Image Signal Processing
Professor A.N. Rajagopalan
Department of Electrical Engineering
Indian Institute of Technology, Madras
Lecture 42
Kronecker Product

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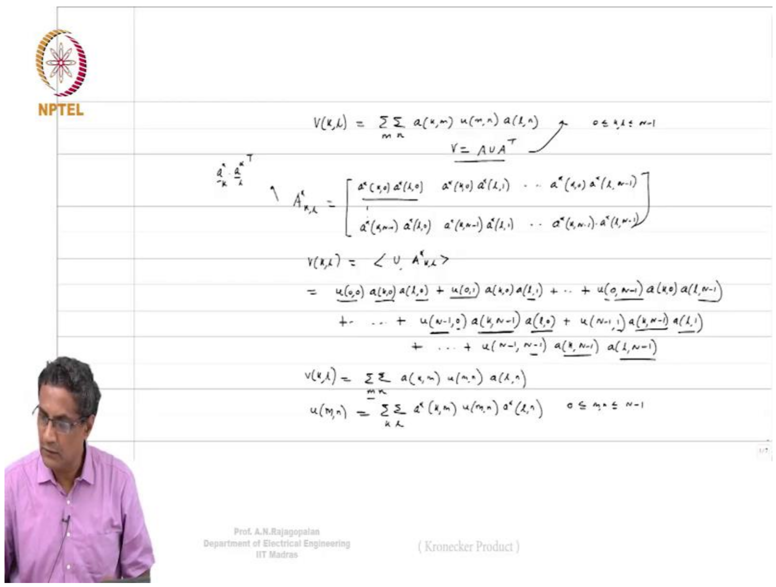

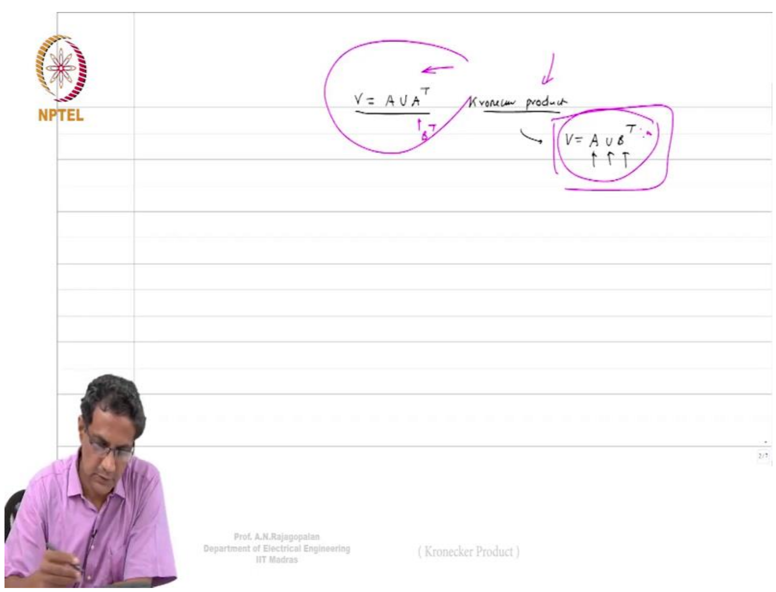

Now, the point is right that this particular thing okay that you have right this expression which is V is equal to au A transpose, right. I know I want you to write. So, we would like to show that there is an alternative way of writing this. Okay, there is an alternative way of writing this kind of a matrix product in terms of using a kronecker product. And it is a kind of a standard result.

Okay, that is actually well known, but since a few people may not have been introduced to a kronecker product or matrices, I thought we will actually derive it, okay.

So, in general, okay, we would like to look at if I had something A U B transpose, okay, we are in No, I do not care what A u and all mean, for example, they do not assume the day is unitary and all of that. Some arbitrary matrix A is given some arbitrary matrix U is given Of course, we should be able to multiply them okay that compatibility should be there, but otherwise we do not care what they are.

So, B could be something else okay suppose with this with this come out of this come out of this image transforms completely and simply look at a situation where I have a product like this, it is somewhere in some application and we do not encounter some A u B transpose then there is a there is a way to actually write this using a Kronecker product and that is valid for any A u B Okay, it does not have anything to do with image transforms by the way, and then if you have that result right then you can then you can invoke it for your problem okay.

