Nonlinear System Analysis
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Lecture – 31
Center Manifold Theorem

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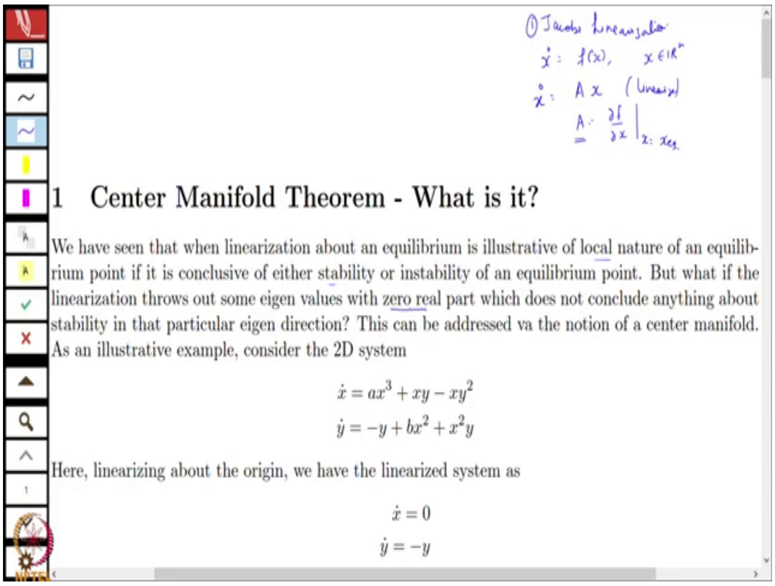

Hi everybody welcome to this lecture series on Non-linear Systems Analysis. We are in week 9 and we have done a lot of things so far now say starting from properties of non-linear differential equations still bifurcation still stability and so on. So, what I will talk of is one of the concluding lectures of stability analysis before we proceed to other topics.

What we have seen so far when we talk of stability is first we started looking and stability in terms of local linearization which is also called the Jacobi linearization or the Lyapunov's first method or the Lyapunov's direct method. There we just look at eigenvalues of the linearized system around a particular equilibrium point and remember that stability was always defined with respect to a particular equilibrium point.

So, I do not have just either systems stable or not was defined with respect to a particular equilibrium right for example, the downward position of a pendulum was a stable equilibrium whereas, the upright position was an unstable equilibrium right.

So, we talked about linearization, we talked about defining explicit Lyapunov functions right, so and then we had some methods to derive Lyapunov functions especially the basic ones could be like an energy function or a quadratic Lyapunov function then we had the LaSalle's invariance principle.

A few converse theorems about a Lyapunov stability for non-linear systems we also saw extensively of in the previous lectures of this week on explicit Lyapunov base analysis of linear systems. And different notions of stability and how exponential stability and asymptotic stability are in a way the same concepts when it is a linear time invariant system and what exactly a Lyapunov construction of a Lyapunov function means.

So, in all this analysis or even while during the bifurcation analysis we were interested or we kind of postponed some analysis to the later part, namely when I was looking at what happens when the linearization has a 0 eigenvalue right.

So, as a parameter varied we were looking at the change of or a bifurcation was defined when you are you know when you cross the imaginary axis as your parameter varies when the linearized version or the system a matrix a crosses the has a it causes a 0 eigenvalue right.

So, what to do with those 0 eigenvalues so we will I will briefly give you an introduction to the analysis which is called the center manifold theory, which will give you a method to analyse stability of a non-linear system when the linearized version has a 0 eigenvalue. And of course, a lot of these will be used in for design purposes too, we will not explicitly look at design this course is mostly based on analysis and then the design principles will follow right.

But a lot of systems before you even do the design a lot of time is spent in analysing the systems right. So, we will briefly do the center manifold theorem ok. So, first we did the Jacobi linearization right, so we start with a non-linear system let me take it autonomous x dot equal to f of x s usually a state vector in R n.

And around an equilibrium point x dot I just use the same A x was A x is the linearized version. And then A, was computed as the d f by d x evaluated as x equal to some equilibrium points. And then the nature of A told me a lot of things about local stability of the non-linear of the original non-linear system ok. So, most of it was the local nature of an equilibrium point.

