Nonlinear System Analysis
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Lecture – 38
Relation between storage function and Lyapunov function - PR Lemma

So, earlier I had said that the storage function is a bit like a Lyapunov function; that means, if you set the input to 0 and then you look at the system. Then the storage function plays the role of the Lyapunov function. Now the Lyapunov function of course, has to satisfy certain conditions, that is one of the conditions being that the Lyapunov function must be a positive definite function and its derivative must be negative definite.

Now, the fact that the derivative is negative definite we will get satisfied because of the dissipation inequality. So, just recall that the dissipation inequality gives us something like this u transpose y is greater than equal to d dt of E; where E is the storage function ok.

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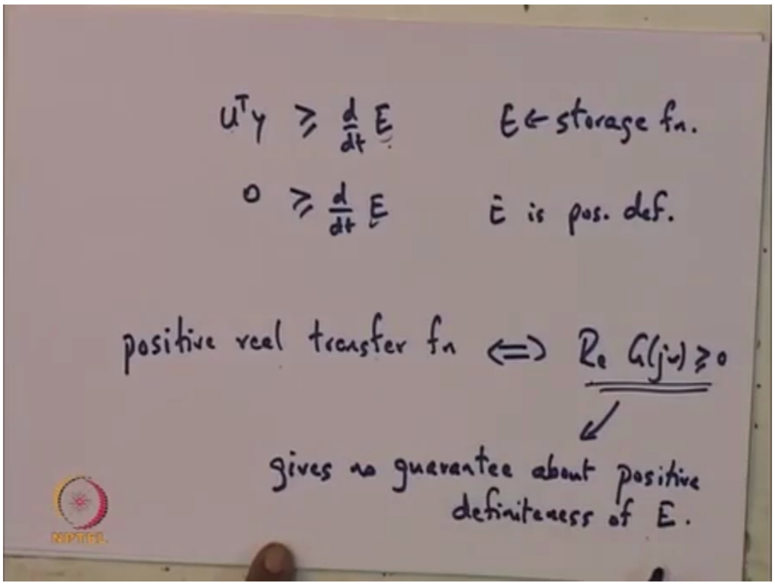

But now, what does this mean? If the input is 0 of course, this quantity is 0 and so, 0 is greater than d dt of E; which means for the system with 0 input. This E can act like a Lyapunov function provided of course, E is positive definite. So, if E is positive definite, here this thing essentially tells us that the derivative of E is negative semi definite and as a result the resulting system the system that you get by setting the input to 0 is a system which is stable. Therefore, it is important that this storage function that you get should be positive definite.

Now, when we use the definition of positive real transfer function, to be equivalent to the real part of G j omega greater than equal to 0, then this condition alone gives us no guarantee about the positive definiteness of the storage function. So, this condition gives no guarantee about positive definiteness of E ok. So, that is why a many people would like to call a transfer function positive real, when not only this condition is satisfied but, it also guarantees that the storage function that would result in the system is a positive definite storage function ok.

Now, in fact the guarantee for that is given by a famous lemma, which is called the positive real lemma and so, let me state the positive real lemma.

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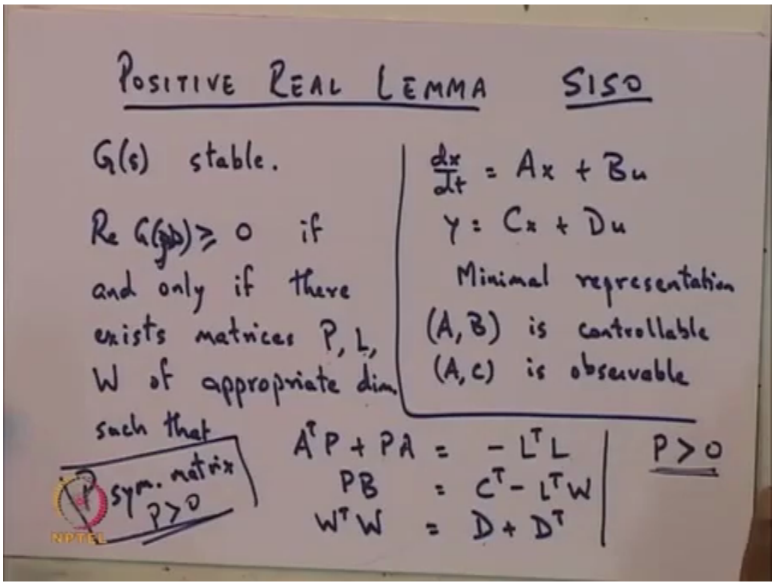

So, the positive real lemma. So, let us take a transfer function G of s which is stable and let us assume that this transfer function is written in as a state space model. And so, we you have equations dx dt equal to A x plus B u y equal to C x plus D u and let us assume that this representation is a minimal representation.

