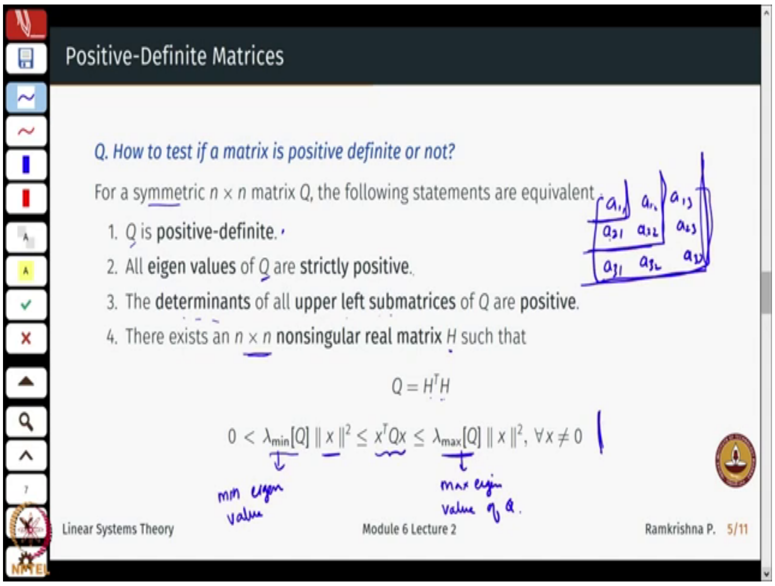

Now, how to check computationally if the given Q matrix square matrix is it positive definite or not I cannot compute all x transpose Q x for all x in R n that will take me infinitely long time, but I will just see that alternative methods of doing that. So, that is what is in the next slide.

So, Q being positive definite is equal to saying that all eigen values of Q are strictly positive right ok. Here, I am all talking of symmetric matrices only right. So, this should be stressed ok. For a symmetric n cross n matrix Q, when I whenever I say Q is positive definite, it also means that all eigen values of Q are strictly positive ok. Not only that this is I can another way of testing is a determinant of all upper left sub matrices of Q or positive.

So, let us say I am just looking at a matrix of 3 cross 3 ok. So, what is what does this mean that ok. First a 1 1 should be 0 and if I compute the determinant of the 2 cross 2 elements starting from here this determinant should also be positive. So, a 1 1 should be greater than 0, the determinant of these four elements a 1 1, a 1 2, a 2 1, a 2 2 should be positive and this also should be positive. The entire determinant of the matrix.

So, these are all the left upper left sub matrices of Q, this one, this one and of course, the additional one by itself ok. Is there another test? Well, yes, there exists. So, Q is positive definite, if there exists an n cross n, non singular matrix H such that Q can be represented as H transpose times H ok. So, again this head should all will also be invertible. Final property which again I will not do the proof, but we will use this extensively is that given a matrix Q.

So, x transpose Q x right so, which is again which will be greater than 0 when it is a positive definite matrix when Q is a positive definite matrix. So, this will be lower bounded by lambda min Q times the norm of x square and upper bounded by lambda max. So, this is the minimum eigen value and this is the maximum eigen value of Q ok. This will again be easy to verify and I will not do the proof of this. So, this is what we will use in our stability proofs ok.

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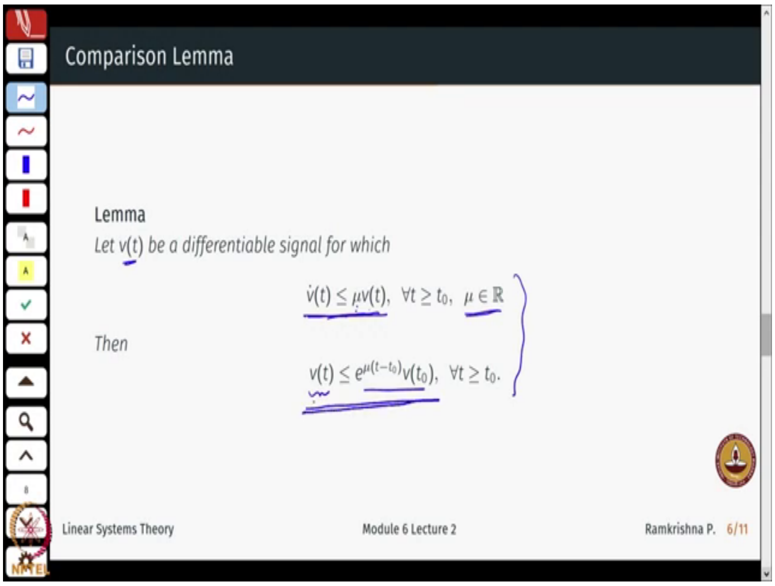

Now, another result that we will use later in the proofs is the following and I will do this the proof of the comparison lemma in a supplementary lecture after I finish the main lectures in this week 6.

So, start with a differentiable function v t be a differentiable signal for which I have some kind of a differential equation involving v that v dot of t is less than or equal to mu times v t for all times t greater than t is 0 and some mu being some scalar quantity. If this holds it turns out that v of t is less than or equal to the signal. So, e power mu t minus t 0 times v at t 0. It sounds intuitively it looks a little ok, but we will actually prove this later on, but this will be another year result that we will use extensively in this lecture ok.