So, for example, I mean in your case also you can write this is B transpose if your if your image is not is not square, correct if it is not square then of course, we will write it as a AUB transpose which will be the same which will be of this kind right so, we want to know, know, what is this form, right that we can that we can utilize, okay, which is a standard result That we would like to hear that we would like to write utilize in fact the way long these people go to higher dimensions actually using this result only okay since you may not have seen it or seen it earlier I thought we will show it here okay.

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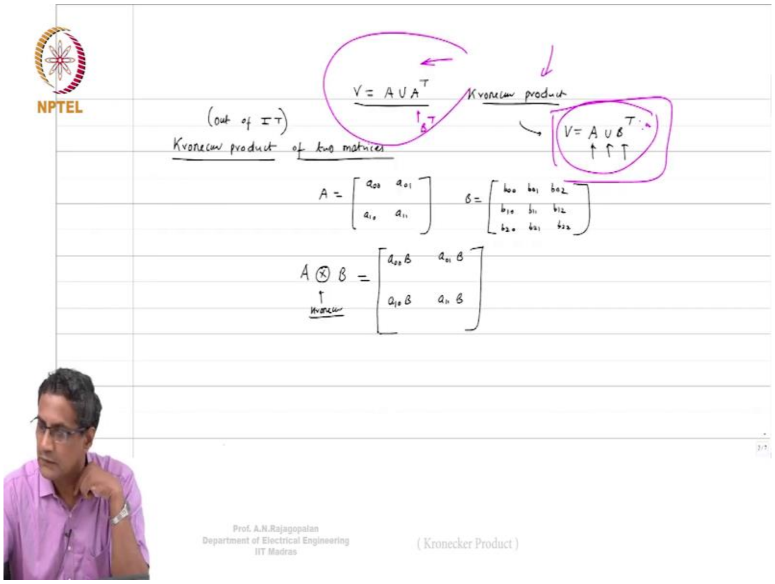

So, prior to that let just talk about what really is this guy a kronecker product actually this is an interesting product you know, i can leave it to you to find out more about it because all that we will need are some elementary things but no it is actually very very interesting sort of sort of a product okay kronecker product of two matrices.

Some of you right who are who are more interested in this kind of matrix operations and all that can can go and can go and then go on and off sort of investigated more, but you know, it works like this. Suppose I had you know, A is equal to let us say some A okay and now we are walking away from image transforms Okay.

Now whatever I write is some arbitrary matrix, none of these unitary and all is there okay. So, this is out of, out of image transforms. Okay, we are out of it. And it is looking at just matrices. Okay, so let us say Ais equal to A 00, A01, A10, A11, suppose I have this suppose I have another matrix right B, which is let us say B00, B01, B02 to something like you know 3 cross 3, it could be B 1 0 B 1 1 B 1 2 and then B 2 0 B 2 1 B2 2, okay.

So, If I have something like this, then when when you kind of talk about a kronecker product, the symbol is a is a cross an inscribed in a circle. And when you say A cross A, this one, okay, so this is this is a Kronecker symbol. Okay, those are those are standard simple, okay, A kronecker B is now the new matrix that is A 00 B, A 01 B, A 10 B, and A11 B.

Because it simply means to take the first entry of A multiply, multiply it with all of B and then take the second entry, multiply it with all of B, third entry whatever Right. So, an operation like this, it might actually wonder so what. I mean, if one can do this, then how does it impact us? It does impact us in a big way. Okay, we will see that. Okay, in a minute. But, we then prior to that, it may also list out a few things about a you know kronecker product kronecker product. Oops. Know what happens?

