Image Signal Processing
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Lecture No. 39
Extending 1D Unitary Transform to 2D - Motivation

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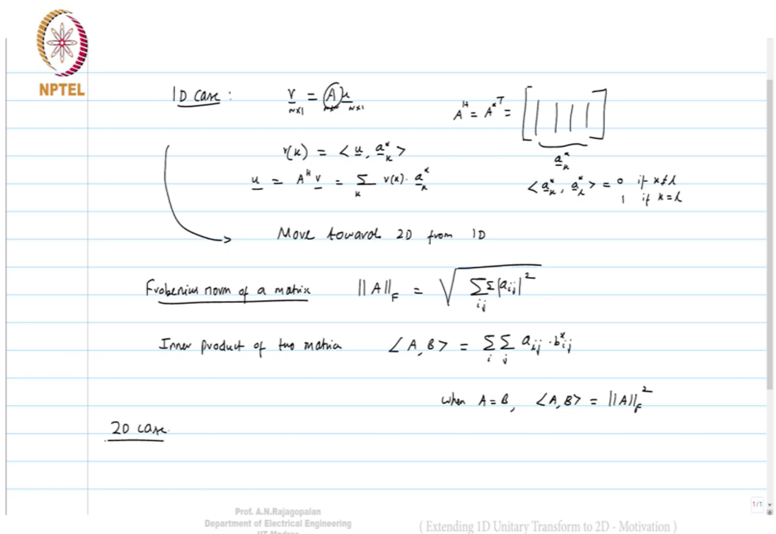

So, we started with actually a 1D 1D case we start we start with 1D case, and this 1D has actually a unitary matrix A, which we say will act on some sequence u. And u is of length n then A is n cross n. So, this is like n cross n acting on n cross 1 to give n cross 1. And then we said that each of these coefficients.

So, you can you can you can express them as, in a product of the sequence u, with a k star and where this a k star come from A hermitian which is suppose A complex conjugate transpose and we have taken some size n cross n and these columns where your a k star. And he said now that is a k star of course, the key thing about this a k star is that they are now also normal, they form an orthogonal basis a k star if you take the inner product with a n star then this is 0 if k equal or not, this is.

If k not equal to l, 1 if k equal to l and and then we said that you can expand u. And of course, one thing is u is A hermitian v but that also can be expressed in terms of the summation has sum over k, v k a k star all of this is. So, in terms of 1D and we would like to, so the idea is to kind of, is to move towards towards 2D situation, move towards 2D from 1D, towards 2D rather than straight away there is a jump into 2D situation we, you know.

So, we want to see if there is a kind of a natural progression that takes you from 1D to, so move towards towards 2D from 1D. And we would like, in fact utilize the same A in fact we would like to see whether we can utilize the same A in order to people to go there. If is a 2D, for example for an image if you wanted to find a transform then we would like to see whether this 1D A itself can help you get there or or enough if it is some other sort of transform which you call 2D unitary transform. Then we would still want to know what kind of a relationship it has been with this guy.

And and the other thing it not immediately clear as to how does how does one write down, for example, if it is a 1D 1D case, then it is clear I will have an n cross 1 kind of a vector for you and then add A on that and then I get E as an cross n, if I add an image it is not clear how how I should be writing it.

I should write it as a kind of n square by 1 vector then what would it multiply and why would I know that it is, is it 1D 1D A that is acting on it or is it really a 2D A, 2D you know unitary transform acting on it so those things do not obvious. But then as we kind of go along those laws become obvious hopefully. Now, prior to going to this, going to what are called the basis images, we will first in this look at this, frobenius norm then I talked about a matrix.

So we need some (())(03:55). So, we all know about it right. So, I just mentioned it here this is just for the sake of people that may not be aware. So, so the frobenius norm of a matrix and therefore all know a matrix has several kinds of norms. One of the commonly used norms is frobenius norm and what that really means is so also so you can have norm of A where in order to indicate the frobenius we will typically write it as A and this is nothing but who does now summation of let us say, cause entries of A as aij, then you take a double some overall increase ij then take the magnitude square root of that.

