**Nonlinear System Analysis**
**Prof. Harish K Pillai**
**Department of Electrical Engineering**
**Indian Institute of Technology, Madras**

**Lecture - 46**
**Sector Nonlinearities and need for generalizing KYP Lemma**

In the last lecture what we saw was that if you took a linear plant which is positive real. And the definition of positive real has these ambiguities, but what we effectively would mean, is that the Nyquist plot lies in the right half plane and the system the given system is linear. Now, if you take such a plant and then such a plant is of course, passive. Now, if you now take non-linearity, such that the characteristic of the non-linearity lies in the 0 to infinity sector, then one can think of that non-linearity also as being a passive system.

As a result what happens is when you interconnect the linear system given by a transfer function which is positive real with a non-linearity which is in the 0 infinity sector, then you end up with a resulting system which when you do not give any input is asymptotically I mean of course, the origin is an equilibrium point of this system without inputs and this system is asymptotically stable.

In fact, when the transfer function is taken to be strictly positive real, then you can say more in fact, you could say that the resulting system is exponentially stable ok. So, let me just reiterate what I have just said in terms of some diagrams, so that you it would be clear to you what I am trying to say. So, what we are doing is the following.

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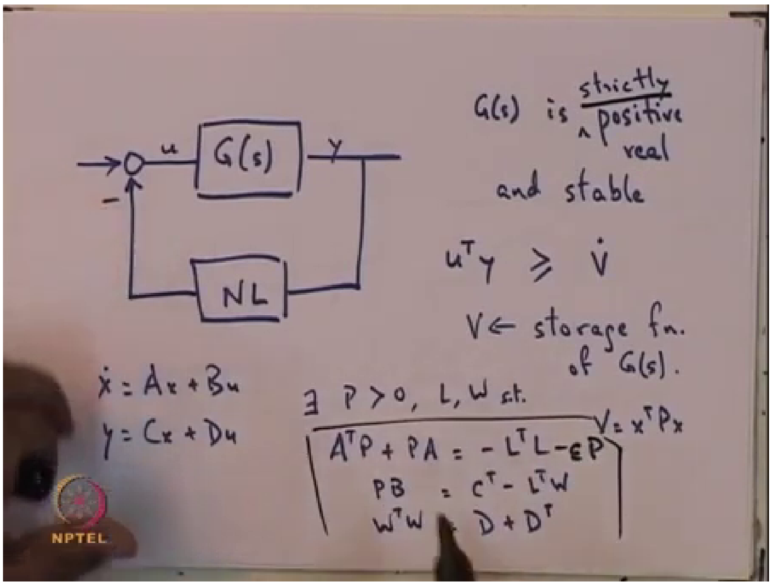


So, you have a linear plant G s and you have this feedback connection and you have this non-linearity. So, this negative feedback. Now, this linear plant that you have we are saying that this linear plant is positive real. Of course in the last class I had given various interpretations of what we mean by positive real, but what I would mean by positive real here is that the Nyquist plot of this plant is in the right half plane, and in addition G s is stable ok.

Of course one could also use the definition of positive real to say that G s plus G transpose s star is greater than equal to 0 for all real for all s whose real part is positive that is another equivalent definition ok. Now, by making this assumption ok so let me write down G s is positive real and stable ok. Now, if I call the input of this G s as u and the output of G s as y, then what it means when you say that g s is positive real and stable, is that u transpose y is greater than equal to V dot where, V is the storage function of G s.

We have already talked about what the storage function could be and in order to find out what this storage function could be we invoke the positive real lemma or the Kalman Yakubovich lemma from where we get this matrix p.

And you know if you recall when you have G s to be positive real and stable then that is equivalent to this set of equations; that means, if you if you take the minimal representation for, G of s to be A x plus B u y equal to C x plus D u, then using these matrices you can write down that there exists P which is positive definite, and two other matrices L and W such that. First of all A transpose P plus PA is equal to minus L transpose L.

And then you have P B is equal to C transpose minus L transpose W, and lastly you have W transpose W is equal to D plus D transpose. And so this P that you get which is positive definite you take the storage function V to be X transpose P X where this P is this P that you get from the positive real lemma ok. Now, what about the non-linearity? Now, the non-linearity that one picks is of the following kind.