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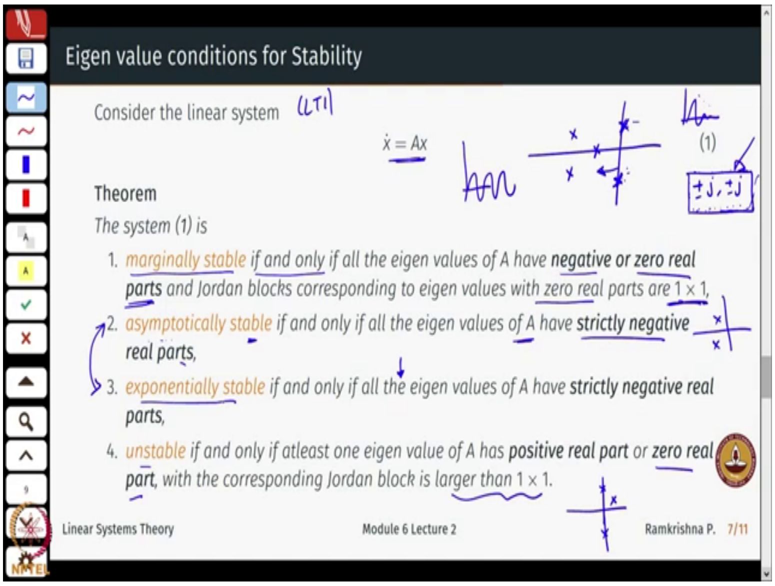

Now, coming back to how do we check for stability? Again the motivation is can I have some better test than by this verifying solutions if they are stable, asymptotically stable, exponentially stable and so on. So, the definitions here would give me some tests based on the eigen values or the properties of the A matrix ok. So, we start with the linear system again this is LTI system; x dot equal to A x. I do not explicitly consider the influence of b at the moment, but this is give you give us a nice exposition to what is coming up.

So, the system one is marginally stable if and only if right. So, it is it works both ways, all eigen values of A have negative or zero real parts. So, like very much to do with the pool see the pools are here, the eigen values are here, here and here. This is all stable ok. And not only this so, this zero real part is comes with one more condition that all Jordan blocks corresponding to the eigen values with zero real parts are 1 cross 1.

So, what does this mean? So, if I have a pulse of the palm plus minus j, plus minus j like repeated poles on the imaginary axis this system shows out to be unstable you can say check for this condition. They said also in any earlier control lectures of basic undergrad control engineering would tell about this side. If they are repeated poles on the imaginary axis then that leads to instability ok.

How does this how the solutions look like saying in this case ah? So, this poles will have exponentially decaying term. So, this will in some sense have some kind of an oscillatory behaviour essentially because of the poles on the imaginary axis. If there are no poles on the imaginary axis, you might actually find that the oscillations might just die down right. Depending on what kind of damping the is exhibited in the system ok.

So, back to definitions of stability or how do we verify with eigen values, then the eigen value should either be on the imaginary axis or to the left of it. The only caveat here is that if they are on the imaginary axis I need to do an additional test corresponding to the Jordan block ok.

Asymptotically stable is easier, this is any that I just have to check if all the eigen values should be on the left half plane, no eigen values on the imaginary axis. So, the definition of the theorem says that the system is asymptotically stable if and only if all eigen values of a have strictly negative real parts. Similar thing holds also for the exponential stability case in the LTI case.

So, that is what we had claimed, not claimed last time, but we had I had mentioned to you that in the case of linear time invariant system asymptotic stability is also equal to exponential stability. It may not be true in other cases of non-linear systems, but we are not interested in that. But we will slowly prove this in this lecture.

Last thing about unstable system; well, system is unstable if and only if well if there is one eigen value on the right half plane then it is unstable. Again if there is there are imaginary eigen values or eigen values with zero real parts, then the corresponding Jordan block should be of size larger than 1.1, for example in this case right.

So, that is the only thing that we must be careful of. What we will prove is not are not things rated to instable or unstable systems. I will just prove conditions related to stability. So, let us first begin by showing that that for LTI systems asymptotic stability is equal to exponential stability ok.

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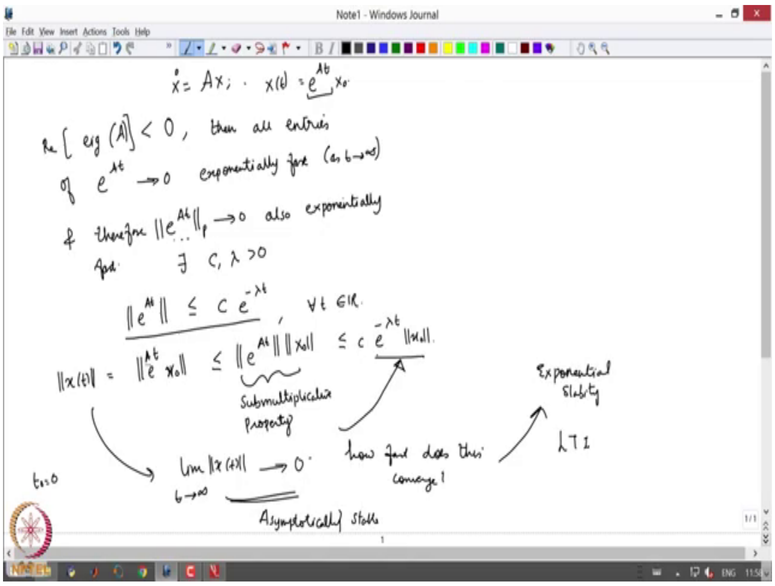