And similarly, that you can also you can also have something like inner product similar to the case of the vectors. You can also talk about inner product of two matrices. So, where we can have some inner product A with B and that will turn out to be summation again, double sum or whatever ij they didn ot square, but they have to be on the same side, whatever you know, same whatever and then aij B star ij.

And of course, when A is equal to B then of course you know clearly inner product A with itself A equal B this will be like norm, frobenius norm square. And later we will also see what is called as spectral norm and all that but now we are going to continue with the, because this is what we need as we go along.

Now, for the time being I mean you just have to kind of go with me in order to go to a 2D case. I am going to say that say that we will do certain operations it might actually look a little odd now, you may wonder why am I doing it. But then along the way, as things unfold, it will be clear why we did what we did ok. But now you might be wondering, why am I doing not required, why am I taking this kind of a product and all that but for the time being. You just have to wait till things unfold.

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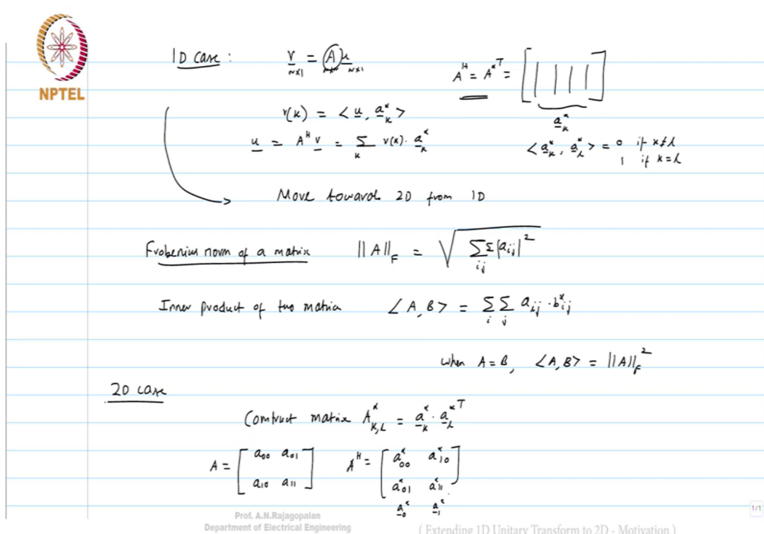

So, what we will do is let us say let us kind of do construction that is construct matrix. I am going to call the call this as a star k l such that it comes as a k star al star that is you see transpose. So, this a k star is the same guy, this is coming from the columns of the sonar mission. These are those these are not some arbitrary AKs and all. These are all, these are all these are the bases that you had here ok for you're a Hermitian. So, these are the columns of, so any a star k l is the matrix now now which looks like this.

Now for example, so look at, so I mean in order to just make this point more clear, we will take a small example where even 3 cross 3 will be slightly messy. Just take take 2 cross 2, so let us call this a00 a01 a10 a11. So, just take A to be that therefore A hermitian becomes that a star 00 a star 01 a star 10 a star 11. So, for us this is like a0 star a1 star. If you had if you had more if you had 3 cross 3, then of course you would have a1 a2 and so on.

So, I am just taking the smallest case for which we can easily write things over here. Rest of the things you can always you know code it in Matab and then you can take whatever size you want and you can show that all that I say for a 2 cross 2 will all good for whatever be the size, you cannot you cannot show all that here it will become messy.

