**Image Signal Processing**
**Professor A.N. Rajagopalan**
**Department of Electrical Engineering**
**Indian Institute of Technology, Madras**
**Lecture 22**
**2D Convolution Part 1**

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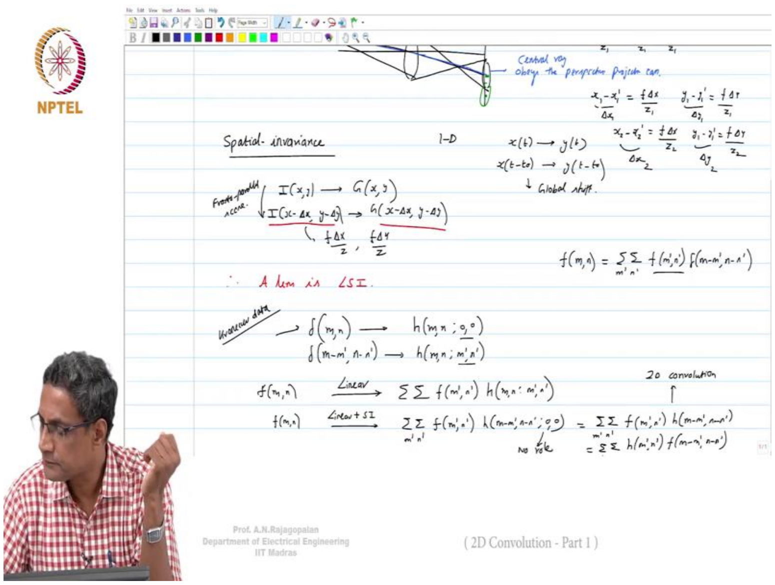


Now, which does that means that you know (())(00:19) we should be able to write down a convolution equation correct, when linear shift invariance that means one should be able to write down the convolution equation, so how do we show that typically so we say that you know if you apply let's assume that we are looking at kind of a discreet case continuous we will mean that you write you said direct delta (())(00:37) instead of that we will go with simply a chronical delta, let say delta m coma n a chronical delta be know right what it is said as a height exist only at m is equal to 0, n equal to 0.

So, delta m n if it go sin, say I mean there is a certain notion there is a small difference between impulse response and actually response to an impulse okay, so impulse response when somebody says (())(1:00) that typically means that you have a system that is linear time invariant and so on. But somebody says response to an impulse it could mean anything it does not mean that this system has shift invariance or time invariance and all that simply this are if I applies this impulse what did I see as a response, so that is response to an impulse.

Impulse response is basically a special sort of special pair of words you know that are actually results only for LTI systems okay, so delta m n lets us say it gives me some output okay which I call as h m n okay and I am going to put a semicolon 0 command 0 just to indicate that this chronical which situated at m is equal to 0 n equal to 0, so the 0 0 is to indicate that is impulse is applied at m equal to 0 n equal to 0. So, this is like an response to an impulse, I see some h m n now and then subscripted or you know as a function of 0 come 0.

Now, suppose I shift this chronical to some m prime n prime and then and as of now I do not assume linearity and noting I assume that let us say right that this system then produces and edge which again depend upon where the impulse is, this is like a response to an impulse, so if you apply m equal to 0 n equal to 0 I see some I see this guy when toy apply that m prime n prime I see let say this guy so this is index by 0 coma 0 this index by m prime coma n prime okay.

Now, in general if you look at any f of m coma n why we talk about impulse because any signal can be represented a scale and shifted version of impulse, so of m n what do we do we can write this as a double sum m prime n prime f of m prime n prime then delta m minus m prime n minus n prime, let me just mention that this is chronical delta because you are looking at discreet case which is called a chronical delta.

So, you can see this no, so m prime minus m minus m prime so we can see that for all values m prime n prime not equal to m n this will be 0 only at m prime equal t m and n prime is equal to n this is 1 and there you get f of m coma n for all of the other values this guy is 0 therefore this is equal to f m coma on the left correct, this is the way we try to express any general signals in terms of impulses.

We can amplitude scale by this number and shift it by m prime n prime. So, if this f m n goes here okay now because I can express it in terms of these impulses and all right, this delta m minus m prime and n minus n prime so suppose I write f m n where I mean f m n generated I that manner okay then what it will mean is the output if I assume that now if I start to assume that I have actually a linear system which means that if all of these were to be were to be put individually or they were be given simultaneously f of m coma n individually each one of them would have give me something.