So, I start with again x dot is A x with solutions of the form x of t is e power A t x 0. x 0 being the initial condition ok. So, well, lot of it depends on e power A t and what we know that if the eigen values of A have strictly negative real parts right. So, the real part ok, let us say the real part of the eigen values are strictly less than 0, then this would mean that all entries of e power A t converge to 0 exponentially fast. That, so, as t goes to infinity. So, this the entries converges exponentially fast and therefore, e power A t converge this to 0 also exponentially fast sorry.

So, it is could be for any matrix norm, one norm, two norm or the infinity norm. And therefore, this would mean that there exists constants c comma lambda greater than 0 such that e power A t is less than equal to c e power minus lambda t and for all t ok. Now how does the solution look like. x of t, the norm is e power t let me assume t 0 to be equal to 0, just for simplicity e power t times x naught. So, this will be less than or so, e power A t. This will be less than or equal to again e power A t x naught. So, here I am using the sub multiplicative property of matrix norms right. And now with this expression, I know that this is c e power minus lambda t with x 0 ok.

So, what does this mean that the solution ok, now check what happens to x t as t goes to infinity ok. First thing I can obviously say that, this convert this to 0 as t goes to infinity again assuming that all the eigen values of A are strictly less than 0, which means e power A t converges exponentially fast and so on ok.

Now how fast does this converge? So, even if I, even before I ask this question so, this is this is already tell me that the system is asymptotically stable ok. How fast this is converged and I go to the right hand side and I see that the solutions can always be upper bounded by some exponential curve here, exponentially decaying function here.

And therefore, this also leads to the equivalence between asymptotic stability and exponential stability for a linear time invariant system. These two are indeed the same concepts right. So, again the steps are pretty neat to follow. So, what we showed here is the following that well that, asymptotic stability would mean that; all eigen values of A have strictly negative real parts not only that whenever A has eigen values which are strictly negative in their real part. This also means that they are naturally exponentially stable right.

So, that is the little proof of the equivalence between asymptotic stability and exponential stability for LTI systems. All the time again computing eigen values may be computationally difficult for me or computationally expensive even though I am doing it on a computer say by a mat lab. For example, for a large sized matrix it might take me a very long time. Now are there another or better kind of conditions that I can check which are maybe computationally efficient for me to verify if the stability if the system is asymptotically stable or not.

Now, I just I am interested in asymptotic stability or I will just use one were either asymptotically stable or exponentially stable and they would mean the same.

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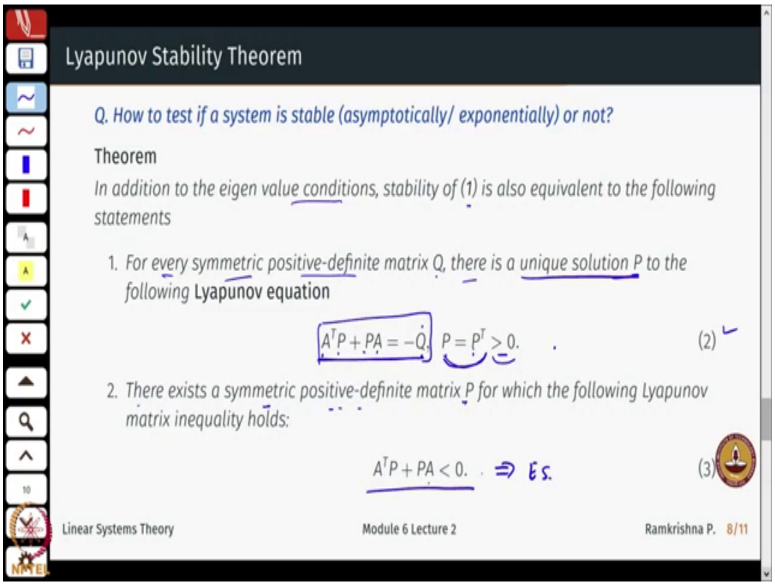

So, these are two nice looking conditions that in addition to those eigen value conditions stability of the LTI system is also equivalent to saying in the following that give me any matrix Q right for every right. So for every symmetric positive definite matrix Q stability would mean that there exists a unique solution P to this following Lyapunov equation right. So, given Q, A is the system matrix, can I find a P? So, if a P exists this and this then this system. So, the solution to this if I if a solution exists to this equation then the system (Refer Time: 28:11) x is exponentially or asymptotically stable.

Moreover the P is such that it is symmetric and positive definite right. So, we will prove this. Second conditions says that there exists again a matrix P which is symmetric positive definite for which this inequality holds.