So, the minimal representation essentially guarantees that A B is controllable and A C is observable ok. So, what we are saying is, we start out with a transfer function of course, I am just restricting myself right now to single input single output case.

So, we are taking a transfer function which is stable. Look at the state space representation and the state space representation that we are looking at is a minimal representation which means the ABCD matrices are such that A B is controllable and A C is observable ok. Then, the real part of G s is greater than equal to 0. So, Gs is already stable that we have assumed. Then real part of Gs is greater than equal to 0 ok. If and only if there exists matrices P, L, W of appropriate dimensions such that ok.

So, what we are saying is, that the real part of G s is going to be greater sorry, the real part of G j omega is greater than equal to 0. That means, the Nyquist plot is going to lie in the first and the and the fourth quadrant, if and only if there exist matrices P L and W of appropriate dimensions such that the following three equations hold ok. And the three equations are A transpose P plus PA is equal to minus L transpose L. P times B is equal to C transpose minus L transpose W and W transpose W is equal to D plus D transpose ok. Let us look at these three equations and think about it for a min.

So, what we are saying is the positive real lemma what its saying is, suppose you start off with a G s which is stable and you look at the state space representation which is a minimal representation, which essentially is equivalent to saying A B is controllable and A C is observable. Then the real part of G j omega is greater than equal to 0 or in other words Nyquist plot lies in the first and the fourth quadrant. If and only if there exists matrices P L and W of appropriate dimensions such that A which is a system matrix here A transpose P plus PA of course, incidentally this matrix P is going to be a symmetric matrix ok.

So, P is going to be a symmetric matrix. And therefore, A transpose P plus PA this whole thing is going to be symmetric and what we are saying is that that this matrix is going to be minus L transpose L. Now, if you take any matrix L and you look at L transpose L that is going to be positive semi definite. I mean its guaranteed to be positive semi definite, it could be positive definite, if L had the full appropriate rank, but it is guaranteed to be positive semi definite.

So, what the first equation is saying is that A transpose there is some matrix symmetric matrix P such that A transpose P plus PA is negative semi definite ok. Now this of course, already connects P and L because P has appeared here L has appeared. The next equation connects P and L and W through B and C, where B and C come from the state representation; it says P times B is equal to C transpose minus L transpose W and this W is something that purely depends on D.

So, if you D plus D transpose this is a this is going to be a symmetric matrix and that symmetric matrix is exactly the same as W transpose W ok. So, the positive real lemma is something that holds, only if you assume G s is stable ok. So, if you take away this assumption; that means, you do not assume G s is stable, then this condition that the real part of G j omega is greater than equal to 0; if and only if these things well this does not hold anymore.

And now, the point is that this P essentially defines essentially defines the it essentially defines the storage the storage function ok. So, yeah so, there is one more condition that I need to add which is there exist matrices P, L and W with P being positive definite. Now if the condition G s is stable is put in then this P is positive definite, but if this condition G s is stable is not put in then this P is not guaranteed to be positive definite.

So, the positive real lemma says that if you take a Gs which is stable and this is the state space representation its in a minimal representation. So, A B is controllable A C is observable, then the real part of G j omega is greater than equal to 0. If and only if there exists matrices P which is a symmetric matrix and positive definite and two other matrices L and W of appropriate dimension such that this all this is true ok. So, P is a symmetric matrix and P is greater than 0, then all these equations are satisfied ok. what does all these mean?

I mean so, I have just stated a lemma called the positive real lemma, but what does this finally, mean ok. What this finally, means is that you could think of the storage function for the system.

