Nonlinear System Analysis

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Lecture – 39
Proof of PR Lemma

So, in the last lecture we I had started talking about the Positive Real Lemma. Now, I did not give the complete proof of the positive real lemma. In fact, what I had said was when you have the positive real lemma it gives an if and only if conditions. One set of conditions is in terms of matrices from the state space representation of the of the system and the other set of conditions is a frequency domain condition which deals with the fact that the given transfer function is positive real and things like that.

But what I showed in the last lecture was that when the set of conditions that is satisfied by the matrices when that set of conditions are satisfied, then passivity takes place in the sense that I showed I mean we had already discussed in the last class this thing about a system being passive is equivalent to the existence of storage function and that the supply minus the rate of change of the storage is equal to dissipation which is a strictly positive function.

And in the last lecture I showed that when the matrix conditions are satisfied then we do get some other matrix which is positive definite and that actually stands for the dissipation function. So, today what I will do is I will start first with positive real lemma and I will give the complete proof of the positive real lemma and then we will yeah carry forward with the rest of the stuff. So, let me recall the statement of the positive real lemma.

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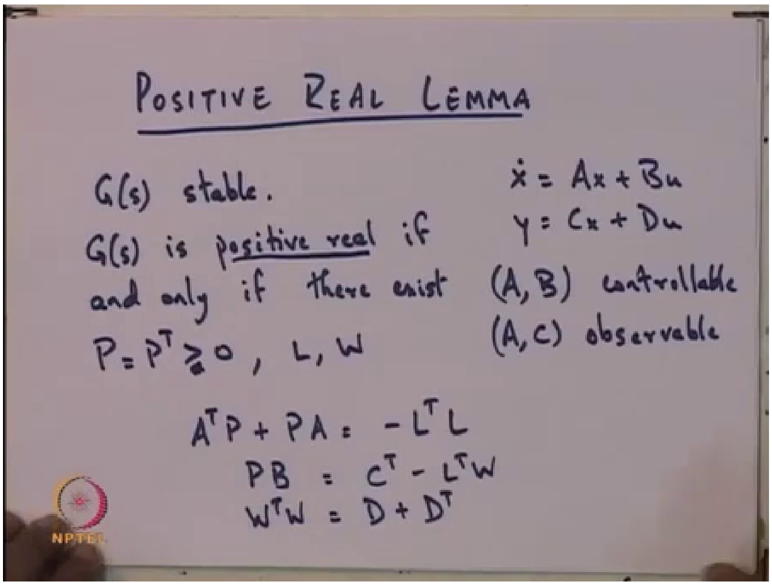

So, the Positive Real Lemma ok. So, let G s be a transfer function and let us assume that there is a minimal state realization of this transfer function which is given by x dot equal to A x plus B u, y equal to C x plus D u. Of course, because there is a minimal state representation what that means, is A B is controllable and A C is observable ok. Now, further we assume that this transfer function, this stable ok. So, G s is stable then the statement says that G s is positive real. If and only if there exists; there exists three matrices.

The first one P is a symmetric positive semi definite matrix ok. In fact, most of the times you are talking about positive matrix a symmetric positive matrix P and two other matrices L and W such that the following three equations are satisfied. And the equations are A transpose P plus P A is equal to minus L transpose L this statement essentially says that using this positive definite matrix P.

And the state matrix say you end up with this Lyapunov equation and the resulting thing is something which is negative semi definite. Now, because L transpose L is going to be positive semi definite and the minus sign will make it negative semi definite, then PB, B being the this matrix B from the state representation is equal to C transpose minus L transpose W and the last equation is W transpose W is equal to D plus D transpose ok.

So, of course, there is also this this ambiguity about what we mean by positive real and what I had said in the earlier lecture is that for the time being at least we will consider a function to be positive real. If Nyquist plot of that particular transfer function lies in the first and the fourth quadrant; that means, the real part; the real part of the Nyquist plot is always positive yeah.