It was concluded we could say if all the eigenvalues of A r on the left half plain then the non-linear system is locally stable right around that equilibrium point. Even if 1 eigenvalue of A is on the right half plain then the non-linear system is also unstable around that equilibrium point and we saw a much of examples. But we really never we never really discussed of what if the linearization has eigenvalues with 0 real part we just not really conclude anything about stability right.

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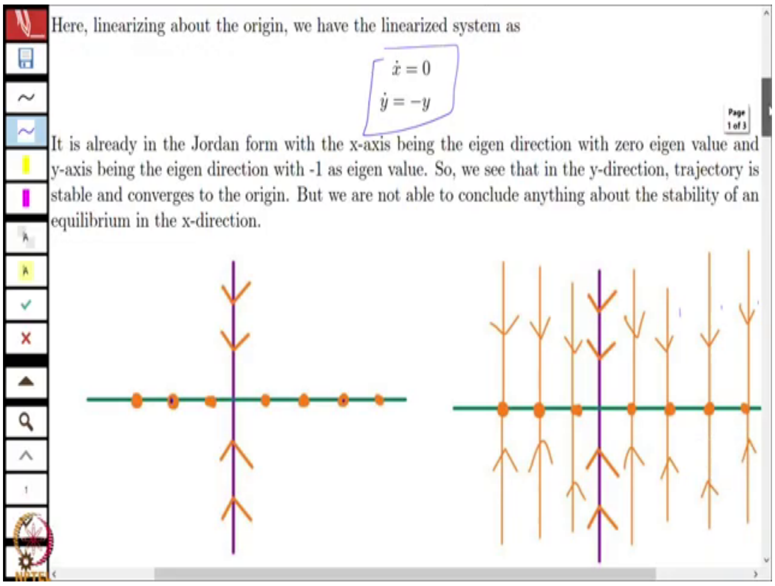

So, we will start motivating with the help of an example and then build up slowly to a theory you will understand what this means as, so now by now we are we know and a to analyse systems in terms of vector fields ok.

So, this nodes and posted on line I did not make proper slides thanks to the lockdown which we are facing here at IIT Madras well almost lockdown ok. So, I start with an example of this form x what is x cube x a example in r 2 and I just linearize about the origin to get something like this right.

So, x dot equal to 0 and y dot is the negative of y right ok, so it is kind of looks in a nice Jordan form ok. So, first looking at y what do I see right, so in the y direction the trajectories are stable and they converge to origin right.

So, y dot equal to minus y will ensure that all at as t goes to infinity y of t goes to 0 right at least the y axis converges to 0 ok. But we really cannot control anything about the stability of an equilibrium in the x direction because if I am just here I really do not know which direction to move if I just here, I am just here right x naught equal to 0 there is no direction which I could go right ok.

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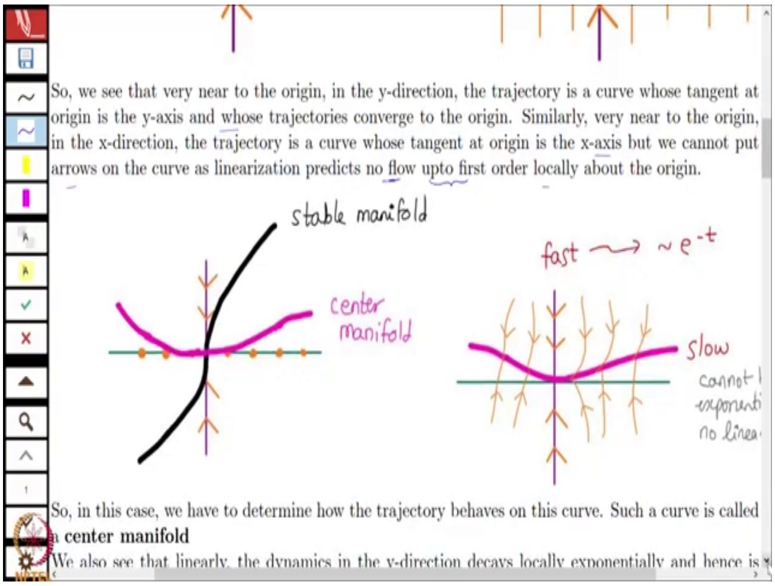

So, little observations here which will help us build up towards the theory right, so near the origin in the y direction the trajectory is a curve whose tangent at the origin is the y axis here right so this is how the y axis would be a right.