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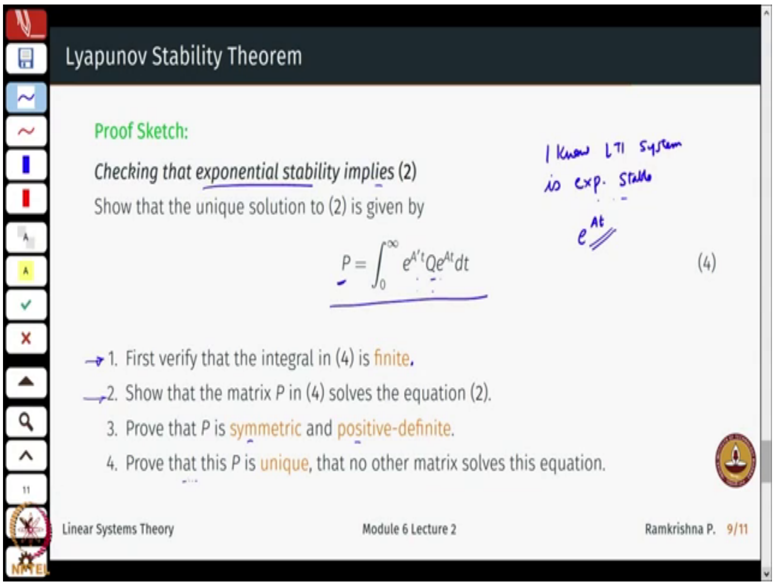

I will give you a little just of the proof and then we will go into the details of it. So, first we need to understand what is given to us and what is to be proven. So, the part one of the proof goes in the following way. So, we will check that when the system is exponentially stable it actually implies this equation ok.

So, we will begin by showing that the unique solution is given by this indefinite integral from 0 to infinity and blah blah blah ok. So, I know the system is exponentially stable right, not that the system, the LTI system is exponentially stable. So, first I need to check the properties of this integral. That is I integral actually converts is that this, this is actually finite ok.

Second; I need to show that if the system is exponentially stable which means e power A t converges exponentially fast and so on that if this is true then this P actually solves this equation ok. Not only that, we also should show this extra properties. At this P whenever it solves that equation is actually symmetric and positive definite.

Lastly; we also should be able to show that this P is unique. Because I am looking here at a unique solution p to the to this what I call as the Lyapunov you know Lyapunov equation ok. So, what does stability mean? Given a matrix Q, can I find P which solves this equation such that this P is unique it is symmetric and positive definite ok. Let us so let us do this steps one by one.

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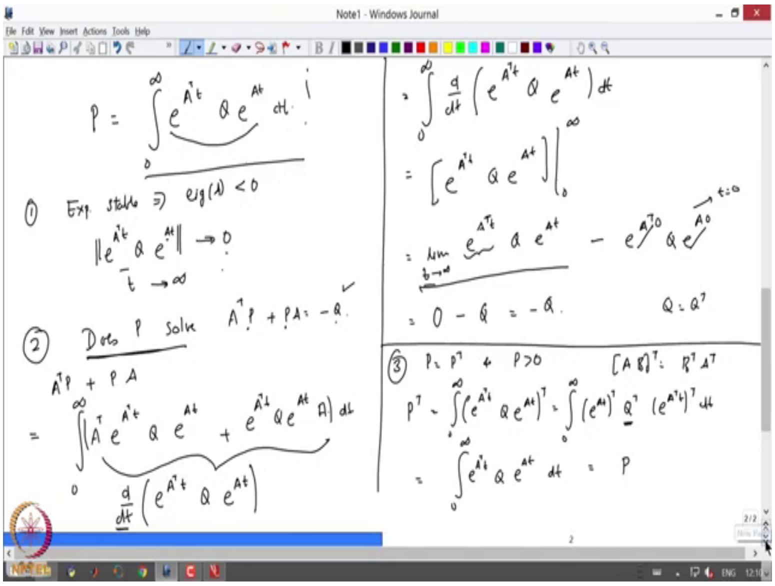

So, I have P 0 to infinity, e power A transpose t, Q e power A t d t ok. First is the integral well defined or does this integral actually exist that at 0 to infinity of some function should not give me infinity ok.

So, what do I know? I know that the system is exponentially stable which also means that I the eigen values of a are strictly less than 0. I am talking of the real parts of eigen values of a being strictly less than 0. And therefore, if I look at this quantity e power A transpose t Q e power A t ok. So, this will converge to 0 exponentially fast as t goes to infinity right. And therefore, I can say that this integral is absolutely convergent or this limit actually exists right that is. So, what am I making use of the proper property I am making use of the property that A A is an exponentially stable matrix and that is what gives me.

So, conversely if A is unstable then this would not go to 0 right. This quantity would possibly blow up and go to infinity as t goes to infinity and in that case this integral is not well defined ok. So, this is the step number 1 is done ok. Step number 2; now does P solve this equation A transpose P plus P A is minus Q ok. So, A transpose P plus P A is ok, I just substitute this and this equation. So, I will substitute for P, it 0 to infinity A transpose. So, what is P? P is e power A transpose t Q e power A t plus P. So, P is again e power A transpose t Q e power A t times A d t so, ok.

Closely look at this quantity inside the bracket. So, this is essentially d by d t of e power A transpose t, Q e power A t ok. Now what am I left with. So, if I replace with the d by d t, I have integral of 0 to infinity d over d t, e power A transpose t, Q e power A t d t ok. Now this is look simple now right. So, I have e power A transpose t Q e power A t with limits from 0 to infinity ok.

