Image Signal Processing
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Lecture 43
Kronecker product (example revisited)

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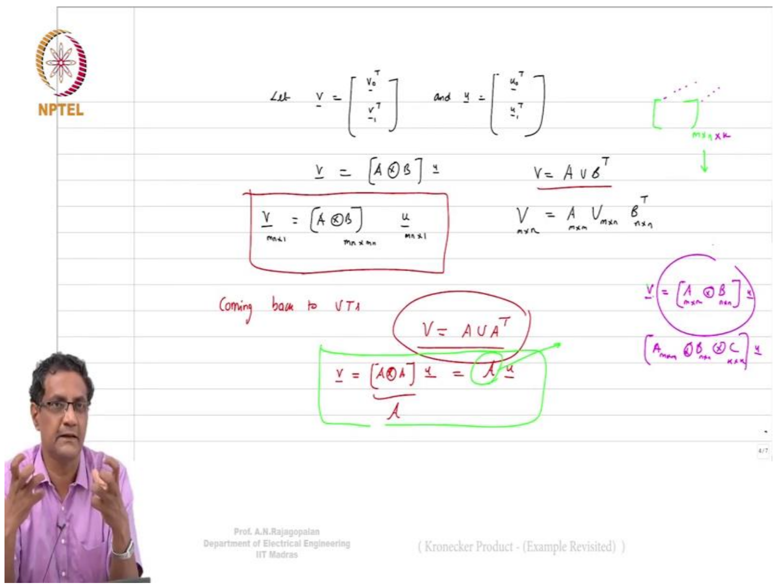

Coming back to our thing, which is about unitary transforms. Coming back to UTs ok. What did we have? We are in a situation somewhat similar, we had V which is some A U A transforms the V to V unitary right U was our image. So, what is this means is that than no because when right when we arrived at this we are arrived independently of anything else right. So, this means that I say I can rewrite this expression as a vector V as A krone as A kronecker A multiplying U. right.

And now suppose I suppose I indicate this as script A right. Now in order to in order to to in order to differentiate it from the original A, that is the that is the one D unitary transform. Then this becomes A times U. So, now if you see right this has the same form as you would have like as a reference in one D divided by V is equal to A U no. ok so where is the u where you have saw the image transform it look like this.

And then if you if you got to work your way around through a kronecker product it turns out that you actually have a form which you are already familiar with but then whatever held for one D right also suddenly holds for and if I had a video right. Suppose I had not this one frame see in this case right this is like this is like you had one D sequence right and from there you went to an image and then you said so forth.

So, your interpretation is that the 2D basis images are all sitting in this guy. In the script A. ok so this so point is right so this script A right contain the basis of images and all. And those basis images themselves are coming from 1D basis. Right because the 1 D basis we know we know right it all sitting in A. And m whatever, A you can basically show that they are sitting in A hermitian or in A whichever way you want to look at. Now, the point is right what is it what is it means for a so far I have told you for example now that instead of a 2D suppose I that extended to that 3D case, what would you do then?

Suppose now suppose the same thing I had an image, let me say which is M cross n what would you what would be formulation now in order to compute I am not let us not even worry about Fourier or something. I have some unitary transform and I want to kind of use that A to go to the transform domain.

So, and so for that A for it go to the higher dimension which is for D to actually 2D. I would say I will do V maybe I should write it from different color. So, V is what would I do? And say A A which is a which is a size whatever right. M cross n kronecker B which is a size, n cross n right multiplying U.

Now, instead of this I had a right if I had right with respect to time had a had a this one a video with respect to time let us say let us say I have a kind of m cross n cross A clips. K frames I have and I want to compute. 3D you know 3D this one unitary transform. What would you do now? Straight forward right mean straight a little then you should be able to see what would you do you do A m cross n then take kronecker B m cross n then take kronecker C of that dimension.

So, you will do A M comma M kronecker B n then kronecker let us say some C k cross k because then you take a video. You cannot assume that you will have as many frames as you have as the size of the image no. That is why I thought we will know till now I said right square as it is easy but then I want to extend it to one more dimension along time I cannot say I can take as many rows as many columns along K you know. You do not have that many frames along K. Right a long time you have fewer frames. You may have 10 frames 20 frames but then you want it in a computer transform right, simply means that you know this entire you know this equipment is available to you.

So, you I mean that is that is this is very elegant ok. Right suddenly you will see that going from lower dimension to higher dimension whatever 4D 5D does not seem the matter at all anymore because starting from that one D right starting from the one D transform everything seems like just you know one just walk away.

Correct so you see the strength of this framework right. If work through summation and all you would not see this strength and all. That is the reason why I said we will spend some time on this is ok right even if this right go through go through this because this way you will be right appreciate this the know structure much more.