And I had also said in the last lecture that of course, this definition of positive real may not necessarily be the definition that you see in all the books. Often the definition of positive real includes the fact that it is already stable and so on, but what those intricacies are I will come to that a bit later and I will explain why I mean of course, there is no agreement as to what exactly the definition of positive real is.

But I will try and explain, so that it is clear to you what are the various notions of positive real that exists and how they all related and within the epsilon neighborhood of one another ok. Now, in the last class, in the last lecture what I did show is the following that suppose we assume, that there is a G s and we assume this is the minimal state representation and we assume that there are matrices P and L and W such that these equations are satisfied.

Then what it means is that the transfer function results in a system which is passive ok. Now, the fact that passive is equivalent to positive real is something that I have talked about, but is not being completely proved. So, in some sense what I showed yesterday along with what I will show today; that means, these two conditions that this is equivalent to this should also prove that when you have; when you have the either these matrix conditions or the fact that something is positive real satisfied, then you have a passive system ok.

So, now let me begin the proof. So, in the beginning let me assume. So, I will prove this way; that means, I will assume that there exist P L and W such that these equations are satisfied and I will show that G s stable and its positive real ok; so that means, a Nyquist plot is in first and the fourth quadrants ok. Now, if you look at this first equation this is the Lyapunov equation.

Now, what we are saying is using this A you write down the Lyapunov equation and with a positive definite matrix P you are ending up with something which is negative semi definite. Now, it is well known that this is only possible if the matrix A is Hurwitz. So, if the matrix A is Hurwitz then the resulting transfer function is stable. So, the first equation already shows us the G s stable ok.

So, all that we have to do further is show that the transfer function that you get is positive real; that means, its Nyquist plot lies in the first and the fourth quadrant ok.

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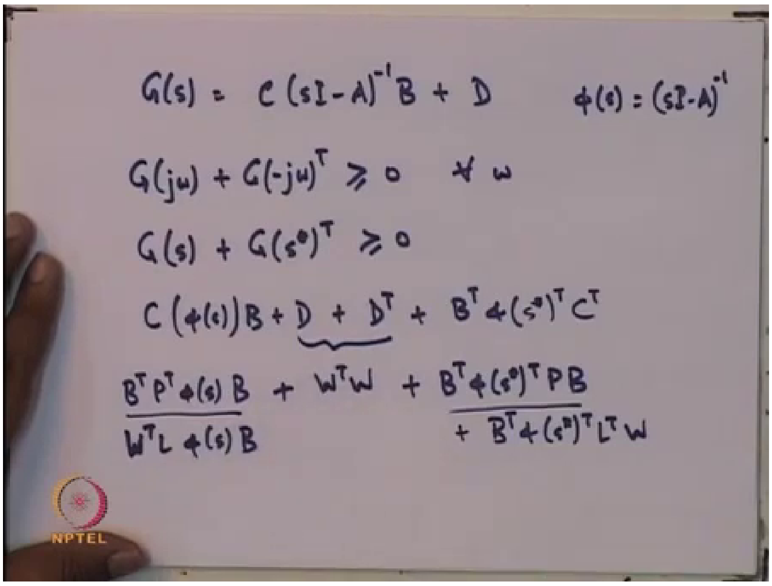

So, let us start trying to do that. So, for that let me first write down what G s is. So, G s is really C s I minus A inverse B plus D ok. Let me use; let me use the symbol phi of s for s I minus A inverse ok.

Now, what I want to show is the following that G of j omega plus G of minus j omega transpose is greater than equal to 0 for all omega. If I show this I would have shown it for the Nyquist plot, but in fact, what I am going to do is something more general than that what I am going to show is G of s plus G of s star transpose is greater than equal to 0 this is roughly what I would be showing ok.

Now, how to show this? So, I will write down the expression for this and I will write down the expression for this, but I will use phi of s instead of instead of writing s I minus A. So, this expression is C phi of s B plus D plus this expression will give me D transpose plus B transpose phi s star transpose phi s star transpose C transpose. And now what I will do is I will make use of some equations that we already have in the positive real lemma.

