Nonlinear System Analysis

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Lecture – 49
Loop Transformations (Part 2)

So, in the last lecture, what we had started doing was what are called Loop Transformations. So, I had started initially with a non-linearity in the k 1 infinity sector and I showed how we can convert that non-linearity or we can think of that non-linearity as a non-linearity in the 0 infinity sector. So, in some way the k 1 infinity sector you expand it out so, that it becomes the 0 infinity sector.

So today I will continue with that, but probably I will also revisit the k 1 infinity and sort of try and wrap it round nicely. So, maybe what we will do is we will continue with what we were doing in the last lecture that is after we had finished the k 1 infinity sector, we had started looking at the 0 k 2 sector. So, let us think of a non-linearity in the 0 k 2 sector.

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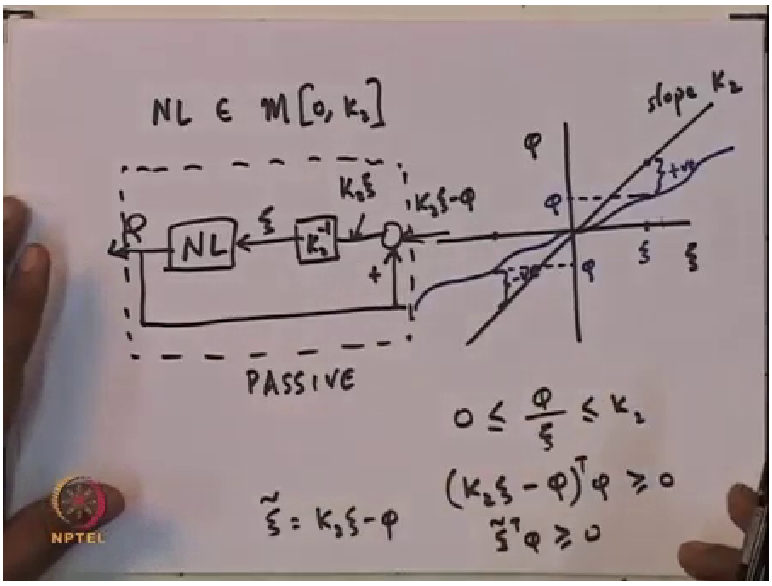

So, we are thinking of a of a non-linearity which belongs to the 0 k 2 sector. So, what I mean by that is So, here Xi is the input to the non-linearity, phi is the output to the non-linearity and we have this line which has slope equal to k 2 and what we are saying is that we have a non-linearity which lies in the 0 k 2 sector. So, it lies something like that. That is a non-linearity.

Now that non-linearity of course, one could think of that non-linearity as phi by xi; that means, at any point you take the phi and divide by Xi you get this this particular sort of triangle. And of course, the slope the tangent of the angle that it subtends is smaller than the angle by give this slope k 2. So, this is less than equal to k 2, but it is greater than equal to 0.

And then we could rewrite any non-linearity in the sector by means of a quadratic form ah. So, the quadratic form that we could rewrite this as is the following. So, you take k 2 Xi k 2

Xi minus phi. So, when you do k 2 Xi minus phi. So, if I am thinking about this particular Xi here, k 2 Xi is here and phi is here. So, this quantity here is positive. If I on the other hand, take a Xi which is negative k 2 Xi is here phi is here and so, this quantity here is negative.

So, if I multiply k 2 Xi minus phi 2 Xi, I would get something positive, but that is not the quadratic form that I am going to take. What the quadratic form I am going to take is you see phi here corresponding to this negative quantity the corresponding phi is also negative.

And here the phi is positive. So, what I am going to do is, I am going to take the quadratic form given by k 2 Xi minus phi transpose phi and this is going to be greater than equal to 0 for any non-linearity that lies in this this particular sector 0 k 2,. Now So, suppose, now you have this original non-linearity and you have the input Xi and the output phi ok.

We want to convert it into a non-linearity which has as its output, it has exactly the same output as the original non-linearity, but its input is modified and this new input Xi tilde transpose phi greater than equal to 0 this Xi tilde is given by k 2 Xi minus phi ok. So, how to modify this? So, that the Xi tilde becomes k 2 Xi minus phi.