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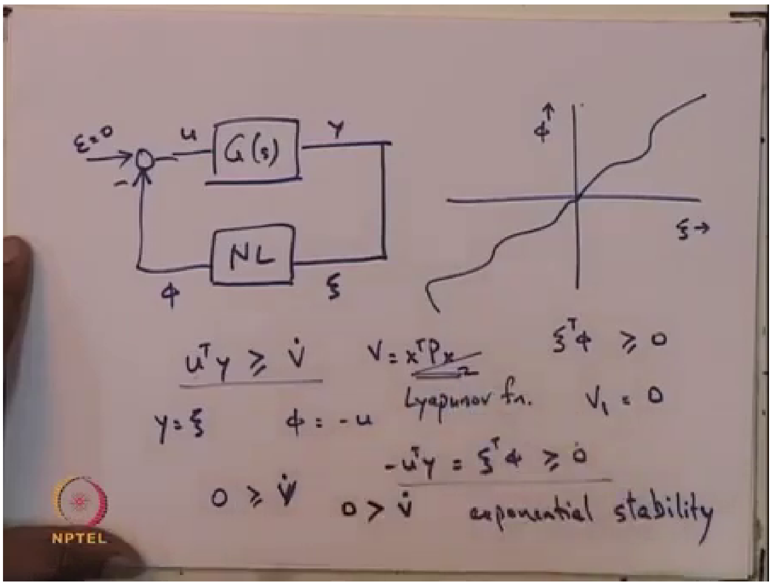


So, the non-linearity, so let me draw that diagram once more. So, here you have the non-linearity, we are calling this output of the linear thing u. Sorry, the output y and the input u ok. Let me call the input of the non-linearity xi and let me call the output of non-linearity phi ok. Now, the non-linearity is such that if you draw the characteristics of the non-linearity.

So, you have xi on this axis and you have phi on this axis, then the non-linearity is something that lies in the first and the third quadrant. Now, if it lies in the first and the third quadrant then it is very clear that if you multiply the input of the non-linearity which is xi to the output of the non-linearity which is phi, then xi or xi transpose phi. If one is considering the vectorial the vector situation; that means, multi input multi output situation this is clearly greater than equal to 0 ok.

Now, since this is greater than equal to 0 one can say that this non-linearity has a storage function V 1 which is the 0 storage function. Now, earlier in the in the last slide I had said that you have u transpose y is greater than equal to V dot where this V was given as x transpose P x with this P coming from the positive real lemma.

Now, if you look here there are these interconnection equations which is y is equal to xi and phi is equal to minus phi is equal to minus u. Therefore, this xi transpose phi in fact xi transpose phi is really equal to minus u transpose y and this is greater than equal to 0. So, if you add this equation and this equation you end up with 0 is greater than equal to V dot.

Now, what this means is that if you use this V that is x transpose P x as a Lyapunov function, for this closed loop system assuming that this input the external input is 0. So, if you have the external input to be 0 this is like a system with no inputs and this system with no inputs will have these conditions satisfied, and if you have these conditions satisfied then you have 0 is greater than equal to V dot. And if you take V to be x transpose P x and that is, then that can act like the Lyapunov function of for this closed loop system, and then what we have here is that its derivative is less than equal to 0, so in fact from that you can conclude that this resulting system is stable.

Now, if you remember we had also talked about the Kalman Yakubovich Popov lemma; what we had said was that if you take this thing to be strictly positive real ok. So, maybe I will mark that with black. Then the change in the equations are that instead of this you will get minus epsilon P also into this equation into this first equation, all the other equations remain as they are.

But what this means is when you substitute in here instead of v dot being less than equal to 0 you will get 0 is strictly greater than V dot. And as a result because of that epsilon; epsilon P which is there, using the Kalman Yakubovich lemma if this G s was taken to be strictly positive real, then we would we can conclude exponential stability ok.

So, what we are going to do in this class is, we will now explore what new conclusions we can draw using this rather powerful result. Maybe before we do that I would like to use this V as which we have got from this positive definite matrix P as a Lyapunov function and show that this resulting system is asymptotically stable.

Well in order to show that I essentially only have to show that u transpose y is greater than equal to V dot and then the rest of it we have already seen; that means, if you take u transpose y greater than equal to V dot. And you already know when you take a non-linearity like this that xi transpose phi is greater than equal to 0 and when you sum both of this you get 0 is greater than equal to V dot.

So, all you have to show is that u transpose y is greater than equal to V dot ok. One small adjustment of a constant I have to do, I should not be taking V as x transpose P x, but I should be taking this as x transpose P x by 2, your half.

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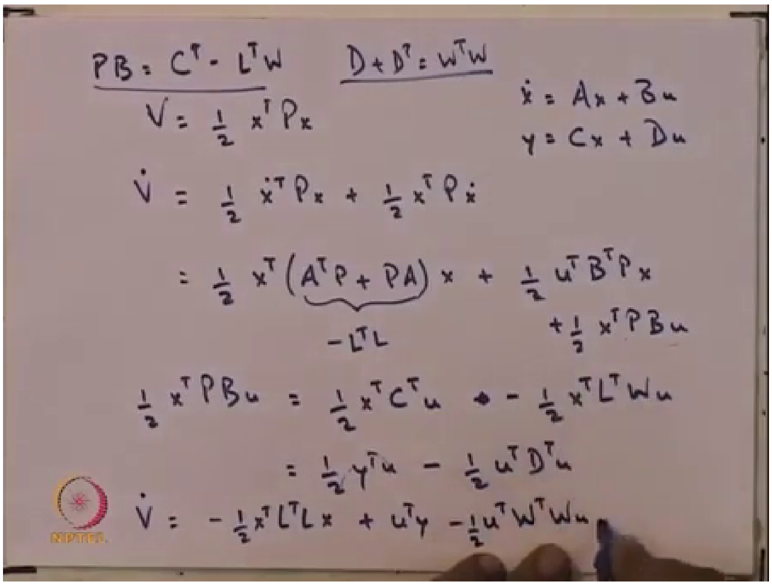