And similarly near to the origin the tragic in the x direction the trajectory is a curve whose tangent at origin is the x axis. But we cannot put arrows on the curve as the linearization predicts no flow up to the first order locally about an erosion, and this is after first order because this is how the how we do the approximation right.

So, we need to determine how trajectories behave on this curve right, so and that is what will be called as the center manifold. And we will use mathematically will write down these two expressions and check what they actually mean ok.

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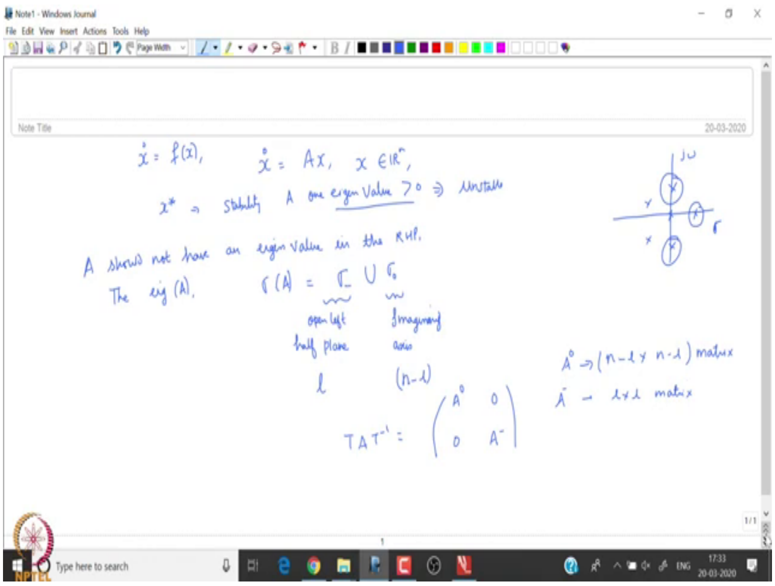

So, let us start in general with the system of the form x dot is f x and say that the linearization of this system x dot equal to A x again x being an n dimensional state vector. So, around this linearization let us say that as some equilibrium point x star for stability ok. So if A has at least or a 1 eigenvalue greater than 0 then we know that the system is unstable ok.

Then it does not really matter if there are otherwise some of the eigen values lie on the origin or say see if I have a system like this which throws up a linear is in this way. I do not really worry about what do I do with this 0 eigenvalues right or maybe there is an eigenvalue here. As because there is already unstable eigenvalue here right, so this is the sigma and the j omega axis ok.

So, we will we will rule out these cases where the system where the linearization throws up a natural instability in terms of an eigenvalue on the right half plane ok. So, therefore you know once I rule out that possibility that A should not have an eigenvalue in the right half plain then the eigenvalues of A can be written as sum disjoint union of all the negative eigenvalues and eigenvalues on the imaginary axis.

So, these are the eigenvalues in the open left half plain then these are the eigenvalues on the imaginary axis ok. So, now, say in r n let us assume that there are l eigenvalues in the open left half plane right then there will be n minus l eigenvalues which are on the imaginary axis ok.

Now what we know that well we can do some similarity transformation on A of the form T A T inverse which will split my A matrix exactly in the form which has I can explicitly write down the eigenvalues on the imaginary axis. And another component which has sorry, eigenvalues in the open left half plane right z A naught is n minus l cross n minus l matrix and A minus is a l cross l matrix ok.

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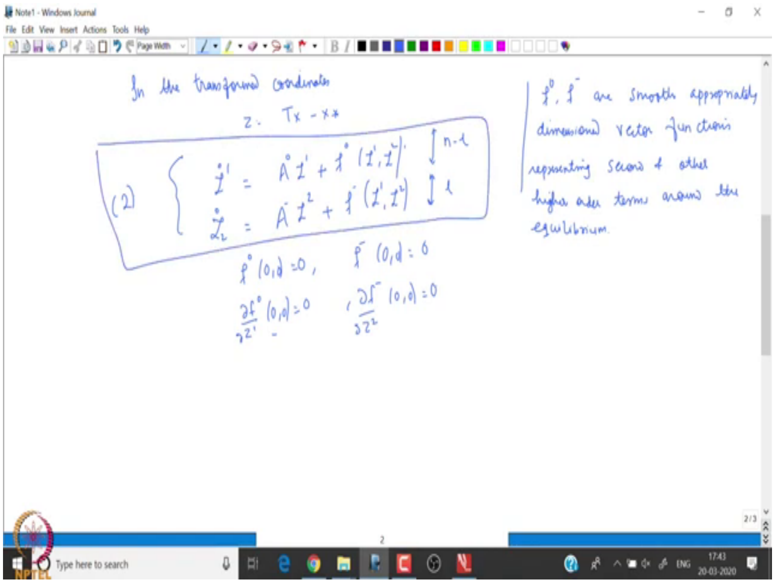