So, the way we are going to do it is the following. So, let me put again here which is k 2 inverse. So, if I put again here k 2 inverse then what should have been here is k 2 times Xi. So, k 2 the Xi multiplying k 2 inverse will give mi Xi. Now I have to get this k 2 Xi and so, what I do is, I take this phi ok, and I feed it back here with a positive sign.

So, I am adding phi to something so, that I get k 2 Xi. And so, the something to which I have to add phi to get k 2 Xi is k 2 Xi minus phi. And now if I think of this non-linearity in the box, which is this non-linearity with this gain and this feedback, then this non-linearity has as its input k 2 Xi minus phi and it has as its output phi.

And therefore, this non-linearity by this equation or this equation is therefore, a passive non-linearity ok. So, the transformation that I do for something in the 0 k 2 sector is in this way and when I do the transformation in this way, I get this new non-linearity which has the original non-linearity as Xi as an input and phi as the output, the new non-linearity continues to have this phi as the output.

But it has k 2 Xi minus phi as the input.

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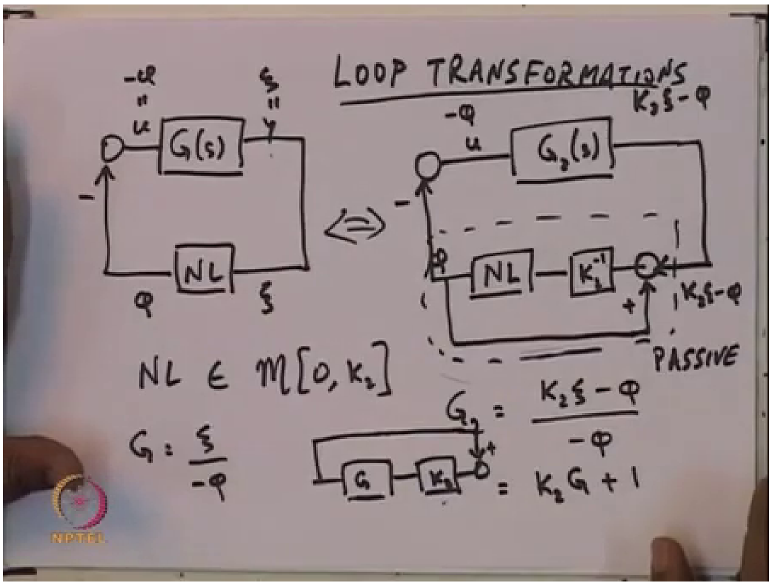

Now, if we are now looking at the feedback structure, where you have a plant G s and you have this non-linearity and this non-linearity is in the 0 k 2 sector. And we are looking at this feedback connection between the linear plant and this non-linearity. We modify this non-linearity to that new thing so, the new the new non-linearity is obtained in this way, you have a gain here k 2 inverse and whatever is the output you feedback a unity feedback with a positive sign.

So that what you have here is really k 2 Xi minus phi and the output remains the same phi ok. So, here you have Xi, phi and this is u the input of the linear plant and the output y. So, let me call this new plant that I have here which would be a modified version of this plant let me call that G 2 s,. So, the G 2 s has as its input the same as the input of G s. But it has as its output this k 2 Xi minus phi. So, let me instead of calling this u, let me just call this minus phi because this u is equal to minus phi and here y is equal to Xi.

And here what I should have is k 2 Xi minus phi. Therefore, the new plant that you get their G 2 is output by input. So, it is k 2 Xi minus phi divided by minus phi. From here, you know that G is Xi divided by minus phi. So, if you now evaluate this, this turns out to be k 2 times G, this transfer function plus 1, ok. So, now, if you had a linear system with this non-linearity in the 0 k 2 sector and you look at this closed loop system. This is exactly the same as looking at this modified non-linearity along with this plant G 2 s, but this G 2 s is really k 2 times G plus 1.

Of course, how do we realize k 2 times g plus 1? If you had the original G, you can put a gain of k 2 in series with it ok, and this plus 1 you can get by having a unity feed forward. So, G 2 s is really this net transfer function. So, if you see just like in the last case, you have some sort of a symmetry because in the non-linearity, you are putting this k 2 inverse and then you are having this feedback which is a positive feedback and therefore, G s will also get modified.

