Nonlinear System Analysis
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Lecture - 40
Proof (contn'd) using spectral factorization theorem

The exact definition of what positive real is for multi input, multi output matrices, transfer functions which are matrices, I have not yet given, but the definition is such that it simplifies to exactly the definition that we had I mean the kind of definition that we had about the Nyquist plot. That kind of definition or about the right half plane mapping to the first and the fourth quadrant. Well, the same kind of thing does appear, but that I would give just after finishing the proof of this positive real lemma ok.

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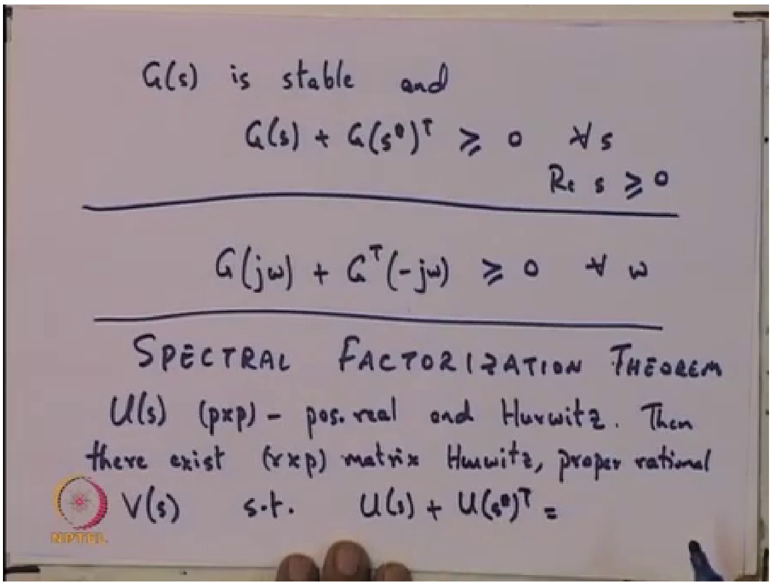

So, let us finish the proof of the positive real lemma. So, coming back here. So, this is what it means and at this point, I would invoke the spectral factorization theorem ok. So, what does the spectral factorization theorem? Say, well the factor spectral factorization theorem says the following. That suppose you have u s, which is ok. Let us assume this us is a p cross p matrix, which is positive real ok.

What we mean by a positive real matrix? Is something that I have not yet defined, but I will define just after this after the complete proof of the positive real lemma. But just assume, I mean you could listen to this portion by just thinking of u s as a scalar single input, single output case and that would be good enough. So, you assume u s is a p cross p positive real and Hurwitz; that means, stable.

Then, there exists r cross p matrix which is an r cross p matrix, which is Hurwitz it is proper rational. So, then there exists r cross p matrix which is Hurwitz proper rational, let me call it V of s, such that u of s plus u s star transpose is equal to oh sorry yeah perhaps, I will just use a new sheet.

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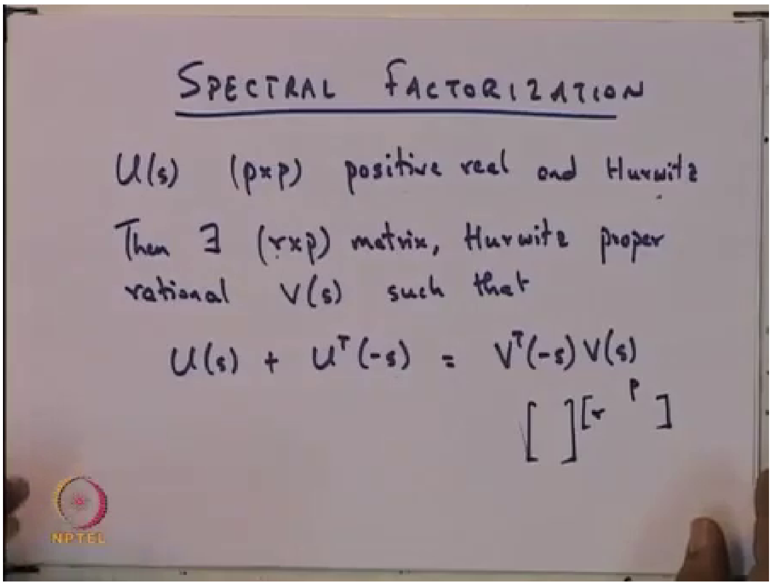

So, we are talking about spectral factorization. So, suppose you have a u of s, which is a p cross p positive real and Hurwitz matrix. Then there exists an r cross p matrix, which is Hurwitz the main stable proper rational, proper rational and let us call this matrix V of s. Such that, u of s plus u transpose of minus s is equal to V transpose of minus s times V of s ok. So, let me explain what is going on here. So, we are talking about u of s which is a p cross p positive real matrix, positive real well I will give you the definition shortly, but its a positive real u.