Now, suppose I put all of them together but if I assume now that my system is linear then you will get f of m prime coma n prime into h of m n semicolon m prime coma n prime this is only if you assume linearity. On top of this so f m n goes in and then this is what comes out, now if f m n goes in and if I say this linear and shit invariance also okay then what will it mean so if I shift m minus if I shift my impulse to m prime n prime then my output should be just same this guy accept the issue become m minus m prime n minus n prime semicolon 0 coma 0.

So, this should give me summation f of m prime m coma n prime h of m minus m prime n minus prime semicolon 0 coma 0 so 0 coma 0 has no implications anymore it has no more meaning only when it was no when the shift invariance notion was not there this has to be exclusively put every time but now that if you assume that it is not only linear but also shift invariant then f m n will lead to so this is like showing an image which could be focused or whatever a pin hole focused image then the output will look like f of m prime coma n prime into h f so this is all summed up over m prime n prime.

So, because of the face that 0 coma 0 has no role so we will simply drop it and just write is as double sum prime m prime n prime and f of m prime coma n prime h f m minus m prime n minus n prime so once you have shift invariance this has no role and therefore can be dropped or this is same as h of m prime coma n prime f m minus m prime.

So, this is the famous sum over okay whatever m prime n prime h m prime coma n prime all of that holds whatever for the for the 1D situation, it h convolution f is same as f convolution at all that and all of that will still hold. So, this is called actually a 2D convolution it is very simple to implement you just now okay earlier to that prior to that we will kind of look at some more things about just h itself.

And then we are going to look at but you can imagine whatever you do with 1D in 1D what you do suppose I gave f the signal and then I gave you h the impulse response let say impulse response is now let us assume that we kind of flip the impulse response right, so we will do this f of h minus m sorry minus whatever minus prime in this case so you do flip it and then you slide it and for every slide you compute whatever is lapped multiply add that is your convolution.

Same thing you will do here also, you have an image you have this h the only thing is you have to flip it both about x and y that is only about one dimension here it as we flipped about both x and y dimensions you get the flipped edge and now you just had to slap it on f and slide it all over and wherever you keep it correct there is the image has some underlying values this h will have some values multiply both of them add them there will be the value at that point shift keep doing this right as simple as that, so whatever you do for 1D the same convolution operation just that you will have instead of one dimensional kind of what you called kernel you will have it 2D kernel.

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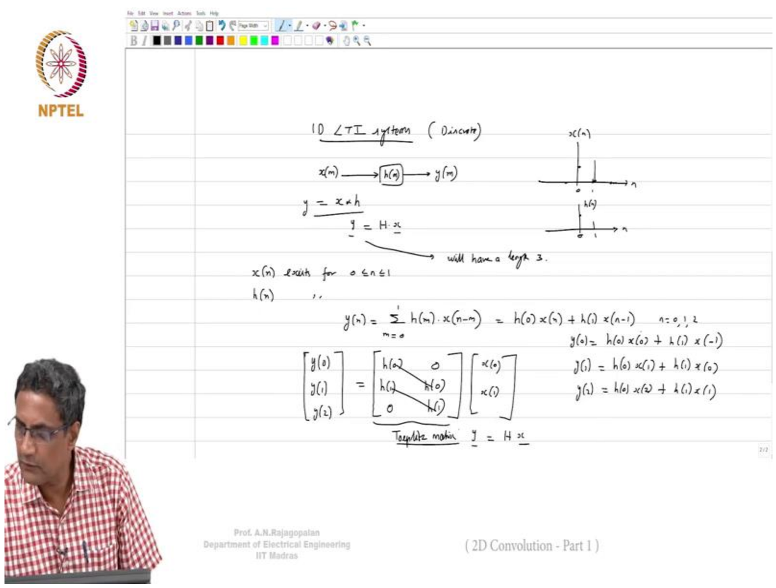


Now, at this point of time I want to take a small diagration to just look at those 1D LTI systems again okay because there are some nice things that we would like to borrow from there again we even when we do transforms and all we will typically look at 1D and come to 2D because 1D is something that you all are familiar with and then it will be important to see how this would generalize now when actually look at imaging system, so look at a 1D LTA system, let us say that I have a system there which takes whatever h m h n or whatever you know x m goes n and then y n comes out correct.