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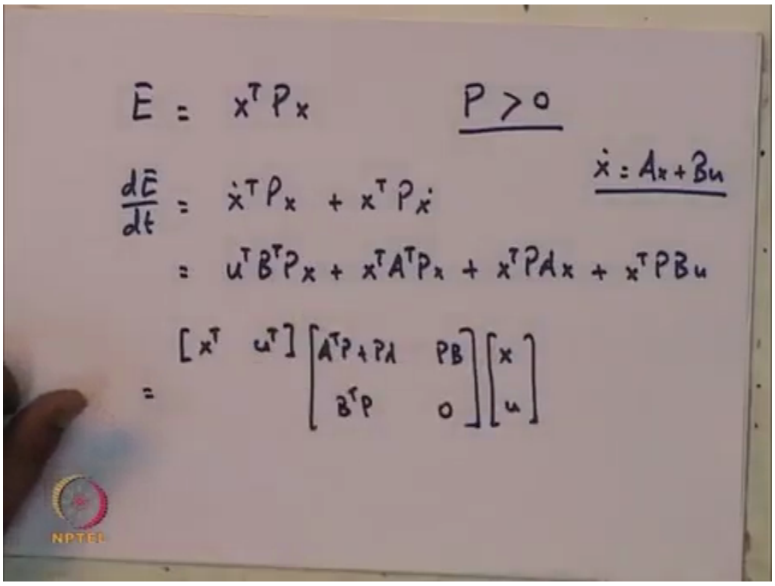

So, the storage function E one could think of the storage function as x transpose P x, where P came from that positive from the earlier lemma. So, P is greater than 0 then you are talking about so, if P is positive definite, then you are talking about a storage function which is positive definite.

So, now if you have the storage function and you look at d E dt. d E dt is equal to x dot transpose Px plus x transpose P x dot. For x dot you substitute A transpose B because you have the state equation saying x dot is A x plus B u. So, you substitute this in for the x dot and you would end up with u transpose B transpose P x plus, x transpose A transpose P x plus, x transpose P A x plus x transpose P B u ok. I will write this in a more compact form, x transpose u transpose and then I have this matrix here. And then I have x and u and out here; I would have A transpose P plus PA then I would have PB here I would have B transpose P here and 0 here.

So, this expression is the same as this. So, what we have got is an expression for d E dt by just using the state space equation and using that P that had appeared in the positive real lemma and I get this.

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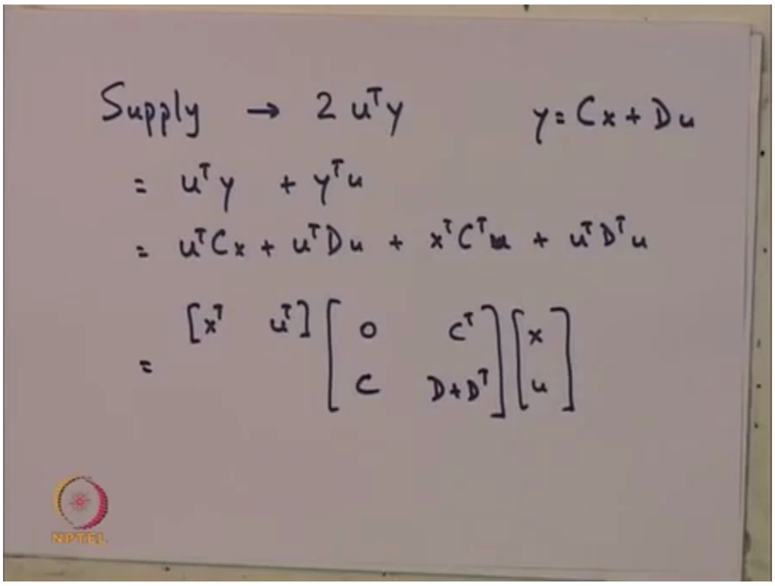

Now throughout we have been saying that the supply is given by u transpose y yeah. So, a moment here, you see I have taken this storage function to be x transpose P x. Ideally this is a quadratic one would put a half here and so, what would happen is there would be a half appearing in all these terms because of the half that I put there ok.

So, I will compensate for that by taking the supply to be 2 times u transpose y and you would see that this is this is the just a play with constants it really does not matter all that much ok. Now, so, the supply is 2 times u transpose y, but from the state space equations, we know y is C x plus D u. So, the supply which is u transpose y I write it in a symmetric form.

So, I will write it as u transpose y plus y transpose u and so, I plug in this C x plus D u in here. And I would end up getting u transpose C x plus u transpose D u plus x transpose C transpose x u plus u transpose D transpose u. Which again just like earlier I would write down as x transpose u transpose some matrix x u, there is no x transpose that term is in there. I have x transpose C transpose u and I have C here and here I have D plus D transpose.

So, the supply turns out to be this expression here.