But this time the gain that you have in series with G s is going to be positive k 2, I mean it is going to be k 2 whereas, with the non-linearity you had k 2 inverse and here in this loop, in this portion in the non-linear portion, this was a feedback. So, out here it is a feed forward. If you recall in the k 1 infinity sector, we had used a feed forward in the non-linearity as a result of which you had a feedback in the in the linear plant, here in the non-linearity you are using a feedback.

Therefore, in the linear part you will have a feed forward. Now, this kind of transformations go under the name of loop transformations, ok. Now, by doing this loop transformation, you have got a new non-linearity here and this non-linearity is passive. this whole thing that I am marking this whole non-linearity is passive therefore, this G 2 s, this modified linear plant if this modified linear plant is strictly positive real and it is stable, then the original plant along with this non-linearity of course, if this is strictly positive real and is stable then this resulting feedback system is going to be asymptotically stable. And that is the same as saying that this particular system is asymptotically stable.

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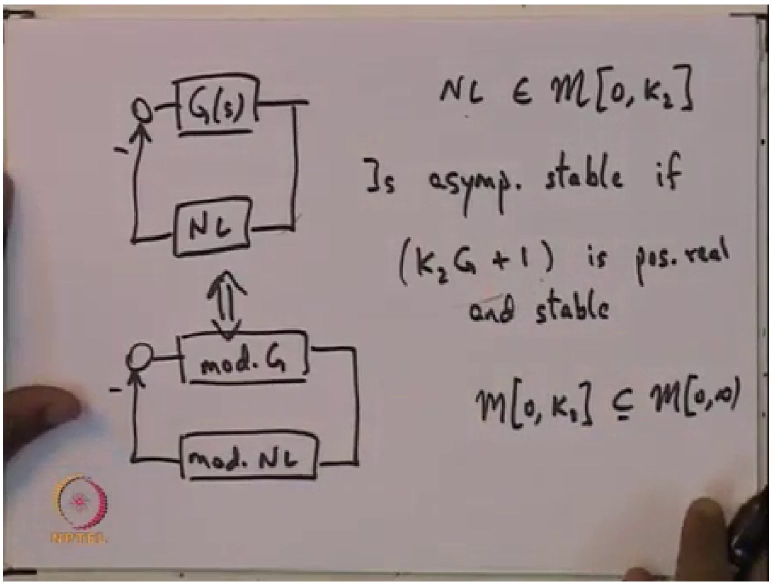

So therefore, from this what we can conclude is, if you have a non-linearity and you have a linear plant G s and you are looking at this feedback structure. And if the non-linearity lies in the 0 k 2 sector, then this resulting system is asymptotically stable, if k 2 times G plus 1, this transfer function which is a modified transfer function that G s becomes; because you have modified non-linearity, these two are equivalent and the modified G is k 2 G plus 1.

This is positive real and stable. So, what we are really doing is, when we are looking at nonlinearities which are in sectors which are really in some sense subsets of the earlier situation means initially, the passive theorem in a passive lemma and so on. Were proved for nonlinearities in the 0 infinity sector, now we are looking at something in the 0 k 2 sector where k 2 is strictly smaller than infinity.

Then one would expect more transfer functions to be interconnected to this non-linearity resulting in in something which is asymptotically stable. Of course, if you have a non-linearity in the 0 k 2 sector, you see this this is true that all the non-linearities in the 0 k 2 sector. This is a subset of the non-linearities in the 0 infinity sector. So, of course, if you take a plant here which is positive real and stable, then the resulting system is anyway going to be asymptotically stable.

But what this result tells you is that, you need not necessarily take G s which is strictly positive real and stable. But you could take a G s such that k 2 times G s plus 1, this resulting transfer function is positive, real and stable. So, we could have a G which is not positive real or stable and you could have k 2 G plus 1 resulting in something which is positive real and stable. And if that is true then that G along with the original non-linearity, that will again result in a system which is asymptotically stable.

Now, I had I had used some sort of quadratic forms. So, let me just revisit these quadratic forms and there are I mean, depending on the tastes of people there are the new additional definitions given to many of these systems.