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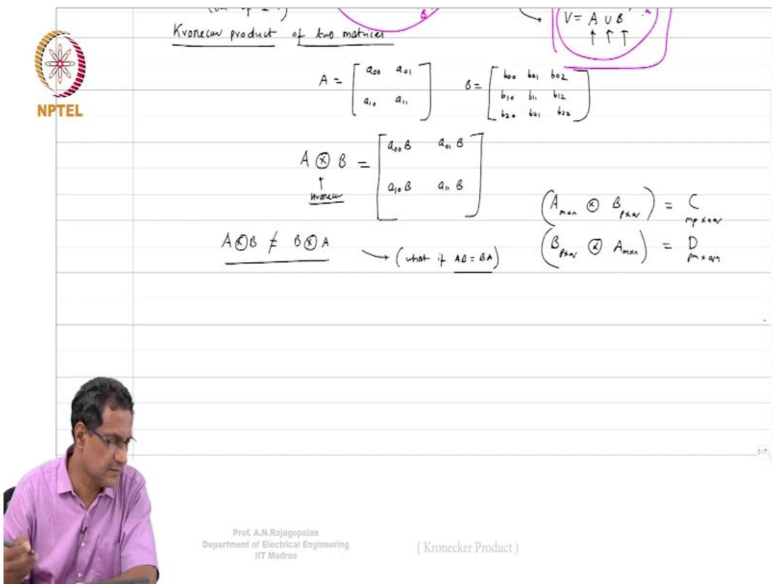

Okay. So, okay, so it actually means that so for example, in general in general, right, do you think A kronecker B, can it be could be equal to B kronecker A right in general local, right? It is not true. But you can actually think of some interesting trivial cases where you can show that this will happen, but in general, it will not happen. A kronecker B is typically what about the size limit? Can we say that the both will have the same size A kronecker B, true, because the way is, let us say, m cross n. And then if we are taking its kronecker product with B, let us say if It is p cross Q. It does not matter, they can be whatever size they are.

So, then if you take this Kronecker product, and suppose you call that C, then C will have MP cross NQ, right? Because it is like saying that, so it is like saying the rate and every time you multiply A 00, okay with this with this B matrix, right, then you already have BSP rows, then for for every element, you have P rows, and you have got like an M number of m number of rows in A. So, to be like P times m, that will be the number of rows. And similarly in terms of the columns, once you multiply you will get like, you know, the as many as many columns that B has and then multiply it by the number of columns a day has right.

So, you will have MP cross MQ and if you do B kronecker A right that will also be the same, right. If you do this, this will also give you some other matrix probably D and then the we see that that is pm cross QN and right size will remain the same but these two matrices right will typically not be the same. What is right what is Let us say A B is equal to B A and right as you can leave it to you to check what your first impression is like what a gut feeling. What if A B is equal to B A. What if it is A and B are such that right A B is A into B revealed to a gut feeling would be what? Yes, vignesh

Student: (())(06:29)

Professor: Well so fine so so but um, but I am asking right you know if this is true, then do you expect that A kronecker B might turn out to be to B kronecker A or you seem to feel that okay, that may not, that may not impact it. Anyone else? Shankara What is your Name? Shrikant, Shrikant What do you think anyway read I can definitely with you. You can think about let me list down a few more properties of this product to the kronecker. Not all are say relevant to us, but I think just this sounds good to know. Okay. You know you would not use all of this. Okay.

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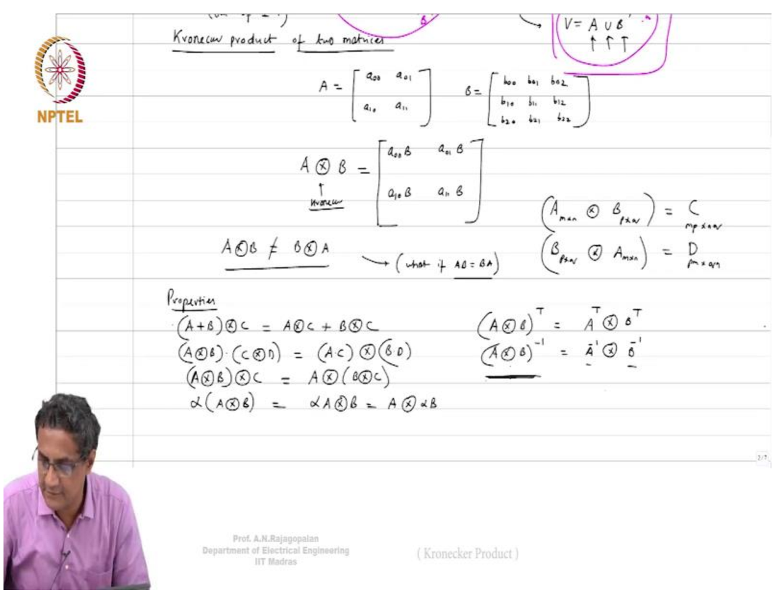

Some more some additional properties. I thought since we are on it right I just I just started, I going to list them. One is A plus B Kronecker C, A Kronecker C plus B kronecker C. Okay, this is one property, then a kronecker B into C Kronecker D is equal to AC Kronecker BD. Yeah, of course I know, in this case, you have to make sure that A and C you can actually multiply right there to be compatible.

