Image Signal Processing

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Lecture - 46

2D DFT Visualization

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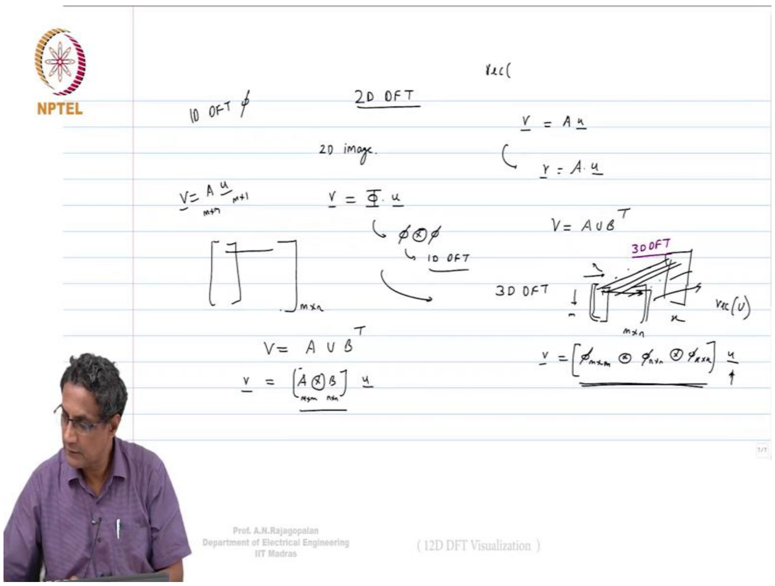

So, 2D DFT. So when you go to, when you want to go to a 2D DFT, so it is like saying that you have an image, a 2D image and you want to take its Fourier transform. So similar to the case that we had, v is equal to A u and then we said moving to 2D will mean v is equal to some script A times u.

Similarly, here what it will mean is we can, so we used phi for 1D DFT. I think I had used this small phi, so what we can say is if we lexicographically order the image, so that is equal to some Phi times u. There this Phi is phi Kronecker phi, so this is the 1D DFT and it follows that. If you have a 3D DFT, I was saying if you have a rectangular image and then not just this and then if you have multiple frames, then the idea was that if you have K number of them and if this was some m cross n.

And then at that time, I think there was a doubt as to how should we write it. So, how should we write it now, because, I think that we should finish it and then move on? How do you, so that day, what did we write? So we said V is equal to, what did we write that day? phi m cross m Kronecker phi n cross n Kronecker phi k cross k. And this will have to act on u.

Now, the point was how should u be stacked now if this is the order that we are taking? So, what do you think should be the answer for this? Which way would you take? So I have this, I have all these guys stacked like this. So my k is running in this direction, my m is downwards, my n is this way.

So, how would you pick the first? See, you have to pick columns and stack, so which column will you pick first? Which will be, which will you pick as your first column? Will you travel along m, will you travel along n, will you travel along k? See the reason is, see if you look at this, if you look at, I mean when you took the 1D case where you did A times u, suppose let us say u was an m cross 1 sequence then this A was actually m cross m.

Now, u is like this 1D. Now, if I add a second dimension to it and then I make it like m cross n now, I have added one more dimension, so it becomes m cross n. So this we said if it is m cross n then it becomes A U B transpose. You cannot write A U A transpose because now it will become A U B transpose, and where B is, of course, B is n cross n, A is m cross m.

And if you write, if you stack it up such that now you go along the rows first, you take the first row put it as a column, take the second row put it as a next column, if that is a way you order your u then your v, of course, will also follow the same order, you will interpret it the same way and this become A Kronecker B times U.

So A is, of course, m cross m, so B is n cross n. So the second dimension that you extended it along it is coming after the first, you initially had m, so it was A m cross m, now you sort of extended it along, like along the columns now, made it m cross n. So you see that if you stack u that way, then it becomes A m cross m Kronecker B n cross n.

I mean if I had instead stacked u the other way, let us say, if I stacked u such that it was the column first, what do you call, yeah. So if I take, no, sorry. If I have taken along the rows first and then take the next row and then stack it up like that then this order would change, it will be basically B Kronecker, B Kronecker A. Correct, I mean then this, then it will not be A Kronecker B, it will become B Kronecker A because you are ordering, ordering u that way.