So, in the positive real lemma we see that D plus D transpose is W transpose W and we assuming these things are satisfied. So, for D plus D transpose I will substitute by W transpose W. So, for this one I can write W transpose W and there are these other two terms and what I would do is I will use this particular second equation for C transpose I will substitute PB minus L transpose W ok.

So, if I do that this particular expression becomes B transpose phi s star transpose times P B that is one expression plus I will have a one more expression which will have B transpose phi s star transpose L transpose W. And now similarly just like what we did for C transpose I can substitute for C it will be the transposes and so, these two guys will appear as transposes here, but instead of phi s, s star transpose I will have phi s for those two terms ok.

So, maybe I should just write them down. So, I will get another term B transpose P transpose phi s B and one more term which is W transpose L phi s B ok. So, I get these five terms. Now, out of these five terms let me concentrate. So, there are these five terms that appeared let me concentrate just on this one and this one ok. So, the let the other three terms be as they are I will just concentrate on these two ok.

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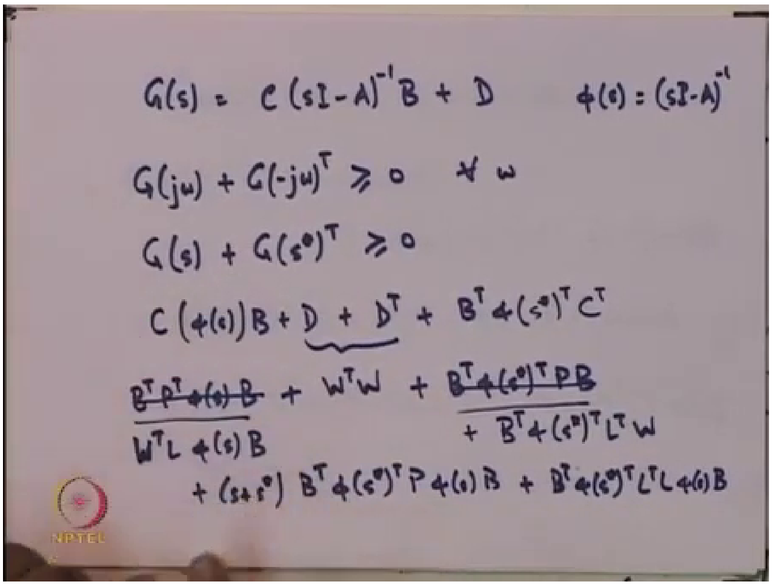

So, just concentrating on those two I have B transpose phi phi s star transpose PB plus B transpose P transpose phi s B ok. Now, I am going to do some simplification of this. So, what I would do is this particular expression I can write it down in the following way B transpose phi s star transpose. And now inside I introduce P times s I minus A and remember that we have already said that this phi of s is the inverse of s I minus A.

So, these two are really inverses. So, I am effectively writing this down, but I have written the first three terms down here and then these two actually cancel followed by B. And I do this same kind of thing for the other term also and so ok. So, sorry probably I should not have done it for this one, but I should have done it for this one I guess phi s no it does not matter. So, now, for this one I can write plus plus.

So, that one is the same as this one plus here I write down B transpose I have phi s trans star transpose and then I have s star I minus A transpose which is really the inverse of this. So, these two can cancel then and then I have P. So, this P transpose because P is a symmetric matrix P transpose is the same as P. So, I am just putting P here and then I have phi s B. Now, if you look at both the terms; both the terms have B transpose phi s star transpose in the left side and phi s B in the right side.

So, what is inside can just be put in together and so then what you have is B transpose phi s star transpose. Now, putting the things inside together you get s plus s star times P minus P A minus A transpose P times phi s times B. And now we again go back to the positive real lemma, the first equation in the positive real lemma say that a transpose P plus P A is minus L transpose L and so we can substitute that in there ok.

So, if you substitute that in there then this particular expression can be written as this particular term will give me s plus s star these are just scalar. So, I can pull them out, B transpose phi s star transpose P phi s B. So, I have just used up this much and then the other portion, so P A plus A transpose P from that first equation that should be equal to minus L transpose L.