So in the transform coordinates I can write is Z is T x minus x star star being the equilibrium. I can write x dot equal to A x sorry x x dot equal to f x of the form say I have I split this into Z 1 dot. I am sorry, this is 1 here Z 1 dot is A naught Z 1 plus f naught Z 1 Z 2 is Z 2 dot is A minus Z 2 plus f with a minus sign.

I mean this is 0 and minus as usual we will denote curve the dynamics corresponding to eigenvalues on the imaginary axis and this will be eigenvalues on the open left half plain ok. So, this will be the first n minus l component this will be the last l components ok.

And what are this f 0 and f naught f 0 and f sorry f f naught and f minus are certain smooth appropriately dimensioned, vector functions representing the second and other higher order terms around the equilibrium. And also in such a way that f 0 of 0 0 equal to 0 f minus equal to 0, then d f 0 by d z 1 to 0 this like the we are computing the gradients also at the audition ok.

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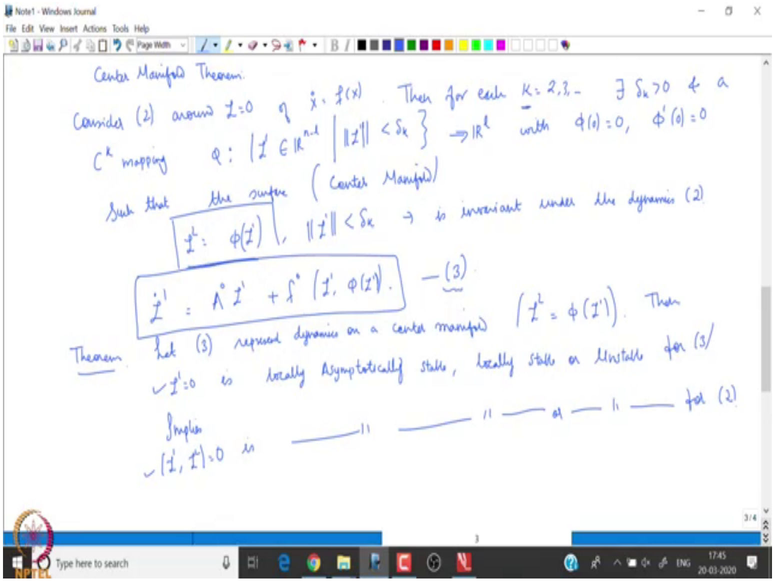

Now given this first is ok, so let us first write on what is the center manifold theorem. So, let me call this some equation say 2 ok, we start with 2 around the equilibrium point z equal to 0 of an autonomous system x dot equal to f x in the original coordinates. Then for each k is from 2 I will tell you why this restriction on 2 and 3 comes that there exists a number delta k greater than 0 and a C k mapping phi which goes from z 1.

Such that since, the normal Z 1 is less than this number delta k 2 R l with phi 0 equal to 0 phi dash the first derivative is equal to 0. Such that the surface which is called as the center manifold, Z 2 is phi of Z 1. So, I am writing all Z 1 Z 2 in terms of Z 1 for all Z 1 less than delta k right and this is such that this is invariant under the dynamics 2 ok.

So, therefore, on this center manifold my dynamics would just now look something like this then Z 1 dot is A naught Z 1 plus f 0 Z 1 phi of Z 1 ok. Now what is the theorem that then we will tell us about stability ok, so if this say call this 3 so you select 3 represent the dynamics on the center manifold this how is a center manifold given this is Z 2 is phi of Z 1.

Then Z 1 equal to 0 is locally asymptotically stable or just stable or unstable for 3 implies that Z 1 Z 2 equal to 0 the equilibrium of the original system is not locally asymptotically stable, locally stable or unstable right for 2 ok. So, I am just looking now at the stability of this equation right and if this is stable then the original system in this original transform system in this coordinates is also stable asymptotically stable or unstable.