Now, we know that specially we are interested in discreet system okay image is typically which deal with discreet greed and so on so we will just stick to a discreet case. Now, we know that we know why of course can be express x convolution h because this is LTI we also know that you now this operation can also be represented in terms of matrix multiplying x, and sometime writing it in this matrix vector form throws a lot more inside into what is going on rather than simply writing this convolution equation.

So, even for actually 2D we wrote a convolution equation so the idea is that if I had to write hat again in a matrix vector form what would I get how would I write it because it looks I have an image here and the output is the image and then when I between I have impulse response so how do I write the whole thing how do I relate the input and the output but prior to going there if we can understand how we do it here and then you can actually borrow so many ideas from here to understand what goes on when you have a 2D case.

Now, during so… in order to show something right out here we will just take a very simple example because anything very complex and all (())(09:50)) will be a writing a lot of stuff here and then it will all scribbled out so we will take a very simple case assume that you know xn exist I am taking a very simple case let me say xn exist for 0 less than equal to 1, else it is 0 and then similar let me say hn also exist for 0 less than or equal to n less ta equal to 2.

So, it is like saying that I have an xn which just has some value at 0 on 1 okay something it has and then afterwards its all 0 equivalently and similarly right xn is something like that so I have a hn okay may be it is some 0 1 and then this is hn something like that. So, we know that when why we will have a length 3 we will have a length 3 because each has a length 2 w will have a length 3, 2 plus 2 minus 1, m plus m minus 1.

So, this guy will have a length 3, so if we start writing out suppose I write this convolution equation that we have here for this very simple case so it will be like y of n is equal to summation over m let m say this is our 0 to 1 let me say hm x of what I am taking x of n minus m, so what do we get we get like let me write this is h0 xn plus h1 xn minus 1 for n equal to 0 1 and of course length 3 and n will exist in this case from  will exist from 0, 1, 2 because okay for this conversation.

Now, If write y 0 instead of writing it in this form I am going to write this as 1 vector so this vector that I wrote here now actually I should put an put okay not this here, I actually put that underscore it is already there so it means that the original y that you have start up as a column vector, so y0 y1 y2 is what you have, y0 y1 y2 and we want to write this in terms of some matrix we are multiplying x0 x1 because of our input is only is basically a non 0 at x0 ad x1, so now we simply we write down so y0 okay so if you see here so y0 from this equation is 0 x0 plus h1 x minus 1 and yi will be h0 x1 plus h1x0 and then y2 as we can see is h0x2 plus h1x1 correct.

So, then we try to fill this same so it is like a h0 and then h1 x minus 1 but x minus 1 is 0 so I will put 0 there then y1 is h0x1 sorry x0h1 so h1 will come here and then h1x0 so does goes there ad y2 is anywhere so x2 is 0, so therefore this drops off and you get h1x1 so you will get 0 and h1.

So, this is for a very simple case that I have kind of say written but then you will immediately identify some structure right in this matrix, so this is what I mean by hx so I have written y in the form hx okay now what do you see about this matrix this has the particular name you all know about it what is it called? I am surprised, I thought by now you would have what is this matrix called? This is the name no?

Student: Topless matrix

Professor: A topless matrix correct, right this is called a topless matrix because if you look at that right I mean so you should look at the off diagonal entries the off diagonal entries will all be the same. So, this is called the topless matrix okay and this come only because you have shift invariant property that is just say time invariance not this linearity if you simply had linearity you would not see this structure because this is an LTI you are seeing a topless matrix otherwise you would still see a matrix whether it would not have a structure.

Now, this structure you may wonder okay what is its significance and so on but right prior to that of course and toe and it does not just arise only with respect to LTA system and all if you had a wide since stationary process rate there also you will see a topless not this kind of a structure. So, I am saying this is just happens to be that for an LTA also it has this property okay.

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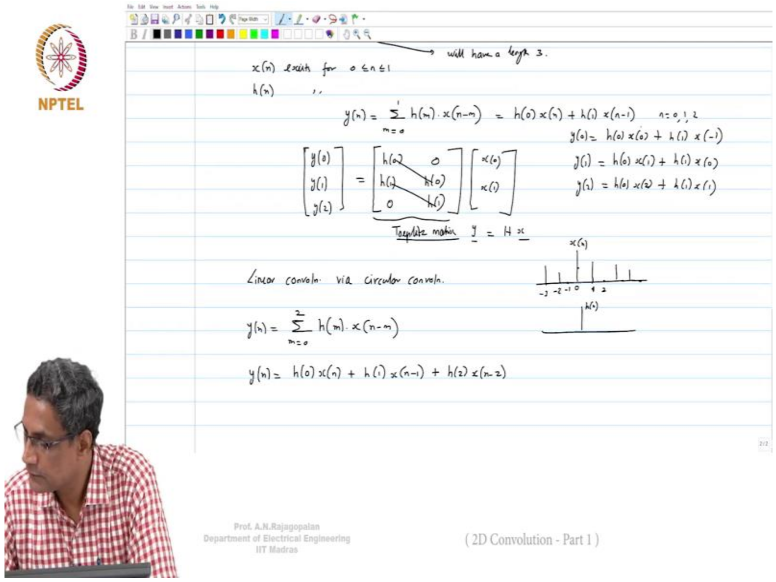


