Nonlinear System Analysis
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Lecture – 41
PR definition for MIMO case

In the last lecture, I spent the whole lecture showing the proof of positive real lemma, but throughout the proof I kept talking about the definition of positive realness and the definition of the matrix situation for positive realness and so on. So, I would in this lecture initially try to spend time and give a sort of a better idea about what this definition of positive reality especially for matrices are.

Now, as far as the positive as far as the definition for positive realness for matrices are concerned, I will be following what is given in the book by Khalil. ah, But, as I was saying earlier for the scalar case, there is no real agreement about what exactly is the definition of positive realness.

So, what I would do is I would just revisit what I had said about positive realness and I would start with this scalar case and then I would give you the definition for the; for the matrix case. And motivate you know what are the advantages and disadvantages of the equivalent definitions which are there for positive realness. So, let me start by revisiting what the definition of positive realness is ok.

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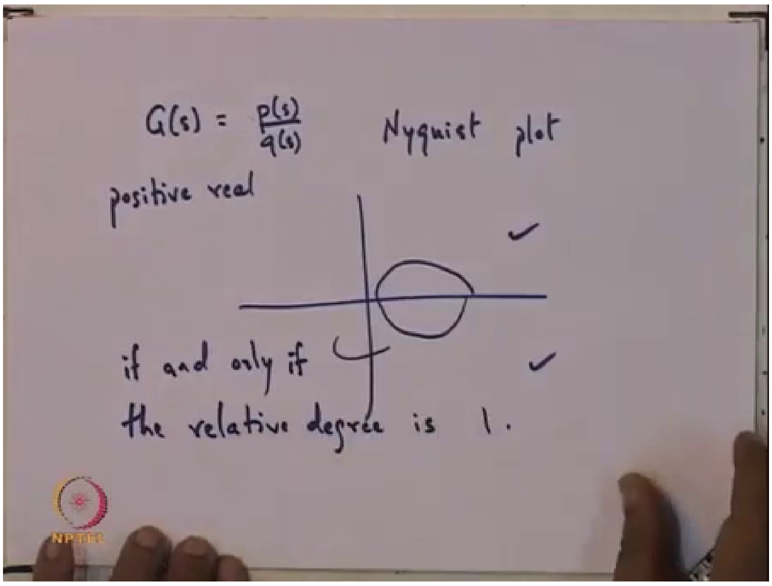

So, given a transfer function G of s; one definition for positive realness is you look at the Nyquist plot and if the Nyquist plot lies in the first and the fourth quadrant. So, if you have a Nyquist plot which looks like that for example, then that transfer function can be called positive real this is one particular definition of positive realness that you could use.

So, if the Nyquist plot is in the first and the fourth quadrant, then you call it positive real. And some time back I had given examples of a of a transfer function, so this is single input single output case. So, it is some polynomial divided by some other polynomial ok.

Now, if you look at if you look at the theory of Nyquist criterion, then you know that if the degree if the relative degree; that means, if p is a degree 3 polynomial and q is a degree 4 polynomial then the relative degree is 1 or minus 1 depending upon your the way you want to look at it. Basically only if the relative degree is 1 or 0 can you expect to how the Nyquist plot restricted to these this half? The reason for that is because if it is more than if the relative degree is more than 1 then it turns out that the angles that is.

So, if the for example, if the denominator is 2 degrees higher than the numerator, then what would happen is finally, it would I mean the Nyquist plot would enter this quadrant. So, one observation that you have for single input single output cases is a Nyquist plot is restricted to this half of the complex plane, if and only if the relative degree is 1. Now, even if the relative degree is 1, we still not guaranteed what we want as far as the positive real lemma is concerned and the reason for that is several ok.

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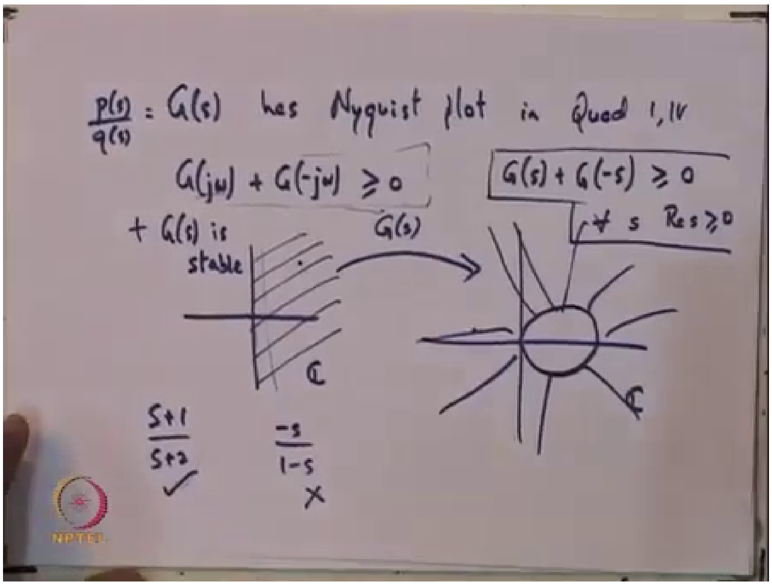