So, this is what so, the first one would be, so I am looking at the limit as t goes to infinity e power A transpose t Q e power A t minus ok. So, when t equal to 0, this will be e power A transpose 0 Q e power a 0. I am just looking in this the second term as t equal to 0 and the first term as t goes to infinity ok.

Now, look at this term carefully, what happens to this term as Q goes, as t goes to infinity. I know that, A is a stable matrix exponentially stable matrix. So, this entries will go to 0 as t goes to infinity. So, I am left with a 0 here, minus so, what is this is the identity this is the identity. So, I am just left with a minus Q ok.

So, what I have shown here is that this P solves this equation right. I substitute for P here and I get Q right. So, I know now that P which I defined it in this way actually solves on my equation. Now is that enough well not really. Third step would be to check if P is P transpose and if P is positive definite ok.

The first step should be easy to check start from here, what is P transpose? P transpose is 0 to infinity, I just have this three matrices and I just invert the transpose formula. So, I think if I write it correctly A B transpose is B transpose A transpose ok. So, this will be e power A transpose t Q e I will just I will be write a little more elaborate steps I have e power A transpose t Q e power A t the transpose of this.

So, this will be 0 to infinity the transpose of e power A t Q transpose e power A transpose t the transpose of it ok. So, the first term would be e power A transpose t. Q is symmetric, symmetric is what Q equal to Q transpose? So, I decide it as Q and the third term will be e power A t d t and P transpose therefore, is equal to P.

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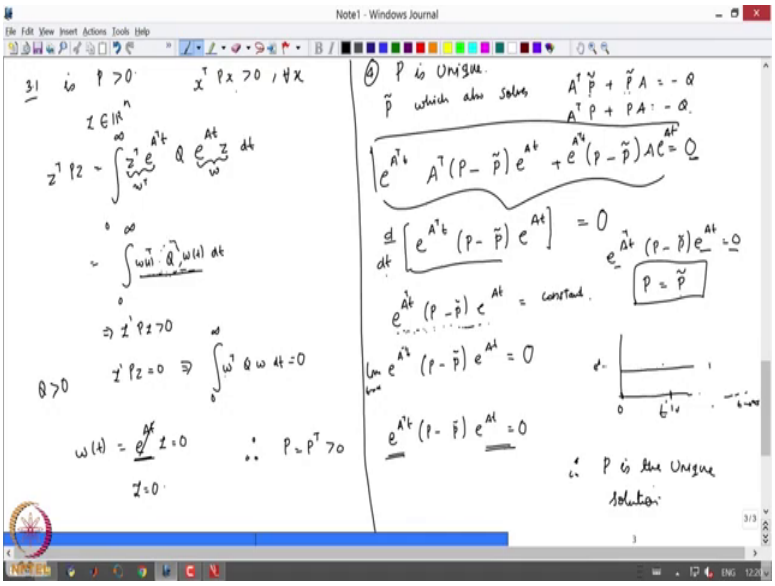

Now, this is done next is say 3.1 is P, a positive definite matrix. What is the definition of a positive definite matrix that x transpose P x should be greater than 0 for all x ok. I do not really do an eigen value test here, but I just do by definitions ok. Just take an arbitrary vectors z in R n and then compute this quantity z transpose P z. That is 0 to infinity z transpose e power A transpose t, which is again the definition of P. e power a t times z times d t. Now this is some vector right, we call it w. And then this therefore, will be w transpose I have sorry, 0 to infinity, some vector w transpose this is again depending on time Q and w t ok.

Now, what do I know? I know that Q is greater than 0 and therefore, if Q is greater than 0, then this quantity should be greater than 0 ok. And therefore, I have that z transpose p z is actually greater than 0. Given that Q is also positive definite and then I just invoke the definition right. So, if I have a vector transpose Q times the same vector that should be greater than 0 and this will hold for all z.

So, I am not I am not defining any particular z here, but this will hold for all z. So, when whenever ok. So, z transpose P said being equal to 0, this would imply that this thing is 0, w transpose Q of w is 0 ok.

Now, when can this happen, this can happen if w t is 0 right. From the definition of the positive definite matrix and this will be 0. So, what was the definition if we recall go back right here. So, anywhere apart from the origin it is greater than 0. So, add when x equal to 0, it is obviously, equal to 0 ok. Which means that this should be 0; that is e power A t times z is equal to 0.

And I know because A, because A is exponentially stable matrix that this guy will not be 0. So, the only option left is z equal to 0 ok. And therefore, we have proven that P is not only symmetric, but is also positive definite right. So, we are done now with this properties that in this step.

So, I just verified that the integral is finite, not only that I also showed that P solves equation 2 right, the which was essentially this equation. Third step I proved that p is symmetric and it is also positive definite. Last one is to show that this p is actually unique that you cannot have some other P 1 or P 2 or P prime which solves this equation ok.