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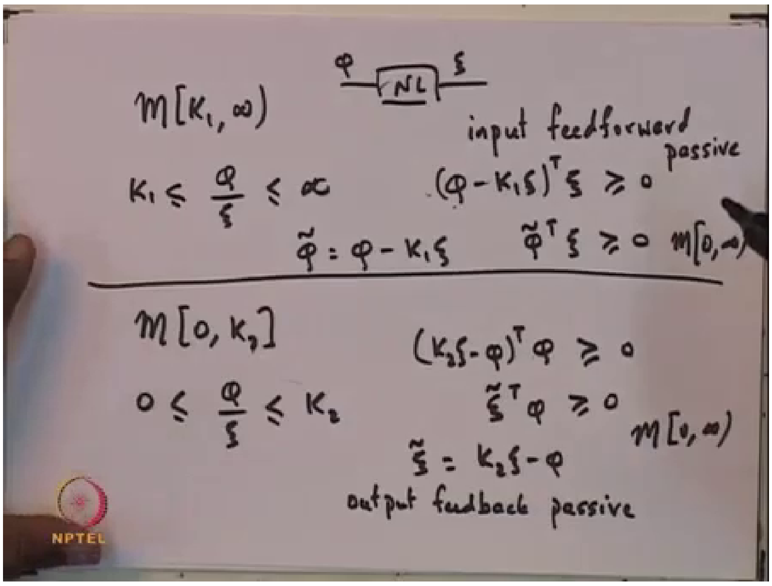

So, when we consider the non-linearity in the k 1 infinity sector, when we had looked at a non-linearity in the k 1 infinity sector, what this means is the output of the non-linearity divided by the input lies between infinity and k 1 ok. And then this I could rewrite as phi minus k 1 Xi transpose Xi being greater than equal to 0.

So this this inequality that I have here of the of non-linearity in this class I could write it in this way and this then I could rewrite in this particular way. And then when I rewrite it in this way, I could think of so the original the original non-linearity was there with inputs Xi and output phi. So, I retain the input as it is, but the output I modified to phi tilde and So, I look at it this way there phi tilde is given as the original phi minus k 1 Xi and then this new modified non-linearity is passive.

Similarly, when you take a non-linearity in the 0 k 2 sector, then the inequality is similar to this, the inequalities that you would get is the output by the input is less than k 2 and is greater than 0. And in this particular case, what I did was, I kept the output the same. So, and I wrote this inequality in quadratic form and so the quadratic form that I wrote was phi and multiplying phi multiplied k 2 Xi minus phi. And this is greater than equal to 0. And this k 2 Xi minus phi, I define that as the new input.

So I kept the output the same, but I modified the input and so Xi transpose phi greater than equal to 0, where Xi tilde was given by k 2 Xi minus phi. So, what I am really doing is, I take a non-linearity in the sector, there are these inequalities which are satisfied, but that is equivalent to saying that it is this particular quadratic form that is satisfied.

And if this quadratic form is satisfied, and I think of a new non-linearity which has phi minus k 1 Xi as the as the new output, I keep the input the same. I change the output, then this new non-linearity will actually be this non new non-linearity would actually be in the 0 infinity sector. Similarly, if I take something in the 0 k 2 sector, I this is one way to define it, but I am redefining this in form in the quadratic form.

And if I redefine this in the quadratic form, I think of this here as the new input so, I keep the output the same, I change the input into this new input and then the resulting system is again in the 0 infinity sector. Now, if you look at this this case, non-linearity in this case, then what we are doing is we have kept the input the same, but we have modified the output.

On the other hand, if you look at a non-linearity in this sector, we have kept the output the same and we modified the input. Now, the way we modify the output in this particular case is we give a feed forward the original plant; we give a feed forward. So, such systems are also sometimes called input feed forward passive.

And similarly, these systems what we did was we kept the output the same, but we gave a feedback from the output and therefore, these are called output feedback passive.

Now, these are all definitions, but the important thing to realize is that whenever you have some non-linearity in some sector like k 1 infinity, you can convert it using this this quadratic form into something which is passive. And similarly, if you have some non-linearity in the 0 k 2 sector, then by modifying the input; I mean, this is the quadratic form that is satisfies and so if you modify the input, then the new non-linearity that you create is in the 0 infinity sector.