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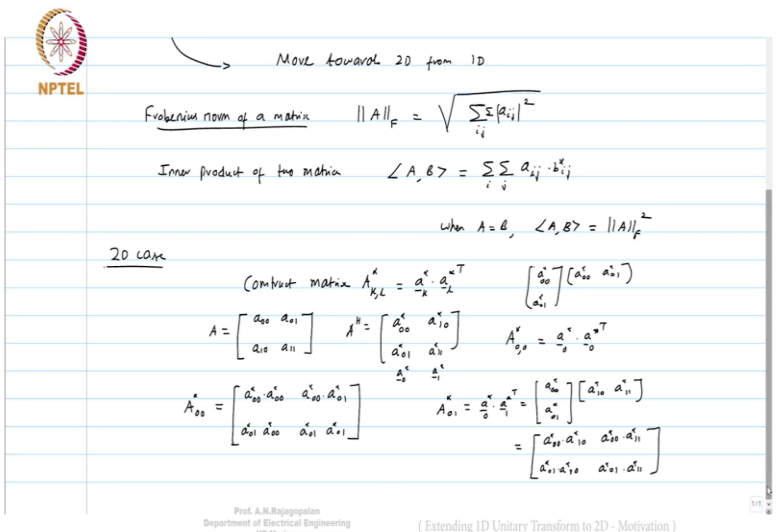

So, let us get also look at a star, so what this means is that then if I look at a star 0 comma 0, if I look at a star 0 comma 0 that is like a0 star into a 0 star transpose. This is a complex a0 star transpose. So, what this means is that my a star 0 0 is equal to, so it is like take this vector and then multiply it with its own row version.

So this will look like a star 0 0 into a star 0 0, a star 0 0 a star 0 1, then a star 0 1 a star 0 0 so it is like this, a star 0 0 a star 0 1 multiplying a star 0 0 a star 0 1, this is what we do. So, a 0 0 star a star 0 0 and then a star 0 1 a star into a star 0 1. And now let us look at let us say one more thing let us say suppose I take a star 0 1 what would that look like? A 1 star into a 1 star transpose.

So, that is a 0 star, a 0 0 star, a 0 1 star multiplying a 1 star transpose, a star 1 0 a star 1 1. So, this gives you a matrix that looks like a star 0 0 into a star 1 0, a star 0 0 into a star 1 1, a star 0 1 into a star 1 0, a star 0 1 into a star 1 1 . And you know similarly you can have a star 1 0 and a star 1 1 you can have 4 such 4 such cases for this a star k l if it is 2 cross 2. If it is 3 cross 3 you have 9 cases to and so on.

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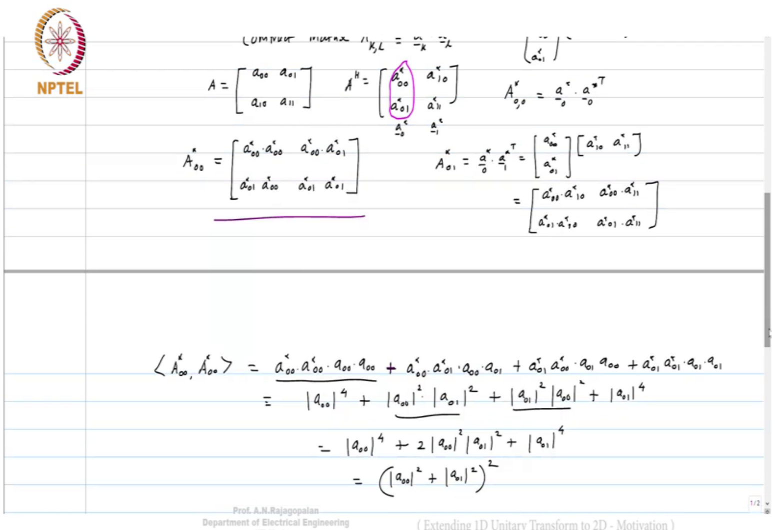

Now, what we are interested in is really is in this, suppose we take the inner product a star and let us say 0 0 with itself. Suppose we do this, we take the inner product as a star 0 0 with its norm, since we have already seen if we have two matrices you have to take the inner product. So, in this case it is going to be a scalar. So, we know this is going to be a star 0 into a star 0 0 into the complex conjugate of that its own entry 0 into a 0 0 0. You have this guy, a star so whatever is this guy 0.

So, we can keep it on top here 0 so it is here. So, then we will have plus plus then the next entry is let us say a star 0 0, a star 0 1, and its complex conjugate a 0 0, a 0 1 plus a star 0 1, a star 0 0 into a 0 1, a 0 0.