Now, what I want to do is you know I want to kind of express the same thing but through another way, let us implement you know linear convolution by a circular convolution lets do the same thing but the I want to come via circular convolution, what does that mean? That means that we should actually 0 u at and you know no how we do that so we 0 up at both x and y x and h to the length of y and cyclically repeat them and then do conversion.

So, what will you do now? So, you will take x and I would say that it exist at 0 it exist at 1 and now we will sort of say that at 2 we will this one by the way okay and then we again repeat it, good goes on and similarly from the left so minus 1 minus 2 minus 3 and then it goes on, so this is like x0 x10 x0 x10x0 x10 goes on similarly y I know hn will also have something like that right same thing, whatever the values will be different of course.

So, now given this we can write yn if we want implement through circular convolution we can write this as a m is equal to 0 to 2 now and we can write this as h at hm x of n minus m, now you see y0 sorry okay now let me just write the expanded in terms of n so yn is equal to h0 xn plus h1 xn minus 1 plus h2 xn minus 2, okay I have taken it does not matter I mean I took  only no alright.

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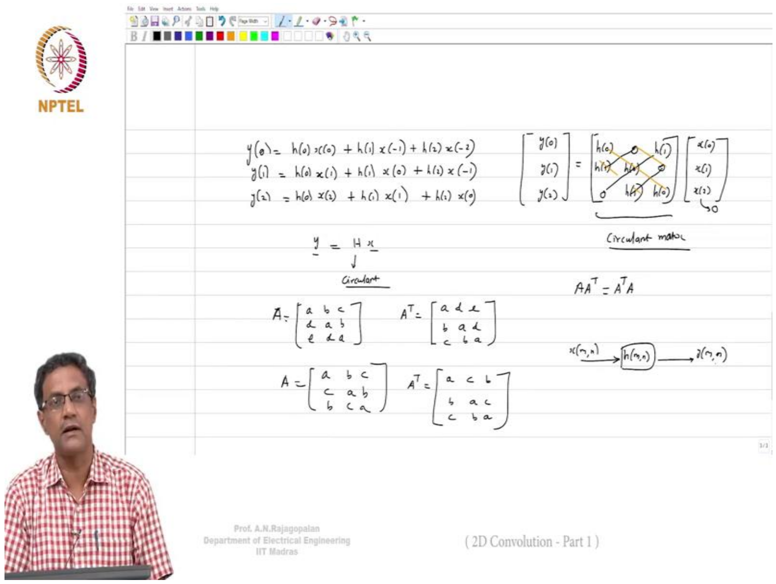


So, let us now expand this write it as the next page we will do so what is it so can you tell yn he if expanded so y0 in fact not yn, so y0 is h0 x0 plus h1 x minus 1 plus h2 x minus 2 y1 instead 0x1 if I make a mistake let me know h1x0 plus which is 2 x of minus 1and y 2 is h0 x2 and there is specific reason why I am writing like this even though both the outputs are exactly identical plus h2 x of 0, and now I am going to write this as y0 y1 y2 is equal to some matrix but now I am going to write here x0 x1 x2i will write but then I know that x2 is 0 this I know is 0 this is simply a 0 paddle input.

So, what do I o now let me write this y0 as h0x0 then h1 okay I think you know what it would have been actually better if you write in terms of okay or else you know you can look here x of minus 1 is what? X of minus 1 is the same as x of 2 so x of 2 will have a h1 I actually we could have written it directly but anyway right so we will write it here then.

Because for x of 2 x of minus 1 is to be interpreted as this one no x of minus 1 must be interpreted as a x of 2 and for x of 2 we have here x of 2 for which it is it will be easy if you write it the other way I think I okay then what about h of 2 into x of minus2 right and x of minus 2 is but x of 1 okay but h of 2 is 0 therefore this become 0 right, then y1 okay h0 what about this one h1 right h0 then x1 h0 plus x of minus 1 so x of minus 1 what we say it is x2 right and therefore for x2 it is h2 but h2 is 0.