So, now if we take that then, so V we are taking to be a half x transpose P x. So, then in that case V dot is half x dot transpose P x plus half x transpose P x dot, but we know that x dot is A x plus B u from the equation of the of the system of the linear system. So, substituting that in here we will get a half of x transpose A transpose P plus from here similarly. So, PA x plus I will get half u transpose B transpose P x plus half x transpose P B u ok.

Now, this from the Kalman Yakubovich lemma or rather the positive real lemma this is the same as minus L transpose L, so I can put that in there. And if I look at these two terms P B we know that one of the equations that we have from the positive real lemma is P B is equal to C transpose minus L transpose W. So, for P B I can substitute C transpose minus L transpose W and similarly for this. So, I will just expand this out and show what that comes to.

So, half x transpose P B u becomes a half x transpose C transpose u plus or rather minus, half x transpose L transpose W u ok. Now, if you look at the output equation of the linear plant you have y is equal to C x plus D u and so the C x, I can substitute this y minus D u.

And once I do that I get a half y transpose u minus a half u transpose D transpose u. So, this is from this x transpose P B u from this thing you would get exactly the transposes of this. So, when you put them all together, then you end up with V dot is equal to minus a half x transpose L transpose L x that accounts for this, and then from both of this you will end up with u transpose y because this half and then the another half coming from there.

And then you have this term and there will be a similar term coming from the other portion and so putting both of them together you will have u transpose D transpose u and you will have u transpose D u and D plus d transpose. One of those equations that you had was D plus D transpose is equal to W transpose W the positive real lemma gave us this equation.

So, using that you will get u transpose W transpose W u a half of that ok. And then these last two terms which ok, so there will be this term and the transpose of this both within a minus sign.

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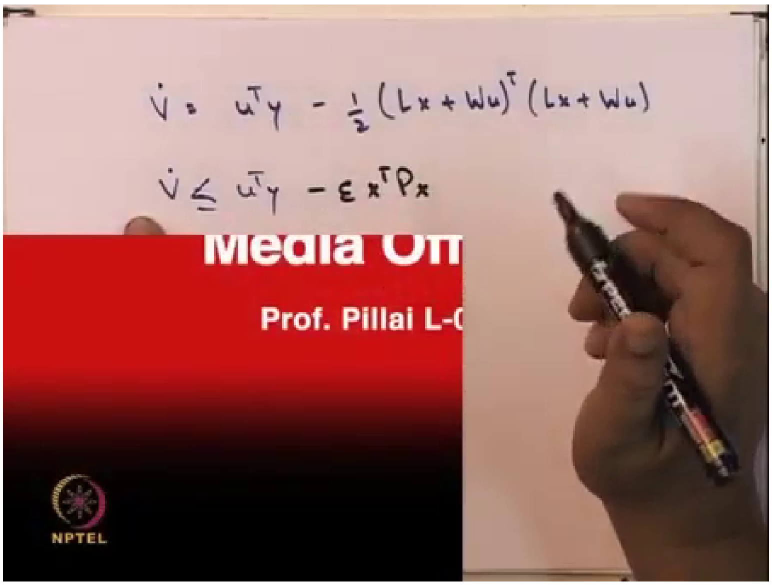


And so if I put all of them together, then what I end up with this V dot is equal to. First of all u transpose y which we had here and then these two terms and the other two terms this and its transpose putting them all together, you will get minus half L x plus W u the whole thing transpose L x plus W u.

And now you see this quantity here is positive quantity and therefore, you can conclude V dot is less than equal to u transpose y and this is essentially what we wanted in order to conclude from this sheet that u transpose y is greater than equal to V dot, 0 is greater than. And you already have this about the non-linearity and when you put both of them together you get 0 is greater than equal to V dot and therefore, you can show; you can show stability.

Now, if one assumes that G of s is strictly positive real, then in that case in this equation here A transpose P plus P A there will be one additional term here and this additional term if I call it epsilon P this additional term. This additional term will finally, end up in the last equation and here you would have minus epsilon x transpose P x and this will now let us prove that the resulting system is in fact, asymptotically stable.