Whatever is the property of the system will be translated to the property of the system ok. Now first is so what do I do with this result and how do I actually conclude whether a given system is stable or not. So, first what we have to look at is so I am essentially looking at this dynamics right.

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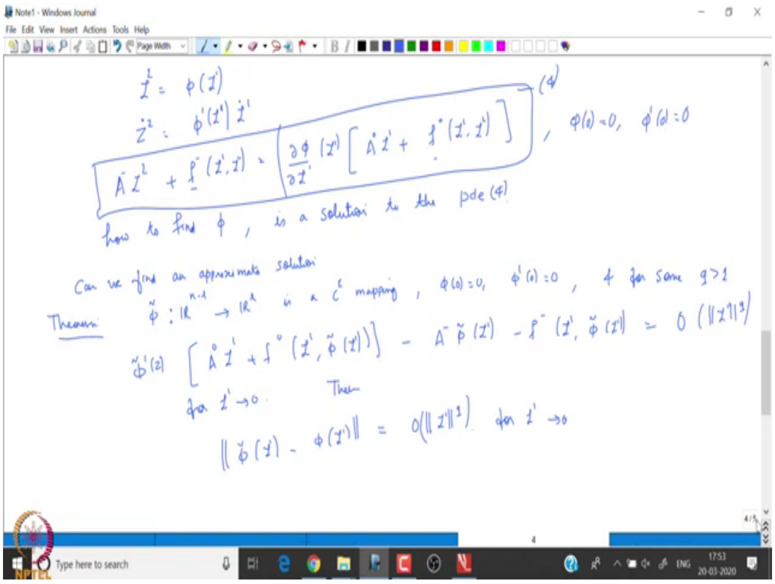

So, let us write down this here Z 2 is phi of Z 1 let us me differentiate this I have Z 2 dot is no this is phi dash of Z 1 comes Z 1 dot ok. So, what is my Z 2 dot if I just substitute here I have A minus with Z 2 plus f minus Z 1 Z 2 is d phi by d Z 1 Z 1 times Z 1 dot what is Z 1 dot, this is a naught Z 1 plus f naught Z 1 Z 2 like this right.

So, to arrive at system 3 right, so why do I want to do this because if I just evaluate stability of the system 3 then it will tell me stability or otherwise of the system 3 it will tell me the properties of the original system which is given by the system with equation 2 right.

So, right so this will again together be with conditions of phi 0 is 0 phi dash 0 is also equal to 0 ok. And then the reason we want this function to be of you know for each k is we are excluding like all this linear functions right. So, which where were Z 1 Z 2 could be like linearly re related we see that also when we when we do an example ok. So, first is now how to find phi right so if I label this as equation 4 so this phi is a solution to the partial differential equation 4.

I just have equation here, so I know what is A? B minus A naught I know possibly know these functions. So, I just could solve a p d e which may sometimes be analytically difficult to solve p d e. So, can we just look at some approximate methods which can give us some phi which is close to the solution or also equal to the solution right ok.

So, can we find an approximate solution ok, so (Refer Time: 26:20) theorem and we will learn how to use this ok. I am skipping the proofs, but we will just as I said we will just know learn how to learn this process ok.

Suppose, I have a instead of phi some phi tilde which goes from R n minus l to R l right and it is a C 2 mapping ok. Which satisfies phi naught equal to phi equal to 0 phi dash at 0 also equal to 0 right and for some q greater than 1 suppose that this phi tilde of Z times plus.

So, it 1 comma this I am writing everything in the new in the approximated phi denoted by phi tilde minus A minus phi tilde Z 1, this is of some order or Z 1 to the power q for Z 1 turn into 0. Then the approximated phi tilde and the additional phi tilde sorry additional phi are related this way.

The order of this one for is that 1 going to 0 ok, so this is so what we are trying to do is try trying to find a way to approximate the file which may sometimes be analytically difficult to compute ok. So, I just quickly do with a help of an example and then well post a couple of examples online so that will help you practice a little more problems.