So, the second property requires request compatibility. It is not automatic, no. Right. So, provided A and C are compatible, B and D are compatible. Then A kronecker, B Kronecker C is equal to you would kind of guess that that should be A kronecker B kronecker C. Then what do you think about this alpha A Kronecker B whether it is alpha is some is a constant, what do you think this might be equal to? Yeah, you.

Student: (())(08:27)

Professor: Correct. So, what do you think it might be like alpha A, alpha A, I can do kronecker B or I can also do A Kronecker alpha B right? That should also be fine. A kronecker alpha B. Then A kronecker B transpose. What do you think it might be like? A transpose kronecker B transpose right. This is unlike the A B transpose where it is B transpose A transpose A transpose Kronecker B transpose.

Then what about a Kronecker B inverse? A inverse kronecker B inverse in this Kronecker B inverse, but this last one right anyway, simply out of interest not that we are here to do a lot of matrix theory but I am just one of these I mean, can one of you tell me what would be the rank of A kronecker B in terms of rank of A and B? Can anybody make an educated guess that, you can see the way this this operation is right a chronicle be the way it multiplies and goes on. Or do you suspect once we the rank of A kronecker B Minimum.

Student: (())(10:03)

Professor: Sorry, no in terms of rank of A and B dubs minimum of both.

Student: (())(10:21)

Professor: Multiplication of both, its rank of A into rank of B rank of A into a rank of B okay. Which will also mean that right I mean, so if you had to write this right, I mean this this, so what do you think that, do you think that if A is not invertible, all let us say B is not invertible Would you be able to write this you cannot write this itself holds only if only both right A and B are invertible. Otherwise, otherwise this inverse itself you cannot you cannot write individually do it all because you can you can see the rank now. Right I mean, if you if you get a write up and if you use this rank result you should be able to show. But again, if we are not showing why the rank is so and so on.

Anyway, right? That is the simply out of interest okay. So, the point is this right? If you have a kronecker product like this Okay, which is what it is now know what we would like some of these we will use, but not all of these properties.

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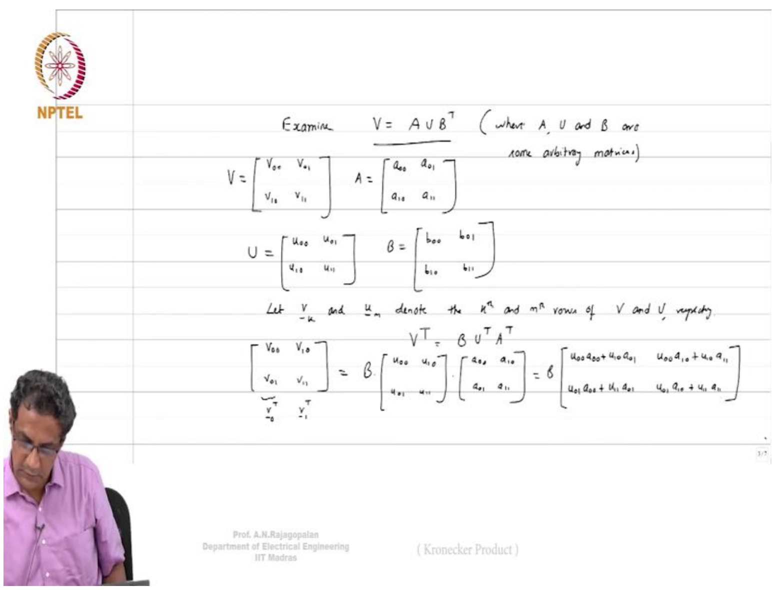

Now, if you go back to that expression that I said that we wanted to struggling this to three okay now, if you kind of now, if you if you simply examine, okay, if you know examine V is equal to A U B transpose okay, where there, now there A U and b are all right, arbitrary made by arbitrary I mean, you know, they do not need to be unitary and all are right, are some arbitrary matrices.