This is all is so very tight, I mean nicely coupled then you do not want to miss all of that. Anyway I wanted to tell this much later but then I thought may be right now is the time to tell it. So, that you just realise that you doing it for a for a reason.

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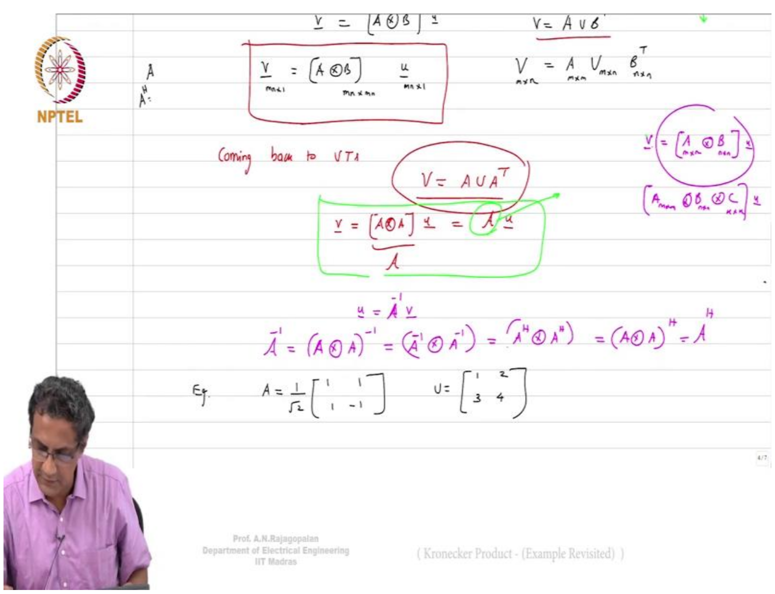

Now, coming back ok now, now suppose I ask you what we can say about A. Now, can I write U as A you know A script inverse V. Does script A inverse exists? How do we show that? Let us say A inverse I mean what is it right it was it was to it was to if it were to exist, this will be A kronecker A right. Because we know that script A A kronecker A inverse. Now, now if you now if you now look at a go back to the kronecker property, this is a A inverse kronecker A inverse.

But then but then A inverse we know we know already A inverse exists because we came from one D. So, that we know this is A hermitia A hermitian kronecker A A hermitian ok. And then again if you go back and use that property A kronecker B if we take A conjugate A kronecker B transpose. This would be simply A kronecker A whole this one hermitia but then A kronecker A is what script A ok. So, it means that so so I mean whatever happens there suddenly seems to be happening in very straightforward way right A hermitian just as you had A hermitian within the inverse of A script A hermitian you know inverse in script A.

And so of course so script A inverse A exists and then and then right you can of course right of course all for you and that a only thing is that u is sitting in slightly kind of in a different form. That is all it does not matter but man it makes us so much easy for us to analyses things. Because once you obtain your U, you just have to stack it back you know take the first n entries put it as a first row take the next one and put it as in the second row and then you are done. Then whatever rectangle square whatever right whatever size it might be ok.

Now, if you go back to that example let us let us kind of go back to example what would we had the other day you know, we did right. So, what was the example so this is solve this example again but now using 2D unitary transform straight away ok. What is the example we had? A we said this 1 D unitary transform given as 1 by root 2 1 1 1 minus 1.

This is and then we said our U is 1 2 3 4. Now, you know what the point is we still we still have not seen where the k star A l are sitting right, where they all vanished? You had the basis images right and there you know it was so very apparent that you actually take the inner product with the basis this all this all very clear.

Now, here it is not clear where they are sitting. Is there somebody who can tell me where they are sitting? Where should they be sitting? Just apply the same analogy now right nothing will change. Whatever happened there will happen here. That is that is the strength of this formulation. What will happen in there where were the where were the 1 D basis image is sitting ok so anyway we will see that now. But we should that no, where that A star k sitting? Where was the where was the basis image? Basis signal one D signal where was it where was it when you did the 1 D. I mean you took the you took the A metrics you took the A hermitian the columns of A hermitian contained the basis no. so that is where we started you know we say A hermitian take A hermitian take the column of A hermitian they form an orthonormal basis.

This is right. In a product A k star A l star is equal to right 1 whatever is K equals to 1 l 0. So, now given that 1 D basis 1 D basis signals if you want to call them K m from the columns of A hermitian then then where should basis image is form 2D transform is come from script A hermitia from the columns of script A hermitia. Same thing should follow no something should be seem less right. So, we will see that ok so that is why we will we will I thought revisit the example ok.