Then third is y2 so x2 okay so here it goes h2 h0 then x1 h1 and x2 h0 but h2 is 0correct, now you see that this is also a topless matrix by the way right, look at all the diagonal and off diagonal elements this is the square matrix by the way, the earlier one was rectangular this 3 cross 3 multiplying 3 cross 1 giving you 3 cross 1 but then this matrix is called actually circulant this has even more structure than the other earlier one this is more structured than topless this one is actually circulant this is circulant matrix.

Circulant in the sense that every row if you take and then if you write shift cyclically so is the h0 h1 right sift cyclically will get h1 h0 0 again right shift cyclically you will get 0 h1 h0 and if you shift that again you will be back to the first row. Now, there are matrices that are called retro circulant in the sense that you have to raped them left towards okay then they become circulant okay but these are not retro these are called circulant.

Similarly, for the other one you have you now the things like retro and so on, in fact just out of interest so if you had the other way around let me just ask if you had these elements to be constant let say this way what would that be called? This if we had a topless this way this is topless, have you seen matrices that has entries this way other identical they would not be similar this way, this way okay now what  mean is like this see this are not equal no only these re equal right see in this case only this are equal correct this way if I go this way it is not equal but there are matrices for which this way things are same okay those are called and they are actually useful in hidden Marko models and so on. So, for a straight…

So, this is like saying y is equal to now same write hx but now this is a circulant matrix okay and when we write it in this form there is one very interesting property, have you heard of normal matrix?

Student: (())(22:34)

Professor: Yeah what is the normal matrix? When do you say the matrix is normal? Let say matrix a how do I check whether it is normal? Normal means not like abnormal verses normal, normal? Just say you have Gaussian normality, right? This normal has a different yes please tell now what is the I thought you said something?

Student: (())(22:58)

Professor: Very good a transpose is equal to A transpose A when this happens you say that write a matrix as normal so for the time being, now what is interesting is that you can check the topless matrix will not satisfy this in general whereas the circulant will, okay I mean I can do this here but you can just check it out just take C3 cross 3 so do something like A is equal to a b c now if I want topless I have to do some let d a b then I may have to write e d a this would be a topless matrix?.

Now, do this check this A transpose so A transpose is what okay a d e, b a d, c b a right now multiply the two so your first element is what a square plus b square plus c square right if you do A A transpose, if you do A transpose A your first element is what? A square plus d square plus e square, so here itself you can see right, whereas if you had something like some like you now some like circulant okay where if you now make it circulant will have what a b c, c a b, b c a.

Now, you can check of course this does not mean that they are symmetric circulant topless will not need in general be symmetrical also you will get like a c b, b a c, c b a right so in general they are not you know symmetric but now you can check that I just leave it to you just verify that you know that circulant matrix is normal every circulant matrix is in fact normal that is (())24:45) by this law and what the implications of this are we will see latter.

Now, for us it just suffices to sort of understand us to now if we can say all of these about 1D LTA system that you can relate it through a circulant matrix right of no in this manner what would happen if we had 2D system now instead of 1D and what does it mean to have, so if you wanted to write input and the output right should write a matrix or should I write put a vector and then have a matrix it will multiply vector then if it is a vector then what is that vector what is its relation to the original image and so on.

Because that is what will happen no now when you are looking now suppose we want to see with respect to 2D what will happen for a 1D LTA system you get a topless matrix or if you write it in this manner you will get a circulant so that has certain properties in fact you are very ability to go and express in term of Fourier coefficient we say now convolution leads to list of products right in the Fourier domain that itself happen because of the circulant nature of this matrix.

It may turns out that dft matrix is the guy which will diagonalize this actually but that is why dft matrix is a special sort of special rapo with a circulant matrices bears a special significance for circulant matrices similarly right we want to understand when you go to 2D I mean that is why all of these will come in but not now but why the circulant and why we are doing this you will realize the significance later but what we are now asking is instead of this suppose I had a 2D system I call this h of m coma n and then x of m coma n goes In and then y of m coma n comes out which is an image now all of this is an image.

Now, I want to see whether I can relate the input and the output to in a kind of a matrix vector orm and if I do that then what will be that matrix what structure will that have can that be related to 1D case and so on, that is the thing that we want to do next.