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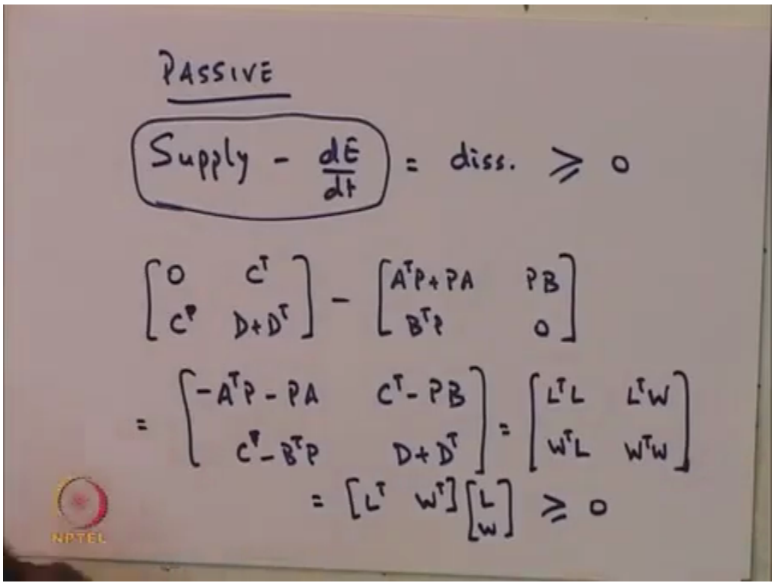

Now, for passive systems, we said supply minus the rate of change of energy this must be equal to dissipation, but this dissipation is of course, going to be greater than equal to 0. Now, we found an expression for the supply yeah. So, its x transpose u transpose with this matrix. So, x u this vector you are using on either side of this matrix. So, I will just use this matrix and sort of suppress this x u ok. So, if I write down the supply matrix I get 0 C transpose C transpose D plus D transpose minus.

Now, d E dt again I had a similar expression using x transpose and u transpose which was this. And so, if d E dt is this particular expression. So, I can subtract this expression from the earlier expression. So, what I would get is minus A transpose P plus PA here I will have PB, here I will have B transpose P and here 0. So, this whole matrix put together pre multiplied by x transpose u transpose and post multiplied by x u this must be so, what I have written down here is supply minus d E dt.

So, this whole thing must be equal to dissipative dissipated energy and so, this matrix must be a positive definite matrices what we should have, if the given system was passive ok. Now, if you put this together then the resulting matrix that you will have the first entry is going to be minus A transpose P minus PA ok.

Let me write it down minus A transpose minus PA then here I will have C transpose minus PB, here I will have C transfer no both of them are not C transpose one of them at the bottom one C so doing a mistake. C minus B transpose P and here I have D plus D transpose ok. So, this particular matrix is the same as this. Now I am going to use the expressions that I derived that I had written down in the positive real lemma. You see minus A transpose P minus PA if we look at the positive real lemma a transpose P plus PA is minus L transpose L.

So, the negative of this is L transpose L. So, for this I could write down L transpose L. Now, if I look at the second equation it says C transpose minus PB is L transpose W. So, for C transpose minus PB I could write down L transpose W and this is just the transpose of that. So, this is W transpose L and then the last equation tells us W transpose W is D plus D transpose.

So, this I could write as W transpose W, but what this matrix is this is nothing but, L transpose W transpose multiplying L W and this is certainly greater than equal to 0 because this is essentially like squaring a matrix. So, so what we are essentially saying is. So, if you go back to the positive real lemma, then you are saying G s is stable and this is true then it says that there exists all these matrices P, which is a symmetric matrix greater than 0 such that these conditions and the other two matrices L and W such that all these conditions are satisfied.

Now, what I have just shown here is if you start by considering E the storage function. If you start by considering the storage function to be x transpose P x where P is obtained from that positive real lemma. Then and if you think of the supply as u transpose y, then when you take supply minus the rate of change of storage then you end up getting this equation and you manipulate that, you get that that thing is going to be great than equal to 0.

So, what we have essentially shown is that in the positive real lemma, if all these conditions hold then if all these conditions hold then you have passivity. But as far as this lemma is concerned, we have not shown either way what we have shown is if these conditions are satisfied then the system is passive. So, now, I seem to be out of time for this lecture. And so, in the next class I would show the positive real lemma and the implication the necessity and the sufficiency, I will prove in the next class.