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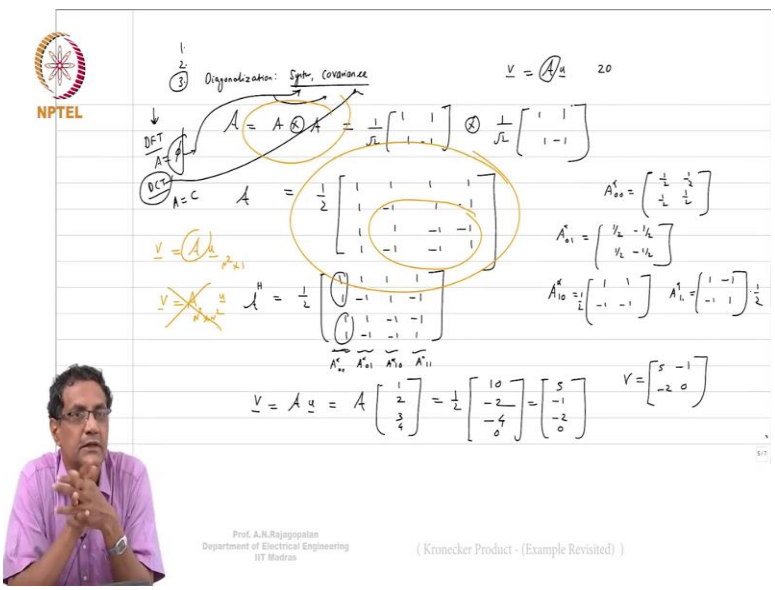

So, let us write down so script A first script A is what A the right A kronecker A, so what do you have A kronecker A is the what you will have so 1 by root 2 1 1 1 minus 1 take the kronecker with 1 by root 2 1 1 1 1 minus 1. Then you will get 1 by 2 we will 1 1 1 minus 1 1 1 1 minus 1 1 1 1 minus 1 minus 1 minus 1 minus 1 1. Ok this is A.

So, A Hermitian in this case, it is symmetric rate it would not matter. This is actually symmetric 1 minus 1 1 minus 1 1 1 minus 1 minus 1 right so it is actually the same. Ok but any way, write it down. So that, for this example it is this case but for some other example it would not be 1 minus 1 1 minus 1 1 1 minus 1 minus 1 minus 1 minus 1 minus 1 1 ok.

So, now if you see right. So, the first column of this one A hermitian ok which is all is the independent of the image itself right because they have a basis now because of because of you know what we are talking about is all data independent transform right. D and A is independent of whatever u you have.

So, if you look at the basis image can you get the can you see check if check your first A star 0 0 which is your first basis image. What was it? It must have been 1 by 2 1 by 2 1 by 2 1 by 2. Can you go and check this is what must have been. Here A star 0 0 ok and that you see sitting right here. Just so you know sitting in the form of a form of a single column a vector but then if you take the first two guys, put it as a first row, take next two guys put it as a second row as A star 0 0.

Then claim is your A star 0 1 should be this guy. 1 by 2 minus 1 by 2 1 by 2 minus 1 by 2. Please check that the second guy should be your A. So, this should be your a star 0 0, this should be A star 0 1, this is A star 1 0, this is matching. Should match with what you had that day no. 1 0 is 1 1 minus 1 minus 1 and then you have A star 1 which is of course 1 minus 1 minus 1 1 right. Sorry 1 by 2 has to be there multiplied by into half.

Correct so, all this all this should be exactly the same with what it was there. Except that you do not have to come you do not have to walk like in that manner. Out of all these 4 or 5 ways that image can write. The most elegant way to write it is this form. V equals to script A times u because whatever was happening there right, all of that is similarly apparent here. So, you can see as a straight mapping between whatever is happening at one D, can happen in 2D, can happen in 3D, can happen in 4D also.

You do not have to struggle to see what will happen if you do not want to extend this transform to the higher dimensions. Similarly, the basis image interpretation, the orthonormal interpretation everything is laid back ok. And you know in fact almost identical to whatever happened with the with the 1D transform. This is what I said all the while that right our focus will be to start from 1D and then not worry about whether looking at D fd Dcd and all.

We create a framework where you just throw in whatever you want and then all of that should just follow. Ok now for example, if I wanted to solve for V which is my transform question, I will do A times u ok so this is my A matrix right so suppose I multiplied it with ok. So, I what is it u again I have to stack it 1, 2, 3, 4 correct so my u go back to the u matrix. Ok so we stack it up as vector. Now what you will get?

So, what do you get A is this guy. So, let us say we just keep 1 by 2 out. So, 1 plus 2 3 10, 1 minus 2 is minus 1 plus 3 is 2 minus 4 is minus 2, then 1 plus 2 is 3 minus 3 0 and then minus 4 and then 1 minus 2 is minus 1, minus 3 is minus 4 plus 4 0. So, then this should be 5 minus 1 minus 2 0 or the Metrix v right drawn to put it in a transform and in the same way right. That you would you would rather probably see it as an as an image transform. It will be 5 minus 1 minus 2 0. Right this is what you would have got.