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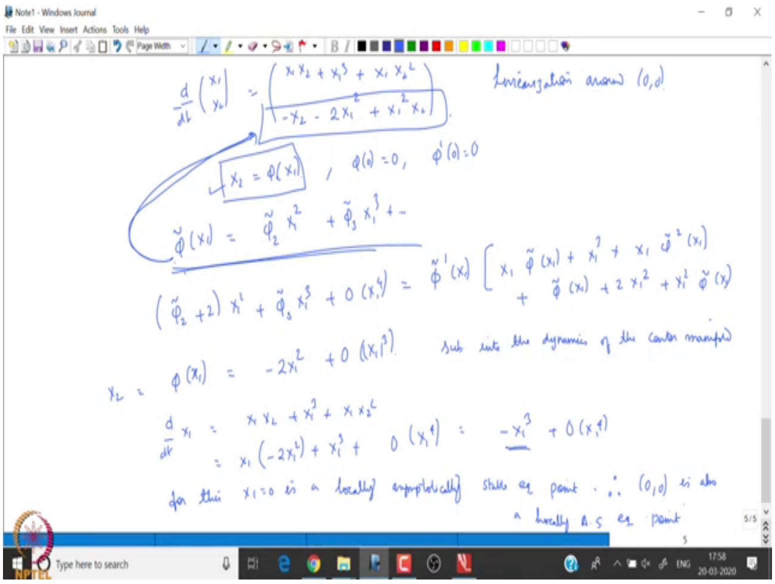

So start with a simple system which again in R 2 I have x 1 x 2 plus x 1 cube plus x 1 x 2 square I have minus x 2 here minus twice x 1 square plus x 1 square x 2 ok. So, I am just looking at linearization around 0 comma 0 and then now you see that it is it has a 0 eigenvalue and I am looking for the center manifold of the form phi 2 is phi of x 1 ok, again with phi 0 is 0 phi dash of 0 also equal to 0 ok. So, let me just say that phi tilde as a function of x 1 is this C 2 function twice differentiable function phi tilde 2 let me call this coefficient as phi tilde 2 x 1 square.

Some phi 3 tilde x 1 cube and so on, so what I do is now take this phi and then write down with this phi tilde this is my approximate phi tilde and write an equation like this ok. So, should be simple right so I will have phi tilde. So, this expression on the right hand side so let us write the expression on the left first, this is phi 2 tilde plus 2 x 1 square plus 5 3 tilde x 1 cube plus some other terms of order 4 ok.

So, what I am doing is I am just taking this phi tilde of x 1 and this substituting it here right and what are the exact dynamics here those are given by this x 2 terms here. And you are substituting this phi approximate phi over here what does this equate to? This equates to this expression on the right here.

So, that will look something like phi tilde dash x 1 times x 1 right I just substitute it here x 1 then x 2 is phi of x 1 what is phi of x 1? Is to write down this approximation here phi tilde x 1 plus I have x 1 cube plus x 1 phi tilde square of x 1 plus ok.

So, this is just so very simple thing of substituting this into this one into or into this equation number 4 as this is a little lengthy process, but it is purely mechanical. So, if you do all once we do all the manipulations what we find is the following at phi of x 1 is simply minus 2 x 1 square plus o of x 1 cube ok. Now if I substitute this into the dynamics the dynamics of the center manifold ok.

So, what do I get is x 1 dot from this expression, so d by d t of x 1 is x 1 times x 2 plus x 1 cube plus x 1 x 2 square right ok. So, what is x 2? Is phi of x 1 and from here so this phi of x 1 with x 2. So, I will have x 1 x 2 is phi of x 1 that will be minus 2 x 1 square plus x 1 cube plus other terms of order 4 that will give me minus x 1 cube plus other terms of order 4 and above ok.

Now, for this right x 1 equal to 0 is because of this minus x 1 cube is locally nothing to declare stable equilibrium point. And therefore, for the original system 0 0 is also locally asymptotically stable equilibrium point ok. So, this is a little illustration of how to analyse systems for which the linearization throws up a 0 eigenvalue or eigenvalues on the imaginary axis y are the center manifold theory.

We do not really go deep into the details of what if the systems are of order 4 and higher, but just to give you a little glimpse of how will you handle systems for which there are eigenvalues on the imaginary axis on of during Jacobi could be a linearization right. So, well put up some problems for you to solve might be there might be helpful for you to answer more solve more problems or questions by yourself and also help you answer the assignment problems.

So you can always post questions online if you have. So, this is about the theory on center manifold a very brief introduction thanks for listening.