So similarly, when you have here A U B transpose, now if you have m cross n cross, so the simplest way to remember if you want to remember the simplest way then you can think of it as initially. Initially, it is along, let us say one-dimension which is m, so you will act A m cross m on it. You extend it along n, then now you make it as A Kronecker, in terms of phi, phi m cross m Kronecker phi n cross n.

But now remember that you are using stacked such that along the dimension that you increased it, you take that first put that as a column and then next guy, put that as a column, I mean if you want to remember it that way. And now, if you put in k now, so that it will become Kronecker phi k cross k wherein now you will take along k the first. So it you will be like, now you will be spanning like this. Go take the first column as this, second column as this, finish of the first row then you will go to, so it will be like you will cover all the K-dimension along k first and then it will mean you will next cover along n and then you will next along m. So, and that will be the order for you.

So what I was going to say was, okay. Now coming back to this 2D, so this was all for 3D DFT and the same notion will also extend if you want to do some nD DFT you just have to see, you just have to see how we are going to order and then accordingly do this. Now going forward, I think if I use this page then there is problem, so I will have to use a next page.

So, going forward. So what we are saying is, if you now have an image I will show you square images, that rectangular I just brought in because if you had a video then the dimensions and all need not match.

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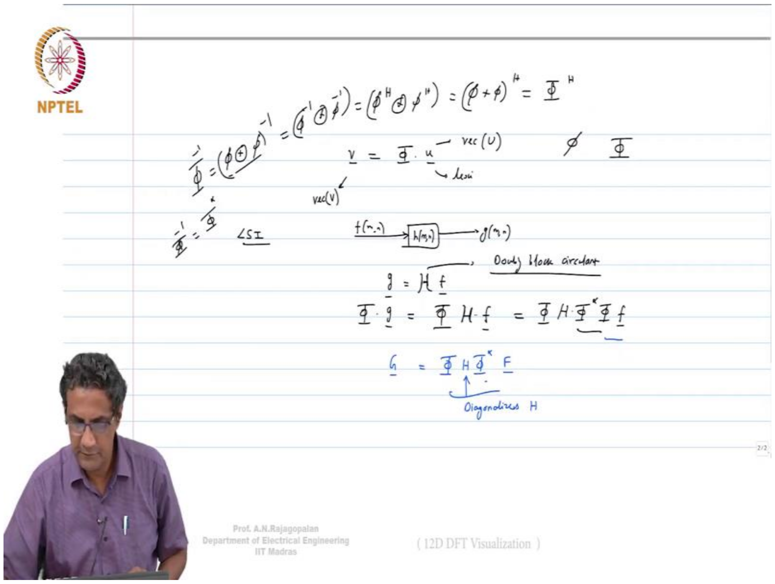

So, what we will have is, so v is equal to Phi acting on u. So this is kind of lexicographically ordered or vec of the capital U. Then this is a vec of capital V. If you want to think about it, this is vec of capital U and then this guy is, of course, and on this has a dimensions. If U is n square by 1, this is n square by n square. So it is very simple.

So for example this is similar to 1D, when we have, if we have linear shift-invariance system and you think of an image going in and then it is getting convolved with some, with a point straight function h m, n to give, let us say, some g m, n; and g in this case if you stack it up again, the same way that we did earlier, then we can write this in a matrix-vector form as H times f.

And this, I will write this as some kind of different H just to denote that this is a 2D, 2D set of a system matrix and this we know has a doubly-block circulant structure, doubly-block circulant structure, that is what it has.

Now, the same thing, now if you want to go to the, go to a transform domain, if you want to take a 2D transform, of course, you should then do multiply on the left g, by Phi and then multiply here also by a, so I have written it as script Phi, this capital Phi so as to, so that we know that it is not the 1D DFT. 1D DFT, I am writing like this. 2D DFT, I am writing like this just to differentiate.