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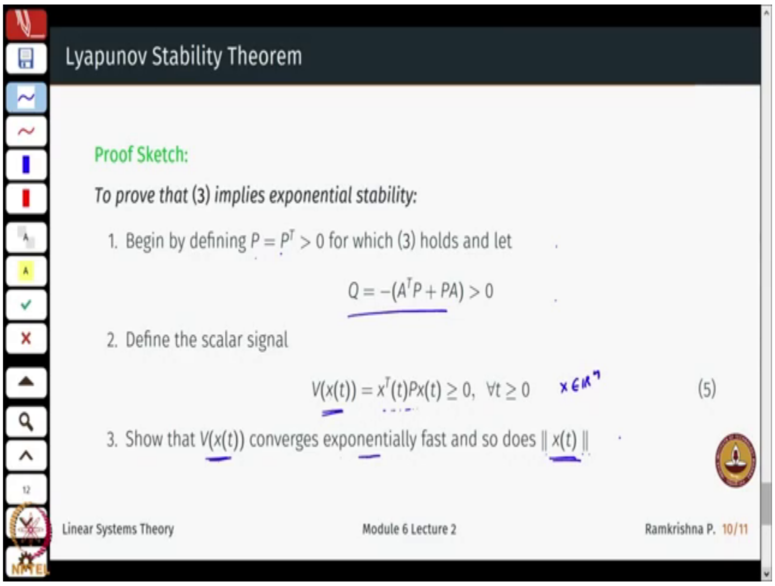

So, how do we go about proving this and much of proof sometimes are also by contradiction. So, I assume or I begin with assuming that let there be some other matrix P 1 which also solves this and see what happens. If there is any other candidate solution let me compare that with my original solution and see does there actually exist another candidate solution to this. That is what how we will prove that P is unique ok.

Now, let me prove just consider any other P tilde de which also solves this thing A transpose P tilde plus P tilde A is minus Q right. What I already know from the first three points here that P is also a solution ok. So, this guy P tilde comes from nowhere and claims that I am also a solution. Now I will just verify what does this mean. So, I just do some manipulations here. So, this is will turn out to be A transpose P just write it a little A transpose P minus P tilde plus P sorry, P minus P tilde A is 0 is by subtracting these equations ok.

So, let me just do a little trick here. I just multiply this on the left by e power A transpose t on the right by e power A t similarly here. So, I am multiplied to the left by e power A transpose t on the right by e power A t. This will still be 0, the right hand side will not change ok.

So, this turns out the quantity inside that d over d t of e power A transpose t P minus this P tilde e power A t right, which is essentially what is above and that actually is equal to 0. d by d t of this entire quantity expands to this big expression here. And what do I know of this expression that this actually is 0 ok. When d by d t of something is 0, I know that e power A t P minus P tilde e power A t is some constant ok.

Now, if I show that this constant is 0, then things will be easier for me because if this goes to 0, if e power A t P minus P tilde. So, this is P transpose here. e power A transpose e power A t is 0, then P will actually be turn out we will turn out to be P tilde, that the other guy who comes and claims to be the solution is actually a solution itself. That is because e power A t is always invertible right ok.

Now, how do I show this that this is actually 0 ok. Look at this expression carefully right this is actually a constant. So, if I were to just plot this with time. So, this function of whatever this is right. So, e power A blah blah blah so, this is constant ok. So, this is constant; whatever value it holds for t equal to 0, should hold for say t equal to 10 to t equal to 100 and all the way to as t tends to infinity ok.

So, if this holds at infinity, what happens to this quantity at infinity? e power A transpose t P minus P tilde e power A t, what is the value of this as t goes to infinity? Because e is because A is exponentially stable that is what I am being exploiting all the while. So, this will be 0 right, because at infinity I know it is 0.

So, it should if it is constant a valued function then the same value should also hold at t equal to 0 same value should also let t equal to 10 and so on. And therefore, this number by itself e or this matrix by itself P minus P tilde e power A t is 0 again because the system is exponentially stable I started with an assumption. Let the system be exponentially stable and then I proved the condition number 4 right.

Now, this is 0, if this is 0, then I know that P is equal to P tilde and therefore, P is not only the solution, it is also the unique solution. In addition to it satisfying the property of it being symmetric and positive definite ok. So, that was about the proof of this ok. Now what should I prove last. So, that stability or asymptotic or exponential stability is also equal to the current through this to this statement. That there exists again A P for which the following matrix inequality holds that A transpose P plus P A is strictly less than 0 ok.

So, what are the proof steps is it is a little might look a little tricky with this steps, but we will do that one by one. So, again define we will show that whenever this holds that whenever this expression holds that this will imply exponential stability because I will assume this to be true ok.

So, I start by defining a P which is again symmetric positive definite for which 3 holds right, this one. And I define this quantity like this, this comes from say some equation number 2 ok. Now I define another quantity a scalar signal right.

So, x comes from r (Refer Time: 48:20) and so, this will be a scalar signal that the if I show that if v of x t converges exponentially fast, the solution by therefore, x of t the norm of x of t will convert exponentially fast and this will also mean that the solution of the our original system converging actually fast resulting in exponential stability ok. We will write that down one by one and then check for ourselves ok.