So, I mean instead of this p cross p 1 could just think of 1 cross 1 and so this is just a transfer function and it is positive real Hurwitz. Then there exists a r cross p well this r cross p business and the p cross p if this only appears in multi input, multi output in single input single output of course, this r will be p is 1 and r will then also therefore, be 1. 1 cross 1 there exists a matrix which is Hurwitz proper rational call it V of s such that the this matrix u of s plus u transpose of minus s. The sum of these 2 matrices is equal to the product of V s with V transpose of minus s.

And the r, comes in the matrix case essentially because when you add these 2 matrices they are p cross p square matrices it has a certain rank, and that rank is this r. And therefore, this matrix b is not really a square matrix, but it is, so V is more like this and so V transpose is like that ok. And so the and the product of these 2 matrices is a square matrix. So, so this has p columns and r rows, that is what V s is ok. Now, if you see earlier we had said G s is stable and therefore, this thing holds, but you see G of j omega plus, G transpose of minus j omega.

So, if you look at the imaginary axis you saying that in the imaginary axis its positive definite. If you think of this as matrices its positive definite matrix. So, what would happen is, when you look at the sum, this sum will have roots which are on the left half plane and roots which are on the right half plane. And then this V transpose minus s times V s is constructed by using all in the sum using all the roots which are in the left half plane you use that to somehow construct this V s. And then the V transpose minus s comes automatically.

Because of some symmetry which exists in this sum ok. Now, the spectral factorization is of course a very well-known result and use widely for example, in communication theory, but we will invoke this. So, because we have started out with this G s which is stable and has this property. Therefore, now we can use this and invoke the spectral factorization theorem and by invoking the spectral factorization theorem.

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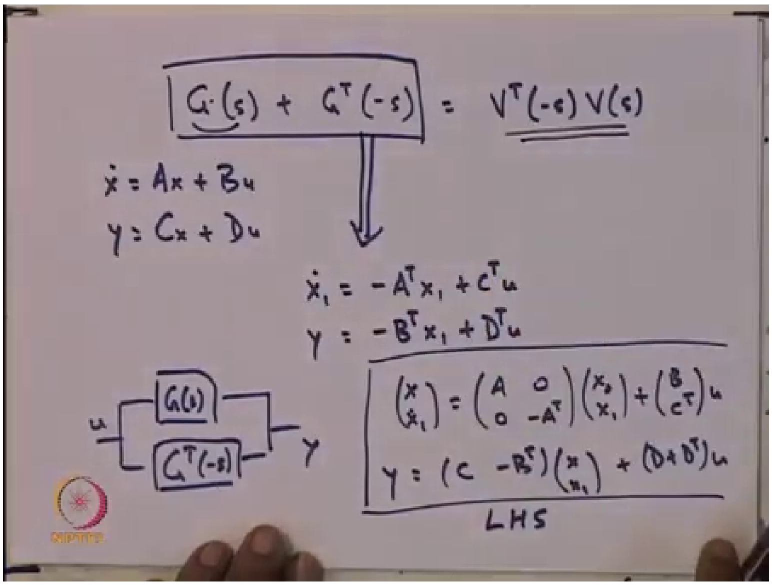

We can say that G of s plus, G transpose of minus s is equal to ok. Let me just continue calling this V, V transpose minus s into V of s. And now, what I am going to do, is I am going to start off by looking at state representations of each of these matrices ok? So, now, if you look at G of s we have already seen what the state representation of this G of s is, so it was x dot equal to Ax plus Bu, y equal to C x plus Du ok.

Now, if this is the state representation for G of s, then from here we can get the state representation of G transpose of minus s. And the state representation for G transpose of minus s would be ok. Let me call the states here x 1. So, x 1 dot is going to be minus A transpose x 1, plus C transpose u and y is going to be minus B transpose x 1 plus D transpose u ok.

Now, here we have G s plus, G transpose minus s. So, it's as if these 2 transfer functions they are adding up which is like so they are 2 systems which are parallel I mean you could think of them as this is G of s, this is G transpose of minus s out here you give u, out here you will get y ok.