So, so let us just take this definition that G s has Nyquist plot in quadrants 1 and IV and this is the same I saying G of j omega plus G of minus j omega is greater than equal to 0; mind you in this particular case I am thinking of G s as a single input single output case. So, it is some polynomial divided by some other polynomial and so this one is greater than equal to 0 yeah.

So; that means, when you evaluate the transfer function along the imaginary axis, that imaginary axis the image of the imaginary axis lies in the right half lies this in the right half of the complex plane ok. Now of course, we have this map from the complex plane to the complex plane. And what we are saying is that the image of the imaginary axis is something that lies in the right half of the complex plane that the image I mean this map here is G s, so any point s here goes to G s there.

Now, there are several examples of such things. So, for example, if you use s plus 1 by s plus 2, this transfer function when you look at this map and you look at where the imaginary axis maps to that will lie completely in in the right half. If you look at minus s upon 1 minus s this was also an example that I used earlier.

And if you look at the map if you take G s to be minus s upon 1 minus s and you look at the map, then again the imaginary axis will map to something here yeah. So, for example, in this particular case it will map to some something like that ok. Now, if positive realness definition is just taken to be this then both of these will turn out to be positive real.

Now, if you take the state space representation of this; state space representation of this and try to use the positive real lemma; the positive real lemma will be applicable to this only because its the denominator has its roots in the left half plane. Whereas, in this case the denominator has its roots not in the left half plane, but in the right half plane and so you cannot apply the positive real lemma. Earlier I had also talked about idea of dissipativity and idea of storage function. So, in both these cases you will get a storage function.

The only difference between these two is that in this case you will because the denominator has all its I mean roots in the left half plane; that means, this is stable transfer function. Therefore, in this particular case the storage function that you get is positive. On the other hand, in this case also you can construct a storage function, but this storage function is not going to be positive, its going to be negative.

Now, in physical systems if you are talking about storage function its some function which stores energy. Now, if you are going to look at a function that stores energy and that function is negative that does not make sense, I mean what does what does it mean to say that it the amount of energy stored in the system is negative yeah. As a result, it does not make sense in this particular case though you can still find a storage function whereas, it makes sense in this case and you can find the storage function.

So, in both cases; so as soon as this equation is satisfied you can find a storage function. But its only when the when the transfer function is Hurwitz that you can find the storage function which is positive so whereas if it is not Hurwitz, then you can find a storage function, but that will not be positive ok.

Now, if you look at it this further; that means, instead of just looking at it this way, if you put the condition that G s plus G of minus s is greater than equal to 0 for all s such that the real part of s is greater than equal to 0. That means, instead of just looking at the imaginary axis you look at where this whole half plane maps to under G s. And this whole half plane should map such that it falls in this right half; if you put this added condition, then because the imaginary part is in the boundary of this thing. That means, if the condition is this condition is satisfied, then this condition is automatically satisfied ok.

But there is something more satisfied; that means, whatever is here now gets mapped to the right half plane, then in that case this function for this transfer function it would be all right, for this transfer function it will not be alright. So, if you take this transfer function and see where this right half maps to what you will get is this whole area outside the curve.

Of course about; that means, some of these points get mapped to points here which is not in the right half, but in the left half. Whereas, if you had taken s plus 1 by s plus 2, then the imaginary axis would have mapped to a curve. And the right half would have mapped to the inside of the curve which means every point here on the right half is getting going to get mapped to the right half plane.

And so this equation captures the fact that this is satisfied plus the storage function is positive. And therefore, in some sense this should be the real definition of positive realness yeah, but for historical reasons this is usually given as the definition for positive realness. And there are some places where this definition for positive realness is used, but if you give this definition for positive realness then such functions also permissible.

And to disallow these functions the additional condition that is given is this condition wholes plus G of s is stable and as it turns out that these two conditions together is equivalent to this condition yeah. So, the various books that you go through might have some mixture of these definitions as a definition for positive realness.

And this is for the scalar single input single output case and I mean depending upon your taste you can adopt any one of them as about you would believe positive realness to be, but they all roughly say the same thing, but there are these subtleties that need to be handled. So, let me now give you the definition for positive realness as far as matrices are concerned ok.