Finally, plus a star 0 1, a star 0 1, a 0 1 into a 0 1. So, now if you want to further simplify this so this first term is what? No, I am just saying the first term. First term is not one, first term is simply magnitude that a00 to the power 4. So, this is like magnitude a00 to the power four. Then this magnitude a00 square into magnitude a 0 1 square then this magnitude a01 square into a00 square plus this is magnitude a00 to the power 4.

So, if you look at it these two are exactly identical, no these two are the same. So, its magnitude a00 to the power 4 plus 2 magnitude of a00 square into a0 square plus a01 to the power 4. This is is nothing but magnitude a00 square plus magnitude a01 square the whole square. But then we know that magnitude a00 square plus magnitude a01 square is but is but the norm of, is but the norm square of a0 star.

Because if you look at this guy a0 star is distinct here, this is what we call as a0 star. This one is a0 star. So, so so if you look at if you look at norm its norm but its magnitude norm square is magnitude a0 square plus magnitude a01 square. But then we know that, we know from this other relation that we had, we know that this we saw here here we know that if you take the if you take norm of each one of them this form and orthonormal case so they all have length 1

So, what this means is that this then this kind of boils down to 1. So the answer is correct, somebody said 1 but after you, so you know the first term is not 1 but ones you ones you collect all the terms turns out that they read inner product as a star 0 0 with itself is 1. What about inner product of, so what was the other matrix that we down that someone 0 something, 0 1.

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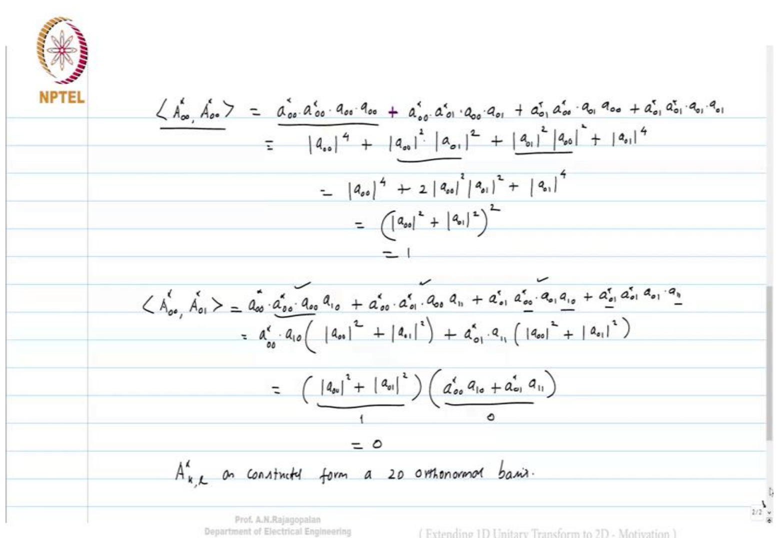

So, let us look at a star 0 0 with a star 0 1. If we had to look at what would what would this give, then let us say go back to that there. So, what is that entitle now a, a0 0 star into 0 star so a 0 1 is here so the a 0 0 into a 1 0 plus a square 0 0, a star 0 1, a 0 0, a 1 1 plus a star 0 1, a star 0 0 into a 0 1, a 0 1 plus a star 0 1, a star 0 1, a 0 1, a 1 1, a 1 1.

Now, we let us try to simplify this. So, if we so if we try to pull out a star 0 0, a 1 0 let us see how can we pull out a star 0 0, a 1 0 if you pull out I mean you can do it in different ways. I am just seeing, let us suppose I do that. You can do it in your own ways. Suppose you do a star 0 0 into a 1 0, take it as common, then this leaves me with magnitude a00 square for this stuff.

And from here, if a pick a star 0 0, a 1 0, then that leaves me with a star magnitude of a01 square plus then what do you have? So these two terms are gone we have counter for them. Again, let us now look at the magnitude a00 so a star 0 1 into a 1 1, suppose I pick into a 1 1 suppose I pick a star 0 1, a 1 1, then I get magnitude a0 square from this 1 plus then from this guy what happens, a star 0 1, a 1 1, so you get magnitude a01 square.