So, if you now decide to look at the state representation of the full transfer function, that you have on the left-hand side. Then that is given by x x 1 dot, equal to A 0 0 minus a transpose x bar sorry x x 1 plus what you have here would be B C transpose u and y will be given by C minus B transpose x x 1 plus D plus D transpose u.

So, what we have got here, is the state representation this is the minimal state because we have said that this is the minimal state representation for G s therefore, this is a minimal state representation for G transpose minus s. And therefore, this is the minimal state representation, for what is on the left hand side. So, let me just call this left-hand side. So, what we have is a minimal state representation for the left-hand side.

Now, in the same way let us consider a minimal state representation for the right-hand side. So, in order to construct a state representation for the right-hand side.

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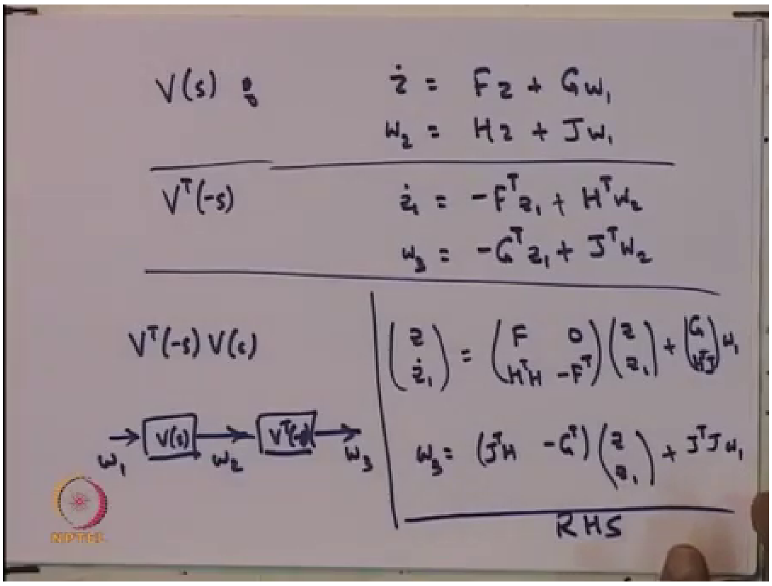

Let us assume a state representation for V of s. So, let the state representation of V of s be given in the following way. So, z dot is equal to let us say F z plus Gw and so w is the input for this V of s and then, ok.

Let me call it w 1. And then w 2 is equal to H z plus J w 1. So, if this is the state representation for V of s. Then V transpose minus s has a state representation, where the matrices involved will be minus F transpose here and here you will have H transpose, here you will have minus G transpose and here you will have J transpose.

But on the left-hand side, we were looking at V transpose minus s times V s. So, this is like saying that w 1 is an input to V of s that way, and the output of that is w 2, but this w 2 is the input to V transpose minus s and the output to that let me call it w 3. So, its a series connection, if you think about it. And so I will appropriately use w 2 as the input for this for the representation of V transpose minus s.

So, the input I will use as w 2, let me call the state that 1 I had call z. So, let me call them z 1. So, z 1 dot equal to minus F transpose, z 1 plus h plus h transpose w 2 and the output let me call it w 3. So, w 3 is equal to minus g transpose z 1, plus j transpose w 2 ok. So, now what is what was there on the right hand side, the net thing its a series thing. So, I will have to put both these state presentations together and I will end up getting a state representation for this thing. And so the state representation I will get for this is going to be z, z 1 dot is equal to ok.

So, F 0, z z 1 plus G w 1 and for z 1 dot I will get minus F transpose z 1 and out here I will get a term because H transpose w 2, but w 2 is H z plus z. So, I would get here H transpose H, and here I would get H transpose J. And the output equation well w 3, is equal to or would I have minus G transpose. So, this I have taken care of and then I have J transpose w 2. So, putting that in there I will have J transpose H times z. And plus, J transpose J times w 1.

So, this now is the minimal state representation for the right-hand side. So, if you recall there was a minimal state representation for the left-hand side that of this equation that we obtained. And we have also obtained minimal state representation of the right-hand side of that equation ok.

And so this left-hand side and the right hand side ideally, if they are equal; that means, these 2 minimal state representations are a similarity transform away ok. So, now, what we would do is, we would use some smart way of manipulating these matrices in such a way that we can show some relation between this and that and ok.

So, one other thing that I wanted to mention is that from the spectral factorization theorem, we get that that G transpose s plus, G s plus G transpose minus s is equal to this product V transpose minus s Vs. And we can always take this V s to be Hurwitz. And if V s is Hurwitz, this matrix F is a Hurwitz matrix.