At that time also you had to A u transform this is what you would have got. So, V equals to A u ok. So, this is A star 1 1 after this what we do is we take up take up like one transform at a time. ok so what I thought is right we would not do all of them right there is too many of them to do right. Because you just want to know how this works. So, we will do we start with DFT and ok as until now I have read one more thing you know see right the other thing which you would again want to see right I had I had said that said that actually 3 properties.

One, two and see the third one right we still have not seen the impact of it see the third one which is about doing diagonalization and all. Till now I have not talk any of it. I have I have only kind of Hovered around 1 and 2. But this third property which is about which is diagonalization property ok. This is a diagonalization this has this implication for both system matrix and for covariance of the data ok for both live implications. This so the point is to see you know this what kind of structure this A acquires.

When you do when you do a 2D transform ok. We would like to for example, you know if you go through a DFT right. Go through DFT and suppose you have this A matrix for that let us call this as phi. Then we will see for example then ok then the idea is that what is the what is the impact of phi on this matrix of the certain kind. Which is let us say which we know for a reason let us say circular in nature and similarly if you think of the covariance which is again circulant in nature.

Have you see may be this kind of random process which is circulant. What is a word we call cyclisationary process and all, somewhere you have heard about it. So, for such matrix data, what are the implications? And the similarly right if you come down to actually A DCT. So, we will first look at a DFT and we will not I am not I would not spend too much time on all the DFT, in the way we do it signals and systems right in every property.

We are not interested in all of that. We just want to see how we can compute it from let us say from 1D to 2D. in some simple properties that now on that some of the many many of them are simply the same what you have in 1D but then 2D there are certain nice things you want to know about. We will talk about it. Then we move on to DCT and then here again we will talk about where it is implications are so DCT has more implications here ok. On what kind of covariance. Will it actually impact and we will see that it right more impactful then phi.

Suppose we call this A equals to C. So, now this matrix have to have to keep changing. Ok as you walk through walk from one transform to that another, this A will start changing. So, for so DFT will be phi so let us say DCT will be A will be C DCT is all real by the way ok. Whereas DFT we know does not generally not clear right. 3 cross 3 not at all real ok and now this one point I want forget to tell.

What will happen see if for example, I had this expression no. I wrote down this V equals to A script A times u. Now, it would have it would have some of you right some of you would have this doubt what if I what if I take the A matrix which is actually which is actually a 1D unitary matrix unitary transform. Suppose I say u minus n square 1. Suppose I take square image. Square by 1 so stack it up as vector.

Suppose I take A to be a size n square by n square. Would it kind of then give me give me a 2D transform? You see you see my point what I am saying as I put script A here. I said right that in turn comes from A but then you can argue what is will take an A which is a 1D transform. I have it you know I simply take it as same size u now. I make it as big as you want it. n square by n square, now I multiply, can I claim that you know well that that is that should be then 2D transform. It cannot be right then it will be wrong then right. That happens then that whole thing falls flat.

And in this example you will be able to see you have to see whether you saw that in this example. What is striking about his, you tell me one thing this guy something is very striking about it you know very small example. Tell me what strikes you there. The same argument you have applied it there what would have happened? If you had 4 cross 1 image, you know you know in this case simply a 2 cross 2 correct. Suppose I taken say this is as DFT DCT they all they all will be the same. At 2 cross 2 but imagine that if I had a u which was 4 cross 1 and suppose I interpreted as 1D sequence and multiplied that with actually DFT matrix now which 4 cross 4, will that entries be real?

You see here all entries are real. Ok so this is not the same as taking A or DFT of that size 4 cross 1 start multiply. It is not that ok so right this is important please remember that you cannot take a 1 DFT and you know make it size equal to this vector size multiply and claim that right that should be then why that is now not you know why that is not 2D is not, that is not the way it will arrive a way 2D transform. 2D transform comes through this this one. Through a kronecker product.

A kronecker of A with A and one striking example is this because you know right 4 cross 4 DFT matrix, you would have already encountered entries which are which are all complex. You would have entered you have entered you would have entries here which is all which will all be complex. Ok the first row and first column, rest of all will all be all be complex. Whereas that is not the case everything that is sitting here is real of course even here it will no be complex. The moment you go for A which is 3 cross 3 things have become complex.

But the point I am trying to say please remember that this is not correct. Cannot take a 1D transform and think of u as a signal 1D signal and start multiplying and claim that I have this one 2D right. Then the then this whole exercise meaningless ok. Let us stop here.