So, this magnitude a00 square plus magnitude a01 square is common and then you get a star 0 0 into a 1 0 plus a star 0 1 into a 1 1. Now this of course is simply the norm of again a0 star. So, so this quantity is just 1. Look at this other one this is like taking the inner product of a0 star with actually a one star because that will be like a star 0 0 into a10 plus a star 0 11 into a11.

And this we know that these two vectors are orthogonal and therefore this is inner product of a0 star with a1 star, therefore this is 0, and therefore this product is 0. So, the same way I will simply leave it to your some exercise to show that this a star kl and as as constructed which is like when they come from come as ak star, a star transpose as constructed form a 2D orthonormal basis.

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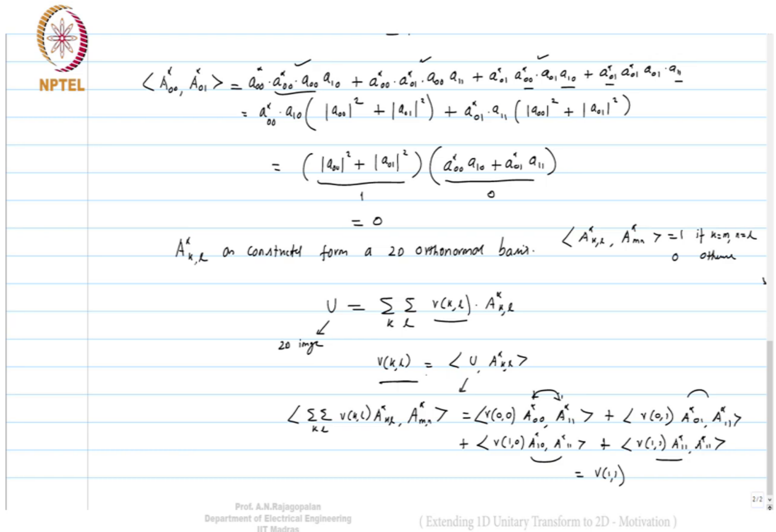

So of course, it is not like these matrices are orthonormal, they form an form an orthonormal basis which then means that you know it should be able to so which means that you can actually expand an image. Suppose I suppose I give you I give you 2D image now so what you can effectively do is we can actually represent it as sum double sum over some v k l. Again the motion of transform coefficient now for a 2D.

Now, see we I mean there are of course, other interpretations for all of this. I will come to them one by one. There is a there is a there is a larger framework under which they all fit in, but then I will go at it step by step. So, v k l into a star k l where this k and l will go over the size of whatever v, v will have the same size as u. And this v k l is again can be found as inner product of u, with you know a star k l that you can show.

Along then if you take this n, you can take this. So, if you use expansion that this double sum over k l, v k l. Suppose I take some a and m s star k l then suppose I take the inner product with some a m n, where you know m is not equal to k, n is not equal to l then you can see that this whole thing will kind of say expand out. So, it will be like v00 I mean you can see that.

Suppose you have let us say 0 to 1, let us just take the simplest case then it will be like v00 a star so you will be like inner product needs a star 0 0. Let us say a star suppose we take whatever now you want to take let us say 1 1. Suppose, then we will have plus you know product v01, a star 01 in a product a star was is it? We took one one plus you know product v 0 1, a star 1 0, a star 1 1 plus inner product v11 a star 1 1.

And we know that out of all this because of the fact that this is same as v00 will come out into inner product of this plus v01 into inner product and we know that inner product of all these case is 0 because they are not identical. So, one thing that made me want to write here is that this a star k l inner product with a start m n is equal to 1, if k equals m, n equals l, 0 otherwise.

So, which means that this inner product this goes to 0 this goes to 0, the only guy that will survive is v11. So, which is why v k l can be found as an inner product u with a star k l and that can be used in turn for expanding expanding u. Now at this point of time it may still not, it may still still looks slightly murky that why is why is why is all this happening. So, this all has to have the sight of suparability and so on let us say down the line.