So, so please remember that this matrix F is a Hurwitz matrix ok. So, first what we are going to do is? We are going to do a transformation on this second state representation that we got and the transformation that we would do on this thing is using a matrix which will help us make the system matrix here, we will convert the system matrix here into a diagonal matrix.

That means, we will try and get rid of this H transpose H ok. Now, the way we get rid of this H transpose H, is the following.

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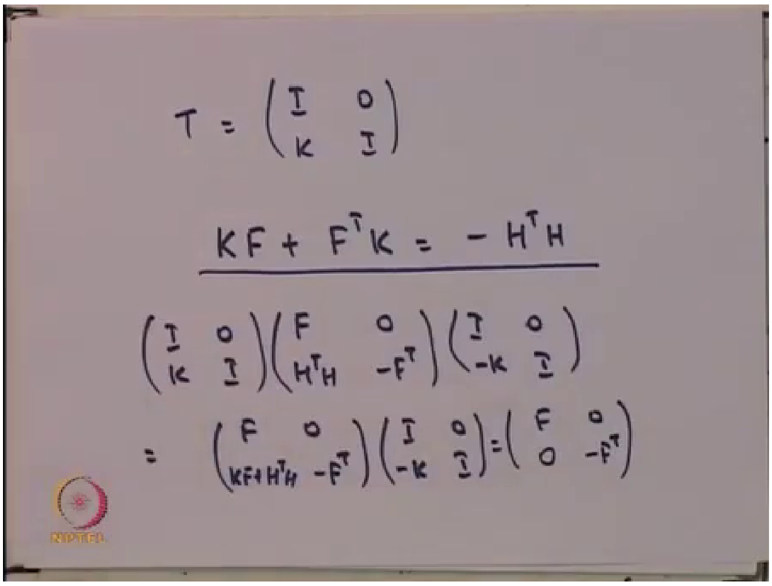

So, we will use for the transformation a matrix T, which is of the form I, K, 0, I. And this K is not any old K, but this K is a K, that would satisfy the Lyapunov equation K F plus F transpose K equal to minus H transpose H ok.

Now, what let me just revisit this is the equation, the minimal state equation that we got for the right-hand side we want to make this matrix diagonal. In order to make this matrix diagonal also remember that this F is a Hurwitz matrix.

Now, if F is a Hurwitz matrix this equation for any positive I mean F is a Hurwitz matrix. If you write down this Lyapunov equation, what you have on the right-hand side well. H transpose H this is always going to be positive semi definite. So, with a negative sign this is going to be negative semi definite. And therefore, you will always get a solution K for this ok.

So, this K is the K that you use in the similarity transformation matrix ok. So, if you use this K in the similarity transformation matrix. So, let us just look at what we would get we have H transpose H, minus F transpose. And we are going to use this, so I, 0, K, I and of course, the inverse of that matrix is I minus K, 0, I.

So, if you multiply this out. So, multiply the first 2 matrices you get F 0, and this one when you multiply you get K times F, plus H, transpose h and the other one you get minus F transpose. Then post multiplied by I 0 minus K I. Incase when you multiply this ok.

When you multiply this, you get F 0 and then you have this thing multiplying this thing. Now, if you see what happens is you have K F plus, H transpose h, plus F transpose K, but that is essentially this Lyapunov equation. So, you get a 0 and then the last one gives you minus F transpose.

So, we have effectively managed to diagonalize that matrix by using a similarity transform T ok. So, now, if we use this T and you diagonalize then ok, then the kind of matrices that you would get what I will do is rather than do the calculations, I just write down what are the matrices that you would get? See these were the original matrices that you had and if you do this transformation using I, K, 0, I.

Then the system matrices that you would get r ok. So, what I will do is I will write down the original system matrices and I will write down what the corresponding system matrices are.