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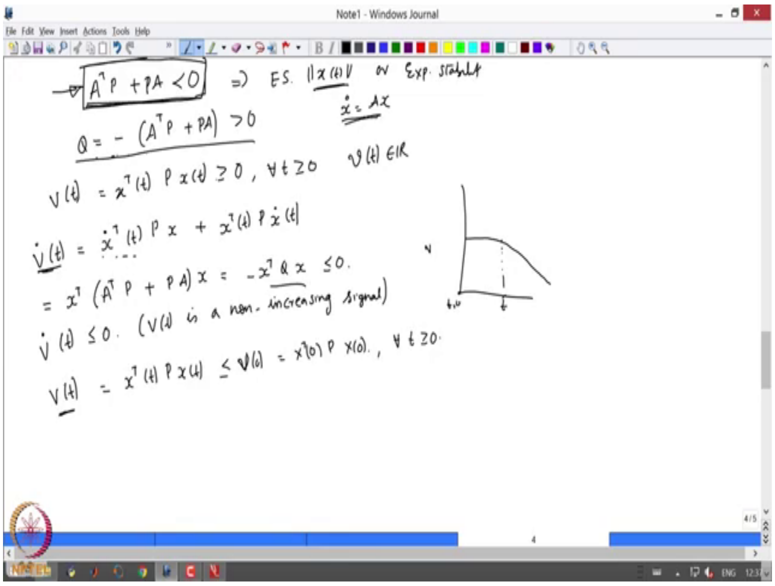

So, the idea is that ok. I should show that if this is less than 0, then the system is exponentially stable or the solution x of t converges exponentially fast ok. So, let us say assume that Q is such that A transpose P plus P A, that this is now greater than 0 ok. So, if a condition like this is holds then, this will be obvious ok.

Now define a function v of t is x transpose t P, x of t. I surly tell you what is the physical significance of this function, but for the moment we will stick to the proof of what we are supposed to do the proof of exponential stability. Now I take the time derivative along the system trajectories. So, this will be computing x transpose t P x plus x transpose t P x dot of t ok.

Now, what do what is x dot? x dot is always A x ok. So, what I get here is this will be x transpose A transpose P plus P A x, what is this? Minus x transpose Q x and this is less than or equal to 0 ok. So, this is what I have now ok. Now just look at this v dot it is just a function right so v dot so, v is a real valued function. And if v dot ok, what is v dot is now less than or equal to 0 ok?

So, what does this mean? And this means that v dot or v of t is a non increasing signal and if v dot is non increasing, then v of t just say I take any arbitrary time t P x of t would be less than or equal to the value of the signal v at 0 is x transpose 0 P x of 0. And this holds for all t greater than or equal to 0.

So, what does this mean that v is a non increasing function. So, v can either it may be constant or it might it might decrease or whatever right. So, the value at any time so, this is t equal to 0. So, the value of v at t equal to 0 will always be greater than the value at some other time t or the value of v at some time t will be either less than or equal to it is value at t equal to 0 ok.

How do we prove this? We prove this by just taking the derivative of v along the system trajectories and assuming we know that this is true. So, the idea here is to show that satisfaction of this kind of matrix inequality leads us to exponential stability ok.

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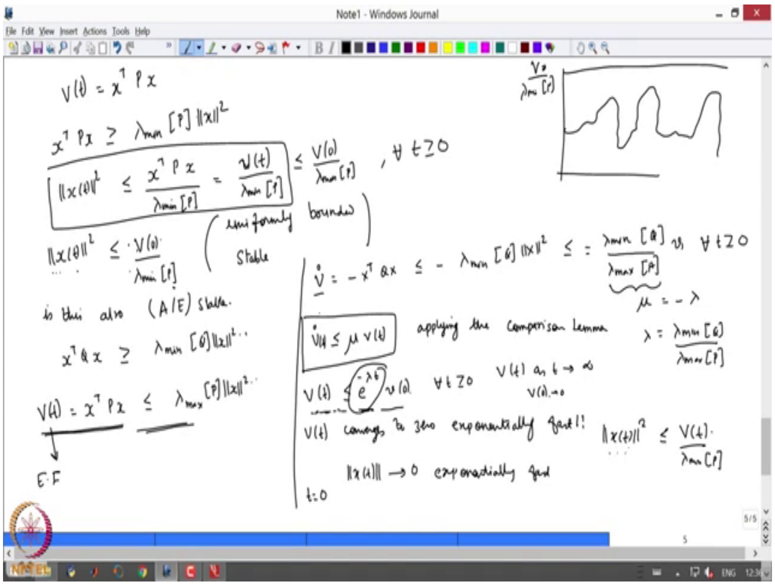

Next, so, v is x transpose P x, now this x transpose P x from the matrix properties which we listed earlier in this lecture is greater than equal to lambda the minimum eigen value of P times x square ok. So, from this what I can conclude? First, I can conclude that x of t square is less than or equal to x transpose P x over lambda min of P ok.

What is x transpose P x? x transpose P x is the function which I defined earlier v of t, this remains the same ok. This is less than or equal to v of 0, because v is a non introducing function as proved just in the previous steps and this is valid now for all times t greater than or equal to 0 ok.

So, what does this mean if I just take these two expressions; v of 0 lambda main of P ok. So, this is this is known to me the value of the function at 0 P is also known to me right because I assume that there exists A P which satisfies this inequality.

Now, what does this tell me about the solution it is tells me that the solution is uniformly bounded sorry, or whatever happens the solutions will now maybe this what here they go here and whatever they are always be bounded by this number here this guy v 0 by lambda main of P. So, this is the this is uniformly bounded and therefore, I can at least now with this step say that the system is stable marginally stable. So, this is peak ok.