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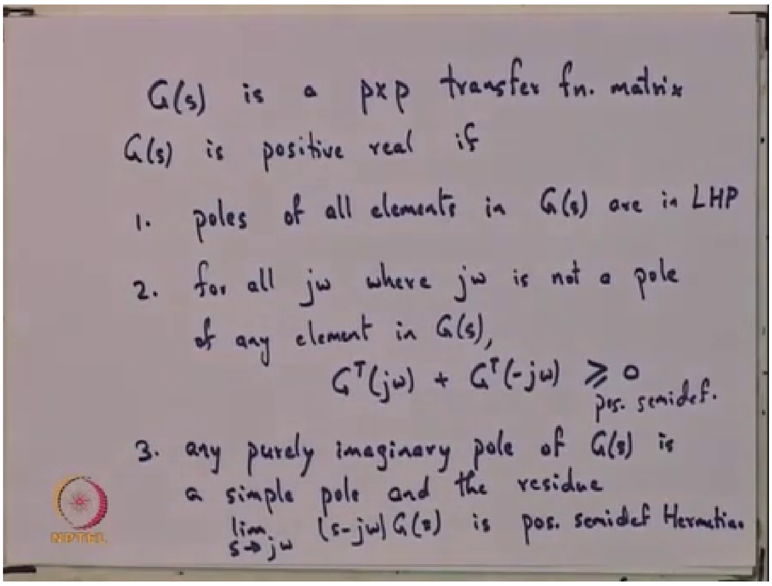

So, let us assume G of s is P cross P transfer function matrix. Now of course, in all these cases this matrix the transfer function matrix has to be a square matrix because the number of inputs have to be equal to the number of outputs.

So, so G of s is a p cross p transfer function matrix then, the following conditions have to be satisfied for G s to be; to be declared as a positive real transfer function. So, G of s is positive real and this particular definition that I am using is the definition that is given in a in the book by Khalil non-linear systems by Khalil.

So, just like what I said in the 1 D case here also in the multiple input multiple output there could be other opinions, but I am just sticking to this for now ok. Its positive real if number 1, the first condition is poles of all elements in G of s are in the left half plane; that means, every element. So, every entry of G of s is a transfer function and every one of those transfer functions is Hurwitz is stable ok. And the second condition is for all j omega I mean imaginary purely imaginary value where j omega is not a pole.

So, the j omega is not a pole of any element in G s; G transpose of j omega plus G transpose of minus j omega is a positive semi definite matrix. So, this condition is precisely like the Nyquist plot condition in the single input, single output case ok. So, what we are saying is for all j omega where j omega is not a pole, if j omega was a pole for some entry in a in the transfer function matrix of course, then this thing will not be very well defined at this there are problems.

So, you remove all those omega which on the imaginary axis which might be a pole of any one entry of G of s and for all the others you will have this, but of course, remember these are matrices. So, the sum of these two matrices; one is claiming is a positive semi definite matrix, a positive semi definite matrix. And there is one more condition the third condition is that any purely imaginary.

So, any purely imaginary root or rather pole purely imaginary pole of G of s is a simple pole; is a simple pole; that means, it does not have multiplicity greater than 1 and the residue; and the residue. So, the way you obtain the residue is limit s tending to j omega of s minus j omega times G s is positive semi definite Hermitian ok.

So, so there are these three conditions, so the third condition is that for any purely imaginary pole of G s. So, any purely imaginary pole of G s is a simple pole and the residue limit as you tend towards a pole of this particular thing that residue of course, here is its a matrix. So, it is in fact, a positive semi definite Hermitian matrix. So, the definition for positive realness is this. So, now, if you just specialize you take G s to be a 1 cross 1 transfer function well the poles are all in the left half plane well as we had said before.

And this is the Nyquist condition Nyquist criterion condition. So, the Nyquist plot on the first and the fourth quadrant and this one is the stability. So, these two conditions are what this is an additional condition that appears in the matrix case in the scalar case this is clearly true, but in the matrix case its a bit more involved and therefore, this condition makes its appearance. So, this is the definition for positive realness as far as matrices are concerned.

And so, now if you take a G cross G P cross P transfer function matrix G of s and you want to talk about it being positive really you put these things these criterion in and you can check whether its positive real. And in the earlier lecture when I talked about the positive real lemma well there G s could just be taken to be positive real with this definition. And as far as a realization is concerned whether it is single input single output or multiple input multiple output the realizations would be in terms of those matrices and so that matrix condition would remain unchanged.