And once things are in the 0 infinity sector, then you know that if you if you put a positive real stable plant in a feedback loop with this such a non-linearity, you get asymptotic stability. So, whatever was the linear plant that you connected that will undergo a transformation to be a new linear plant which you associate with this particular non-linearity and that new linear plant should now be positive real and should be positive real and stable. And similarly, in this case ok.

Now of course, we could also we could also look at a non-linearity, which is in a bound you know k 1, k 2 sector.

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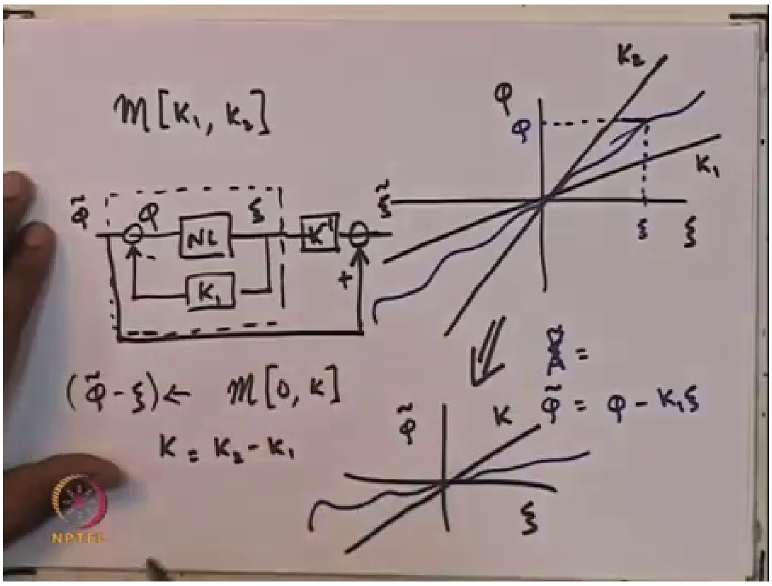

So, one could look at m some non-linearity in the k 1, k 2 sector ok. So, what do we mean by that? This is the input, this is the output let us assume this has slope k 1 and you have another line, which has slope k 2. And what we are saying is that we have a non-linearity which lies in the k 1, k 2 sector, ok. Now, for a non-linearity that lies in the k 1, k 2 sector, we can again write some sort of a quadratic form, but and then use that quadratic form to convert this k 1, k 2 sector into the 0 infinity sector; that means, passive ok.

Now, one way to go about doing this is what one could do is we could convert this k 1, k 2 into a 0 k sector. And how does one convert this into a 0 k sector? Well, you have this non-linearity with inputs Xi and also phi. And now if you think of the input as the Xi the output is this this particular phi ok.

What one does is we modify the input to the new input Xi tilde which is given by, no sorry. What we do is for the same for the same Xi, you have a new output phi tilde which is given by the old phi minus k 1 Xi ok. How to do it here? Well, what we are doing is, the new so, this is the phi tilde. Now what can we say about this new, what can we say about this new non-linearity?

That means, this non-linearity which is there in the box. Well, what we can say is that this new non-linearity whose output is phi tilde and the input is Xi this new non-linearity belongs to the 0 k sector where k is k 2 minus k 1. Now it should be clear why this is k 2 minus k 1 because you see we are talking about non-linearity in the sector.

So, one of the worst cases would be I mean, we could really think of the non-linearity as an linearity which is output is k 2 times the input. Now, when we put this transformation, then the output becomes k 2 times Xi minus k 1 times Xi which is k times Xi. And so, what we have effectively done is we have rotated this round.

And so, the new non-linearity that you get is Xi is the input, phi tilde is the output and you will have slope k, which is k 2 minus k 1 and the non-linearity now will lie like that.

Now, once you have something in the 0 k sector, we can use this particular thing to convert it now into something in the 0 infinity sector ok. So, what do we do? Well, to the non-linearity, you put k inverse here and then whatever is the output you feed it back, yeah and you feed it back with better positive sign I believe. Be sure here ok, and now the phi tilde, but now we have modified the input also and it has become Xi tilde ok. Now, this new non-linearity with Xi tilde and phi tilde is in fact, in the 0 infinity sector, alright.