So, I will substitute that minus L transpose L and therefore, I end up with B transpose phi s star transpose L transpose L phi s B. So, these two terms we picked up and we end up with these two terms this is a term which is sort of symmetric if you like, but multiplied by s plus s star and this is also something which is symmetric, but what you have is that L transpose L. Now, let us go back to that previous thing where you had these five terms and we picked up these two terms and did manipulations to get what we have there.

So, what I will do is into this slide I will cancel these two terms and I will add the two terms that we have got now. The two terms that we have got now are s plus s star times B transpose phi s star transpose P phi s B that is the one term and the other term is B transpose phi s star transpose L transpose L; L transpose L phi s B.

Now, let us forget this particular let me now not think about this one term and let us look at the other four terms and if you look at the other four terms you see this W transpose appearing in two of the terms and W appearing in two of the terms and so you can write this as a sum of squares ok. So, let me just write that as a sum of squares in a fresh slide.

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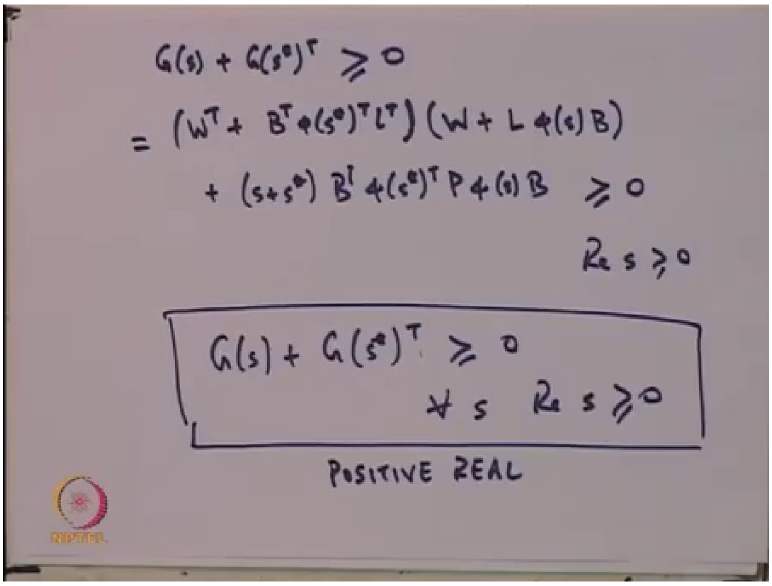

So, as a sum of squares what you get is W transpose plus B transpose phi s star transpose L transpose multiplying W plus L phi s B. So, if you multiply this out you will get W transpose W W transpose L phi s B B transpose phi s star L transpose W and then B transpose phi star L transpose L phi s B. So, you would have got this, this, this and this these four terms. Now, these four terms and then the one other thing that we have is s plus s star B transpose phi s star transpose P phi s B.

So, this is the full expression that you will get and this is the expression mind you when you started out with G s plus G s star transpose this is equal to this whole expression. Now, if you look at this whole expression this is really a square, so if this is a square this is always going to be positive and if you look here P the assumption was that P is a positive definite matrix.

So, whatever is this thing this is something acting on a positive definite matrix and as a result what you have here is something positive. And if you assume that s is such that the real part of s is greater than 0, then s plus s star this is going to be positive. So, this whole thing is going to be positive. So, what we can conclude therefore, is this is greater than equal to 0 and so effectively we have shown that this is greater than equal to 0 ok.

So, what we have done just now, is we have started in the positive real lemma we have started with this assumption and we have gone ahead and shown that this G s is positive real yeah. In fact, what we have shown is whatever I have been using as a definition of positive real it is something slightly more than that. So, what we have really shown is if you look at this slide it is clear what it is that we have really shown we have shown that G s plus G s star transpose is greater than equal to 0 for all s such that the real part of s is greater than 0.