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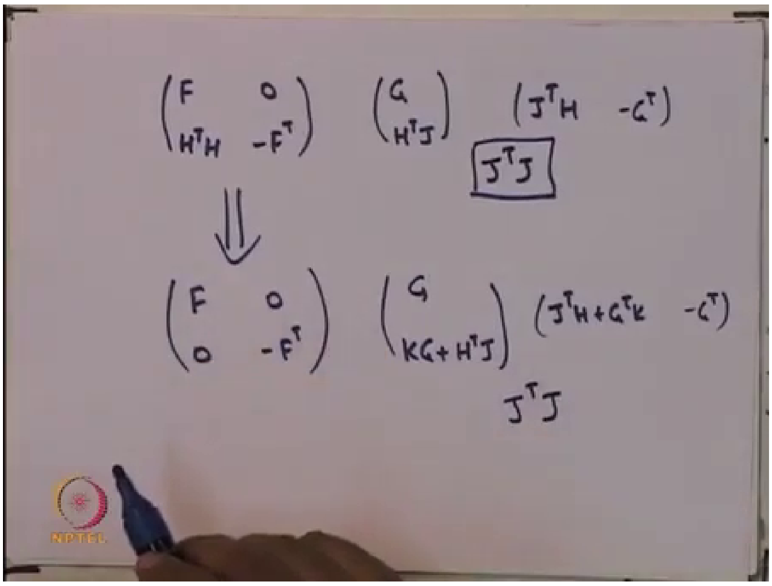

So, the original matrices this is the system matrix that you had and under transformation this goes to F 0 0 minus F transpose ok. Then, then you had the input matrix as G H transpose J and under this similarity transform ok.

Under the similarity transform this will go to this will continue to be G. And here you will have KG plus H transpose J. Then you have the other matrix being J transpose the C matrix of the observable matrix to be this. And that will get transformed to J transpose H plus, G transpose K and here you will have minus G transpose will remain as it is and then the last one which was J transpose J the feed through matrix that will remain J transpose J ok.

Now, this is the new set of matrices that you have and what we will now show is that, you see the right hand side at these matrices and one can show that there is a similarity transform that you can do on these matrices such that you get these 4 matrices ok.

So, this is what we are saying. This left-hand side matrix, this is the right-hand side matrix, there is a similarity transform which will take this matrix to here, this matrix to here, this matrix to here and this matrix here ok. I am not going to go into the details of how to construct the similarity transform, but I would just say that this similarity transform that you use you can show that its a block diagonal matrix and so on.

And now it should be clear that once you do this transform, whatever this gets transformed by, so suppose you have that similarity transform acting on this to make it this. Then whatever this gets transformed by this will turn out be equal to D plus D transpose. So, this J transpose J is really let me pull out the positive real lemma this last equation says D plus D transpose is equal to w transpose w, this is really saying that this matrix is the same as this matrix ok.

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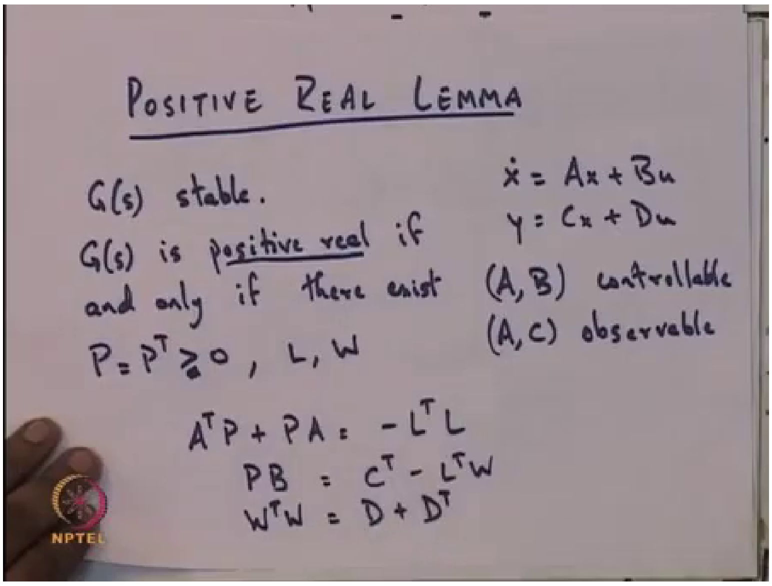

Now, when you do the other transforms then from equating whatever is here to whatever gets transformed here you will get this equation and the other equation about L transpose L, that you will get that you will get from the similarity transform that takes A to F ok.

So, as a result what you have shown from this construction is by this methodology and showing the similarity transform between these 2. You have shown that these equations are going to be satisfied ok. And so that was the converse proof of the positive real lemma ok.

So, it looks like I am out of time and so let me stop this lecture, but what I would do in the next lecture is I would start off by giving the definition of positive real for matrices for the matrix case. We prove the positive real lemma assuming I mean initially I had only given the understanding of what positive realness is for scalar, single input single output system, but of course, it can be extended to matrix case.

The proof of positive real lemma because, the proof was dealing more with state space it really did not matter whether we were looking at the single input, single output or the multi input, multi output case. But what exactly the definition of positive realness is that I would talk about in the next lecture.