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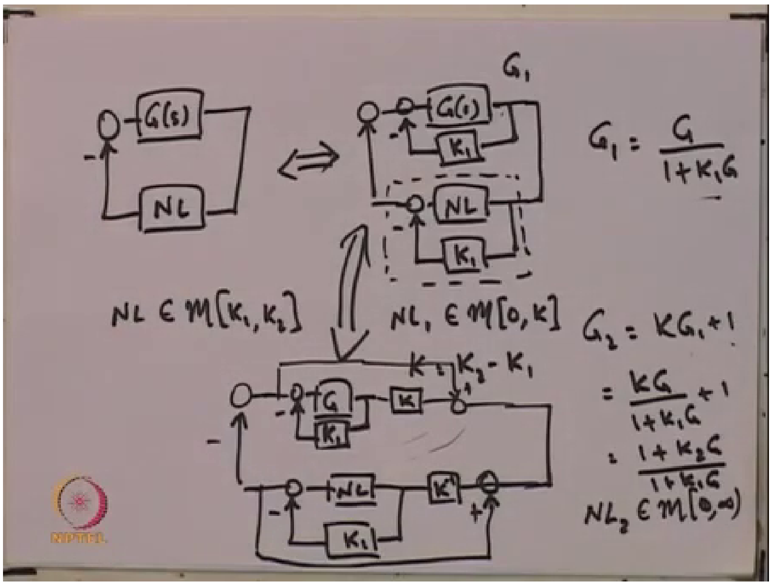

So, here are some transformations that are going around. You have a G s and you have this non-linearity ok, and this non-linearity is in the k 1, k 2 sector. So, you do the first round of transformations and So, you get this new non-linearity. So, the new non-linearity you get is the old non-linearity with a gain of k 1 put here, this is the new non-linearity. Instead of this, we have put this additional thing.

And as a result, this G s gets modified, and how does G s get modified? Well, the G s gets modified by a feedback.

So, now, this is equivalent to this where you have modified the non-linearity and you have modified the plant. This non-linearity which is there in this dotted box that I am talking about, if I call that non-linearity NL 1, this NL 1 belongs to the 0 k sector, the k is equal to k 2 minus k 1. Now, this is further modified now in the following way.

So, you have non-linearity you have this k 1 ok. So, this is a non-linearity and now you put in k inverse here, and you take positive feedback and this new non-linearity that you 1 is having let me call this new non-linearity NL 2, this belongs to the 0 infinity sector.

And of course, when you do this you have to do the corresponding change in the transfer function and the change would be. So, you have the linear plant who had the feedback. And then you do exactly this. But here you multiply by k and then you feed forward, unity feed forward and this is the resulting linear plant that you have. So, this is equivalent to this is equivalent to this.

So, if G s with this non-linearity in the k 1, k 2 sector, is to be stable then this linear plant with this non-linearity this must be asymptotically stable. But what is this? This one could get from this by the transformations that it has to go through. So, if I call this linear plant G 1, then I know G 1 is equal to G upon 1 plus k 1 G and then this G 1 gets converted to this one.

So, if I call this linear plant G 2. So, G 2 I know is k times G 1 plus 1. So, substituting G 1, this is the same as k times G upon 1 plus k 1 G plus 1. This is the same as 1 plus k 2 G upon 1 plus k 1 G. So, if 1 is given non-linearity in the k 1, k 2 sector and we are asked to find out all the linear plants which when interconnected with this non-linearity gives rise to a system which is asymptotically stable. Then, we could do a loop transformation on this k 1, k 2 and convert it into a non-linearity in the 0 k sector.

And that is this thing. But what that would mean is this G will get modified to this G 1 which is G upon 1 plus k 1 G. And then something in the 0 k sector you can convert it into something in the 0 infinity sector; that means, you can make this non-linearity convert this non-linearity into a passive non-linearity by this additional the additional thing that you do here, some output feedback.

Now once you do this the linear plant will also be have will also have to be modified accordingly. And when the linear plant is modified accordingly, then this linear plant is modified to this linear plant, but that is saying that this linear plant G 2 is k times G 1 plus 1. But then G 1 itself was dependent on G.

So, when you put all of them together you get 1 plus k 2 G upon one plus k 1 G. Now so theoretically, if you given this non-linearity in k 1, k 2 sector for any given plant G, you could calculate 1 plus k 2 G upon 1 plus k 1 G and check whether this plant is positive real. And if it is positive real, then the original system is asymptotically stable.