We want to be able to invoke this there is an excel in the article just on this. There is kronecker product article on that I mean, he says on the ubiquity of Kp, read that article. It is a fantastic article for those of you who want to know more about more about this kronecker product what it is what kind is it decompositions has it been used for in fact right the entire in fact there is a whole lot of theory that builds upon all this multi dimensional simple processing on this kronecker product, okay.

So, for those of you are interested, at least read that article, you know, very old article, but it is one of the excellent articles on this when he starts right from the basics and goes all the way up right and how you can exploit it for signal processing applications and so on. Golu Ben van loan okay. So, V is equal to A u B transpose right. So, so the point is okay, now let us say okay, let the so show it for a small example, right two cross two, going to show it and then I leave it to you write a MATLAB code and then show it for any case. Okay.

Now, let V be we have entries, V00, V01, V10, V11. A this we would like to finish today okay at all costs A00 A01 A10, A11, U U00, U01, U10, U11. Then B okay B00, B01, B10, B11. It is what it is okay. Now, let BK and what is this UM denote the Kth and the Mth rows of Mth rows of the V and M V and U respectively Okay.

Now, if you take V transpose, so V transpose is what B u transpose A transpose right. Now, let us just write down V transpose okay V transpose from the above is V00, V01, V10, V11 this is equal to okay. So, suppose we call this V0 transpose we call this V1 transpose okay this column is this Okay.

Then B now U transpose U transposes is U00, U01, U10 U11. And then A transpose which is A00 A01, A10, A11 is what it looks like. So, if you multiply this rate, what do get? You get B U00, A00 plus U10, A01, U00, A10 plus U10, A11, U01, A00 Plus, probably you guys reach (())(15:23) maybe that is the that is your, okay? So, this is what you have right.

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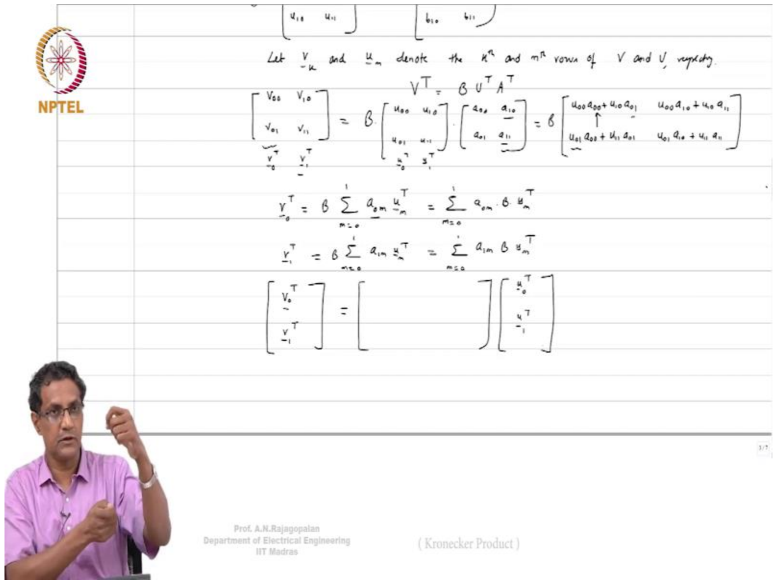

Now, if you if you sort of examine the first column, of V, okay with this your, this is V0 transpose. Okay, now if you examine the first column of V, right? That I can write as B summation in this case I will write it as M equals 0 to 1. See here, right? I mean it is like you see A00, okay, which is actually multiplying the first column. Okay, so let us call this as U0 transpose. Let us call this as U one transpose right? So, if you see, okay, A00 multiply U0 transpose which is the first column of U and then A01 multiplies the second column of U, right? So, this you can write as A0M after all, A00. There is all scalars right?

So A0M, and then UM transpose, because it is next because it is it is of this form, no. Okay, which also means that because A is because right, this guy is after all a scalar. So, we can actually send it inside as a M equal to 0 to 1, A0M B u M transpose right after all A0 is a scalar. So, we can we can push B further in right. So, one thing you have is this, right. So, I am sure you all agree.

Now, V1 transpose will then be what? So, V1 transpose if you see right that is that is the second column that is coming as A10 okay that is coming as the second column of right this guy okay A10 which is which is sitting here, then A11 right which is sitting here, and then it is again multiplying the first column of U and then multiplying the second column of U.