Now, next step to show is this also asymptotically stable, asymptotically or exponentially stable ok. Now let us do this thing ok. Now, this matrix Q which is here, this guy here and here and so on. This matrix Q is also such that x transpose Q x will be greater than or equal to lambda min of the same Q x square, v sorry, this v of t which was x transpose P x ok. The value of this is always upper bounded by this lambda max of P times x square.

Now, look at the expressions for v dot right starting from here, what we can say is the following. That v dot which is minus x transpose Q of x, this will be less than or equal to the negative again of lambda min of Q x square ok. This will be less than or equal to sorry, I just used on top of that this inequality minus lambda min of Q over lambda max of P times v. And this happens for all times t greater than or equal to 0.

So, just a little steps which you can easily verify from all the expressions that we have written here ok. So, let me call this so, this is again this will be a number right. So, the minimum eigen value of Q which is known in the lambda the maximum eigen value P of P which is known. So, this will let me call this as mu. So, I have v dot is less than or equal to some number mu again with v of t. Now say this and this ok.

Now I have a differential equation now in v which looks something like this v dot of t is less than or equal to v of t. Now there was this thing called the comparison lemma which I stated that whenever v which is a differentiable function I know that my v there over there was a differentiable function is less than or equal to mu. Mu was a ratio of those minimum and maximum eigen values of Q and P respectively with a negative sign.

Whenever this happens then v satisfies something like this ok. So, let us exploit that and write down. So, with applying the comparison lemma, now ok. What do I have now? What can I say about v? This v of t is now less than or equal to e power minus so, e power minus of, I will call this lambda let me just say this mu is actually minus lambda of t. Let us say for simplicity t naught is the is just 0, v of 0. For all times t greater than or equal to 0. So, this lambda is lambda min of Q over lambda max of P ok.

So, what does this tell me about v? So, v is a function this lambda will always be greater than 0, because Q and P are positive definite matrices. So, they are all their eigen values will be strictly positive. This negative sign will add up to that and say that this v of t as t goes to infinity ok. We will go to v of 0 right.

Now, v of 0 could possibly be the origin also. So, this and at what rate does it do? This quantity here tells me now that v of t converges to 0 exponentially fast right ok. Now if v of t converges to 0 exponentially fast, how are v and accelerated? v and x are related why are this. So, if v goes to zero exponentially fast then, x of t will also converge to 0 exponentially fast ok.

What is the relation between x and v? So, from here I can say this if I just restrict till here I have that x of t is less than or equal to v of t over lambda min of P ok. v converges exponentially fast and therefore, x converges exponentially fast. And therefore, where did we start with we started by assuming this to be true and if this is true. This now implies that x converges exponentially fast or I have proved exponential stability of x dot equal to x right.

So, this is what a little proofs of how all these statements were equivalent. So, we just invoked a few properties of the matrices that we started off with today we invoked the comparison lemma, we also invoked the relation between the definitions of positive definite matrices and their relations with the minimum and the maximum eigen values. This proofs are nice and very intuitive also. It will be nice for you to just write down the steps for yourself. So, that you get an idea of how proofs in general are done in any control literature.

So, that might be easier for then for you then to understand any other research papers in this area ok.

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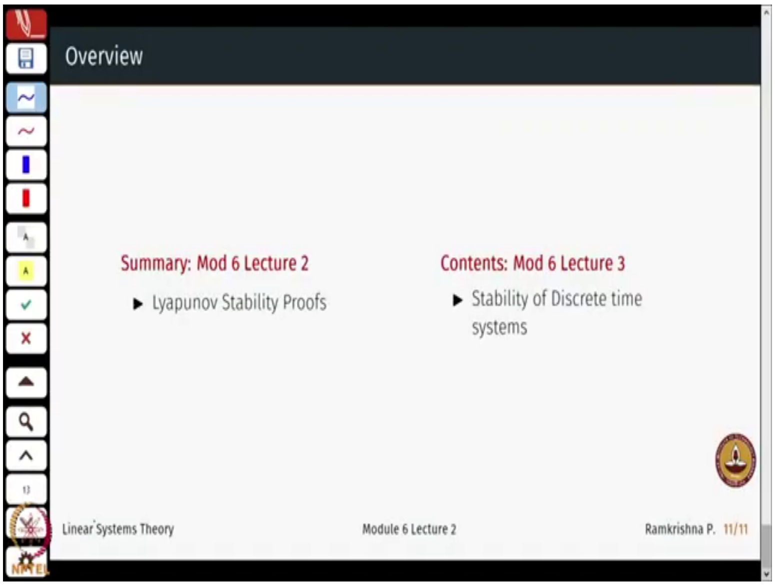

So, just to conclude what we did today was we started off with proving lots of Lyapunov stability proofs started off with giving eigen value conditions for asymptotic stability and then we had some kind of an matrix inequality like conditions to proof prove stability ah. Next lecture will be a little short I will skip all those proofs because the steps will be exactly the same as what we did in this lecture. So, we will essentially deal with discrete time systems in the in the next lecture and that is coming up shortly.

Thanks for listening.