So, phi and then you have your H, this H guy and then into f and then we again play the same trick. So all those things will just follow. So we just do Phi H, then Phi star Phi f, and then all this you can show because if Phi is phi Kronecker phi, then Phi inverse, Phi inverse will be simply Phi star. The same as that what you had. This and all and you can take the inverse of this, take the inverse of this, then that becomes phi inverse Kronecker phi inverse, but phi inverse is phi Hermitian, therefore this become phi Hermitian Kronecker phi Hermitian and that becomes equal to phi Kronecker phi Hermitian and that is nothing but the capital Phi Hermitian.

So all of that holds. That is a nicety about what actually coming that way, and therefore we know that this is identity therefore we can multiply that way. But now we are more interested in equating the Fourier DFT coefficient, therefore we look up on this as let us say some capital G and then this becomes Phi H, Phi H then phi star and then F.

And as you would have expect, it does not, it is not really surprising that this Phi, when it multiplies a doubly-block circulant matrix if you were wondering what will diagonalize a doubly-block circulant matrix when you look upon it as kind of a from a system point of view, turns out that it should be, it should be a 2D DFT. 2D DFT is the one that will diagonalize a doubly-block circulant. So all of this just extends naturally.

For 1D DFT, you had a circulant matrix that was diagonalized by a 1D DFT; for a 2D DFT you have a doubly-block circulant matrix as your system matrix and that gets diagonalized by using the 2D DFT DFT in this manner, Phi H Phi Hermitian of Phi star. Then this will diagonalize it and all those things that we said last time will also hold, this diagonalizes H. And again, the Eigenvalues of H can be interpreted as DFT coefficients, they do not have to be real because H by itself, it may need not be symmetric.

And again the interpretation of all is the same so you interpret the diagonals, the diagonal values as the DFT coefficients and all of that follows. And again as a, in a kind of a statistical sense if you had a 2D random periodic process then this will diagonalize it is covariance, all that, all that will work the same way. Today, I thought I will go ahead and sort of talk about of few other things, about a DFT.

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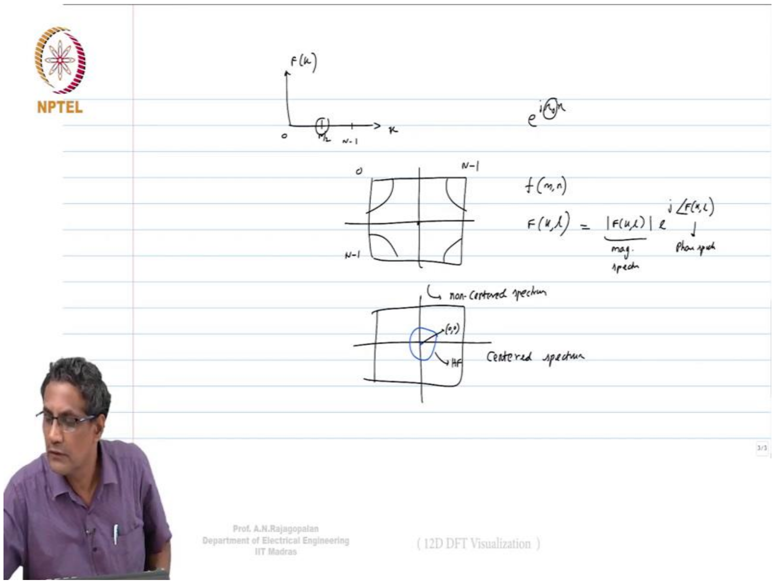

First of all, now, when you plot a 1D DFT, when we plot a 1D DFT, let us say I am plotting a 1D DFT from 0 to n minus 1, this my K. K going from 0 to n minus 1, let us say this is my F of K. So where is the maximum frequency? What is your interpretation? So if I give you a complex exponential, which complex exponential has highest the frequency? Which complex exponential has the highest frequency?

I mean, e j omega naught n, which one? For what value of omega naught do you get the maximum oscillation? Pi, no? We are going to look at pi. Look at it, e j pi n. So n equal to, n equal to 0, it is 1; n equal to 1, it is minus 1; n equal to 2. So it is kind of, that is the fastest that things can oscillate, 1, minus 1; 1, minus 1; rest of it will be a slower variation.

So we sort of interpret n by 2 as where the, I mean, that is how when you plot, you look upon this as, around this are your higher frequency and then as you go towards 0, you have lower frequencies, you go towards n minus 1, you are hitting the lower frequencies, that is way of interpret.