So, following the same thing, the summation M equals 0 to 1. Now, it will be A1M, right because it is A10, A11, no. So, A1M and then again it is uM transpose or this can be pushed in and then you get M equals 0 to 1. Then you will get A1M, B u M transpose agreed right okay.

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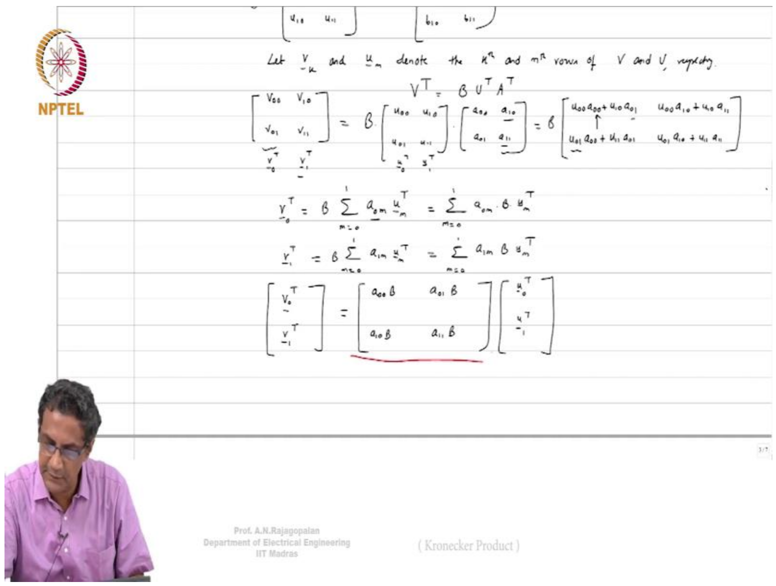

Now, suppose I gonna say stack these two up suppose a stack V0 transpose a stack V1 transpose like that then can I kind of write this as I will stack here U0 transpose stack U1 transpose by stack right i mean that see V0 transpose simply right you take the first row of V and then if you were to make it into a column, then V1 transpose like the second column second row of V right put us a column correct.

So, you are like stacking one column below the other. Then V0 transpose we know is what is A00, B U 0 transpose plus A01B, U1 transpose then V1 transpose is A 10B U0 or transpose, A11 B U1 transpose, now if you know if you if you see this result. This is not it is not something that you have seen before, right? This is simply a kronecher product of A with B back and I just took a two cross two for simplicity, is to show here right on the board, but you can this holds for any size, okay?

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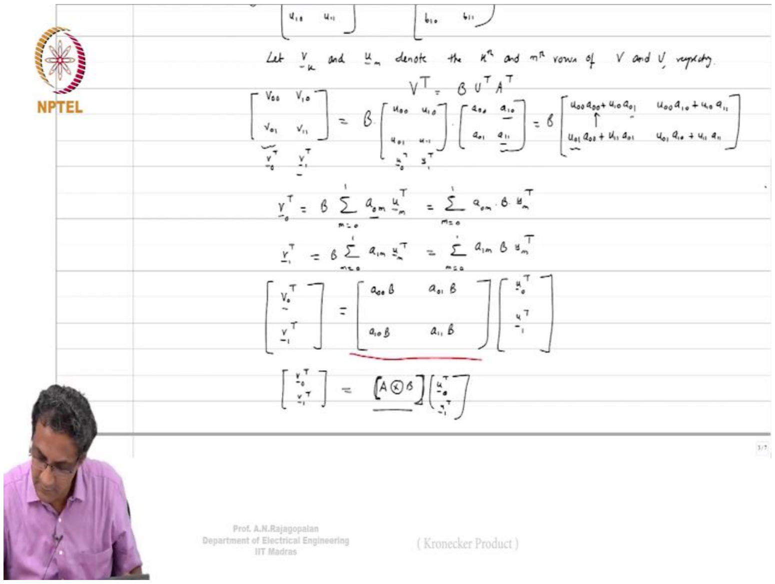

So, what this means is that, then this boils down to V0 transpose V1 transpose is equal to A Kronecker B, okay, and it is A Kronecker B and then multiplying U transpose, okay, this is whatever U1 transpose U0 transpose U1 transpose. It is not like you know, you cannot separate it out. You can do first A cannot do B times u transpose and then do A kronecker, okay. So, U0 oh you What is this You transpose okay. So, so this is the reason why we have enclose this in bracket okay. So, you have to first compute A kronecker B and that should multiply U0 or transpose U1 transpose.