And in fact, this should ideally be taken as the definition of positive real. The definition of positive real in fact, rather than use s greater than equal to rather than use s in the imaginary axis which means you are looking at whether the Nyquist plot is in the first or the fourth quadrant. Rather than use that it is this definition; that means, what we are really saying is that the whole of the right half plane should map under this map G s plus G s star transpose.

When you are taking this map from the complex plane to the complex plane a whole of the right half plane should map to something which is in the first and the fourth quadrant and in that case you can call it positive real yeah. And so if you use this definition of positive reality, then in the positive real lemma in the original statement you do not have to insist the G s stable because it is it has to be stable if this condition has to be satisfied ok.

But you know these are intricacies and there is no general agreement about what is the exact definition, so we will leave it at that. So, what we have now effectively shown is one way of this argument; that means, assuming that these equations are satisfied we have shown that this holds. Now, we want to show the other way; that means, if you assume G s stable then G s is positive real you want to show that these equations are satisfied ok.

Now, in order to do this I will have to invoke some other generic theorems that that are known; one of them is the spectral factorization theorem ok. Now, as we go along I would I would talk about the spectral factorization theorem; the spectral factorization theorem is a theorem which plays a central role in other fields also not just in control theory. So, that is something that we require.

And I would also invoke a lot of; a lot of things that we know about realization theory; that means, given a transfer function how do you realize state space representation ok.

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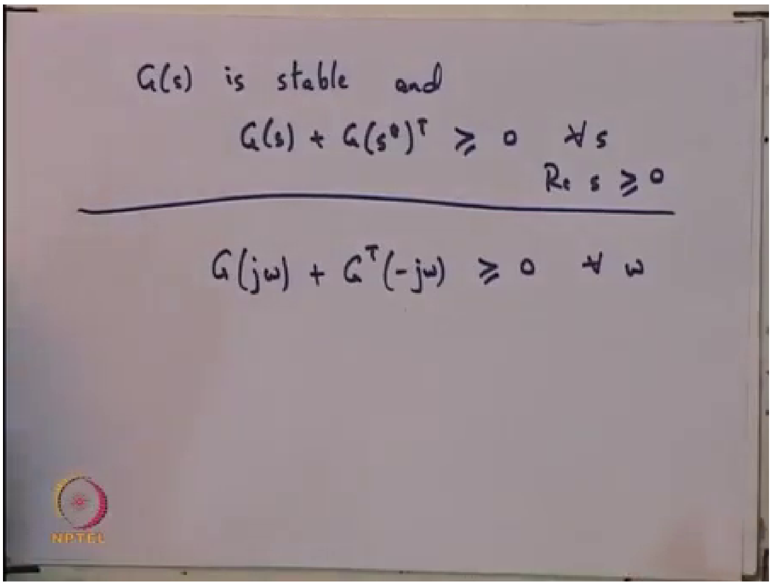

So, let me start. So we are assuming G s is stable and G s plus G s star transpose is greater than equal to 0 for all s such that the real part of s is greater than 0 greater than equal to 0.

So, this is our assumption, let us look at this this particular thing. Now, what this means is if you specifically evaluate this particular matrix on the imaginary axis, then what this tells us is j G G j omega plus G transpose minus j omega is greater than equal to 0 for all omega ok. Now, what this means is that evaluated along the imaginary axis this the this particular quantity on the left hand side is greater than equal to 0.

And of course, even though I have only been talking about transfer functions these things also hold for matrices. So, even though I stated this positive real lemma, but before that I was only talking about transfer functions which are single input, single output. We can carry out I mean the positive real lemma as it stands is valid even for G s s which are not single input single output they only constraint is that this G s must be multiple input multiple output.

So, there is there are square matrices the number of inputs is equal to the number of outputs and G s is positive real well there is a definition of positive reality for the matrices which I have not given, but I will give as soon as we finish the complete proof of positive real ok.

So, whatever I have been showing holds even for the matrix case it is not just for the scalars case even though earlier I have only talked about the scalar situation the single input, single output situation, but you could also look at the multi input multi output situation and exactly the same kind of proofs go through.