Now if you use the same interpretation, I think this is how MATLAB also does it. So if you, if it is directly taken as an array, if you take an f of m comma n as your image and suppose you compute a 2D DFT of that and then if our interpretation is 0 to n minus 1 and 0 to n minus 1. If that is our interpretation then what will happen is, if you plot the spectrum, similar again, now if you talk about a Fourier spectrum in time domain, I have a signal that is varying with time, therefore when I compute the Fourier transform it tells me how the signal is varying with time. So, I get a sense of how quickly it is varying.

But in 2D, when you plot the intensities and when you plot a Fourier transform, then you can again ask for the magnitude spectrum, you can ask for the phase spectrum, similar to 1D. In 1D what do we say, when we have magnitude spectrum? It is simply says, if you have a complex exponential it says what should be the strength of that complex exponential and then, of course, you also know by what phase it should be shifted with respect to the other complex exponential so that when you superpose all of them you will get back your original signal.

Same way here you have kind of a 2D complex exponential and then you will have phases for each one of them, and then you have to kind of align them all properly. Align in the sense that each should have its corresponding phase shift, and then you add all of them up, then you should be able to get back your original signal.

But then here the, here the interpretation of the frequency is what is a the variation along x-coordinate and what is the variation along y-coordinate. So if you have, if you have this leading to a Fourier transform like F of k comma l, then along k you will interpret the variation along x. So, how does the signal vary along it is x-coordinate, how does a signal vary along it is y-coordinate.

So k comma l, it will be like the intensity variation on that grid because there is no longer notion of time. We have only a spatial grid on which the intensity is varying. So if you have everything constant then it means that it has only a DC component; if you have things varying then we understand that now there is variation along maybe x and y, both directions, and then this components and all that you have, that you plot will tell you what is the your component, k component, what is l component, and so on. And similarly, you can have a phase spectrum.

So, typically this is, of course, this is a complex number, so we write this as magnitude F k, l into to e j phi, let us say, angle of F k, l, and that is the phase. So when we plot this, we call this as the magnitude, all of this is similar to what happens in 1D and then this is the phase spectrum for any signal, for an image.

But the only problem is, if you plot like this then what happens is, at the center you will see all the high frequencies because that is how you will interpret. Because this is also complex and it is separable, then all of that, so whatever interpretation you had for 1D, that same interpretation will hold.

Therefore what will happen as we plot some frequency, if I give you an image and if I ask you to plot the spectrum, what will happen is you may actually see something like that. You will not see anything in the middle because very unlikely that you will have a, well, if you still had but then you will not be able to interpret it properly. This is called really a non-centered spectrum.

This is also correct, but then only thing is, I mean, a visualization is tough because it is like saying that the lowest frequencies are here and they are kind of extending this way towards the center whereas what we like to see is something at the center being the (high), being a low frequency and then right as we go away how does it die down, that is what we are kind of more and more accustomed to seeing.

So, therefore, what one does is one kind of swaps these quadrants. So what we do is, so instead of this we would rather plot what is called as centered spectrum, where you simply fold, where you simply fold these guys such that you get something like a centered spectrum where you might actually interpret this as your lowest frequency 0 comma 0 and then things kind of fall off.

Now normally, you expect the, as your frequencies go up, you expect the magnitude spectrum to kind of die off. What will it mean? Actually, you know what, dominant between phase and magnitude even in 1D it is true, in 2D it is much more apparent.

Suppose, if I gave you two sinusoids. Suppose, let us say, you had to add them up to let us say reconstruct the another signal. Now when you add the two sinusoids, suppose I go wrong in the term of amplitude of one, instead of multiplying it by alpha, I do some, let us say 1.5 alpha or something, somewhere I go wrong in the amplitude, then, of course, you will get a signal that is not really what you want, there will be a difference.

But suppose there is a phase shift, let us say the other case is, the other sinusoid it is supposed to be shifted in particular way but then I do not shift it that way, I shifted in an incorrect way and I add the two. Which one do you think will have a more damaging effect on your original signal?

Student: Phase.