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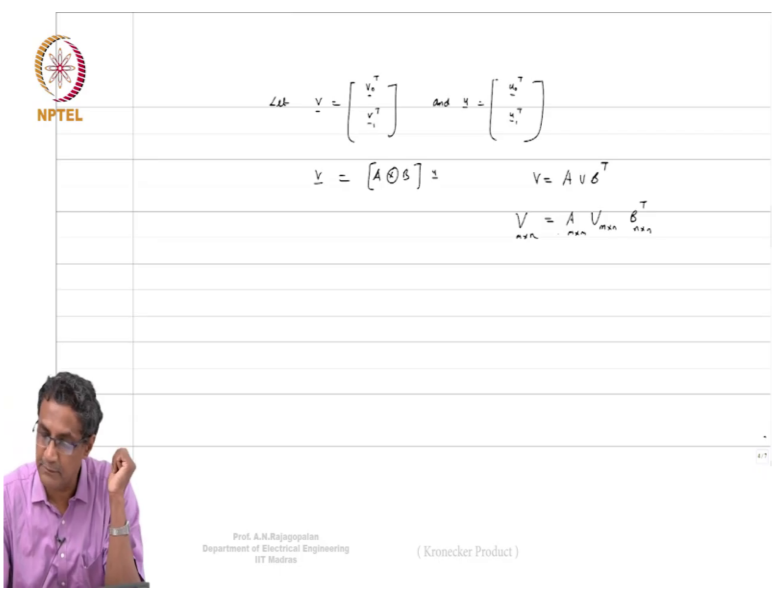

Now if you further right, okay if he writes if let this vector V represent V0 transpose V1 transpose and let this vector U represent U0 transpose U1 transpose, then it is clear that I can write V as A kronecker B times U. Now, if you see right this expression is exactly equal to V is equal to A u B transpose okay and as I said I am not assuming anything about unitary ness or anything right? This is for those in general true. So, for example, if let us say, okay if v is if let us say U is M cross N, okay then B will be N cross N, A will be M cross M and V will be M cross N okay in general.

Which then means that which means if you stack this matrix up right in terms of a column. So, this guy is dimension V will be MN cross 1 correct, you will stack it up as a vector, No, MN cross one and then when you do A kronecker B, that will have a size which is M what happened, which is now wait a minute, A kronecker B is MN cross MN, right MN cross MN and then you multiply it with the U which is MN cross one. So, I took two cross two but it does not matter, right. You can have you can have read something like this.

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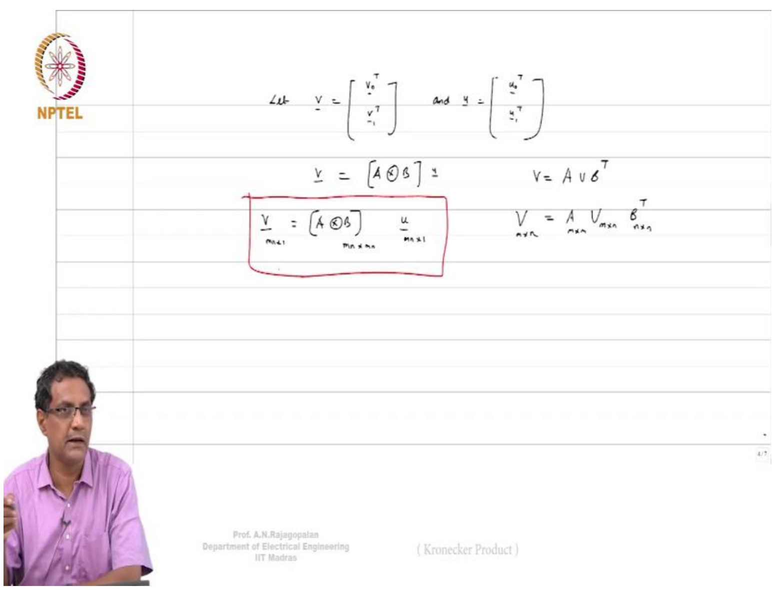

Now, what exactly are the implications for us right us this is the standard result okay. And I know you may you may underestimate it but this is a very very key result okay. Which has been this has been you know used in several places. But now, but no if you come back right