Now, what I am going to do is, I am going to make use of this rather powerful theorem and I am going to look at all kinds of nonlinearities and derive new results about which nonlinearities with when put in a feedback connection with a certain kind of linear plant would result in an asymptotically stable system ok. So, let me conclude about this first.

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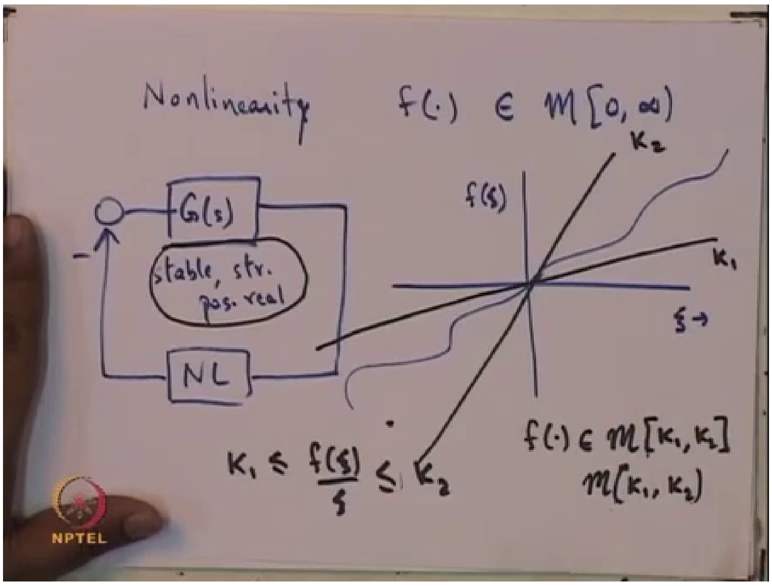


So, what I am saying now is suppose you take a non-linearity. Let me call it f that belongs to the 0 infinity sector.

Now, when I say that a non-linearity belongs to 0 infinity sector, what I mean by that is that, if xi is the input of the non-linearity and this is f of xi the output of the non-linearity, then this non-linearity lies in the 0 to infinity sector; that means, that lies either in the first quadrant or in the third quadrant ok. If you have any such non-linearity; any non-linearity which belongs to this class.

And you take a G of s which is stable and strictly positive real. Then this feedback connection of the strict strictly positive real stable plant with the non-linearity results in something which is asymptotically stable. But suppose now the non-linearity that we consider is this particular non-linearity.

Now, if you consider this particular non-linearity then you can you can see that of course it is true that this non-linearity lies in the 0 infinity sector, but in fact you can say more things about this non-linearity, because suppose I draw slope like this call it K 1 and suppose I draw another slope let us say like that, call it K 2. Then in fact this non-linearity f lies in the K 1, K 2 sector.

So, what I am trying to say is that the non-linearity is such that K 1 is less than equal to f xi by xi, which is less than equal to K 2. So, in some sense by declaring f to be in this sector from 0 to infinity, we are doing overkill because we can get a much tighter bound for the non-linearity and in fact we can say that the non-linearity lies in this sector K 1, K 2. In fact, out here as you can see ok.

So, here the way I have drawn it of course, the K 2 there is some portion here of the characteristic which has the slope K 2, but instead of writing this sometimes people would write something like this K 1, K 2 an open interval K 1, K 2. Now, when you mean a open interval K 1, K 2 or a closed interval K 1, K 2 the difference between them is essentially to do with these inequalities whether they are strict inequalities or just you know non strict inequalities.

So, when you have non strict inequalities then you would put the close bracket, when you have strict inequality you would put the open bracket. Yeah and of course, there would be semi open semi open semi closed kind of intervals also for the non-linearity. Now, fine because this non-linearity lies in the 0 infinity sector therefore, we know that if you take any stable strictly positive; strictly positive real transfer function and connected in this feedback loop with this particular non-linearity you will get an asymptotically stable system.

But since we can put a tighter bound on this non-linearity; that means, this non-linearity actually lies in the K 1, K 2 sector therefore, one would expect that apart from these plants there are other plants also, which you could you could connect with this non-linearity. And these plants may not be belonging to this class; that means, they may not be strictly positive real or stable. But despite that the resulting system is asymptotically stable.

And the reason one, why one would believe that is because it is true that this particular non-linearity is in the 0 infinity sector, but in fact we can say that it is in the K 1, K 2 sector. And so if it is in the K 1, K 2 sector it would be very surprising if there are no extra plants that one can connect to the non-linearity resulting in the system being stable, I mean one would naturally expect that there would be more plants that you can connect to the non-linearity and the resulting system is asymptotically stable ok.

So, now what we are going to do is we are going to explore this situation where you have the non-linearity given in a certain sector. And one wants to characterize all the transfer functions which you can connect in this feedback loop with those nonlinearities such that the resulting system is asymptotically stable.