Professor: Phase, right. So that is what will happen even in so, okay. So the centered spectrum is what you will typically see in the books, this is what they plot and this is called the centered spectrum. So now, these are your high frequencies. This is like a 0 comma 0, so that is your lowest frequency that is like a DC value and so on. So a centered spectrum is much more easy to kind of visualize, let me show you.

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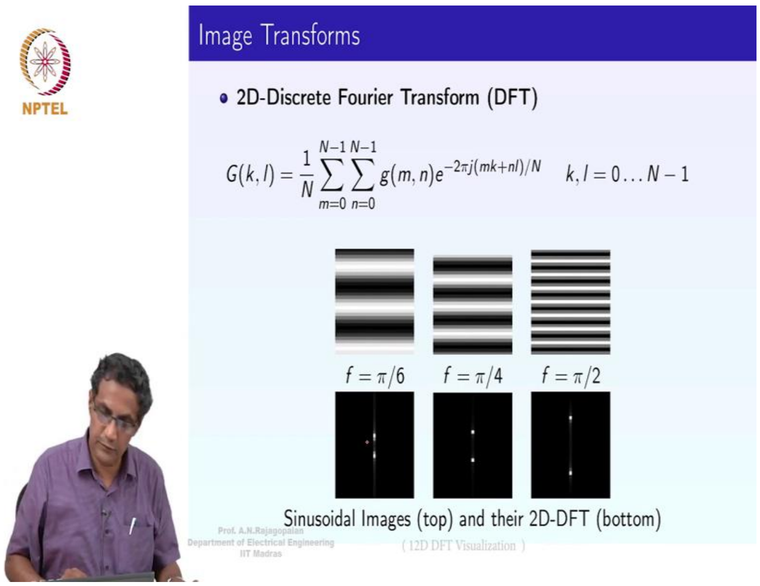

So if you see, so look at this. This is a variation only along, see if you look at these three images, so if you see along x, it does not have any kind of variation. It is a sinusoidal variation along y, it is actually a sinusoidal variation along y, except that from 1 to the, 1 to 2 to 3, you see the frequency is increasing.

So if you notice, f is pi by 6 and then it become pi by 4, and then become pi by 2. And therefore if we compute a DFT because the variation is only along one direction, so it will fall exactly on the, on your L-axis, you get your components and of course, they will go further apart because your frequency is going up. So this is how you will kind of visualize it.

No, G k, l (())(18:06) is down. k (18:08) is down. Its spectrum is down, the image is on the top. Then there is a next one.

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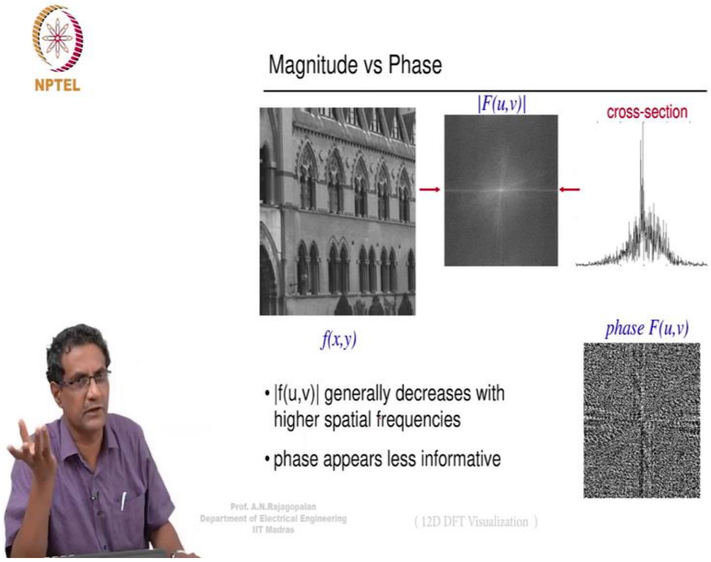

So if you look at something like a building like this, if look at what kind of a magnitude spectrum do you get, so you see the one there. Magnitude, it looks like something but then it is hard to, you cannot interpret what might be the original image and so on. The phase looks great. So that is how it is.

But then you may think that at least looks more organized in the sense that whereas the phase looks like it is, but actually, in reality, it is the phase that as you would expect. Like just now I told you example. So I should take and if you interchange the phase and magnitude.

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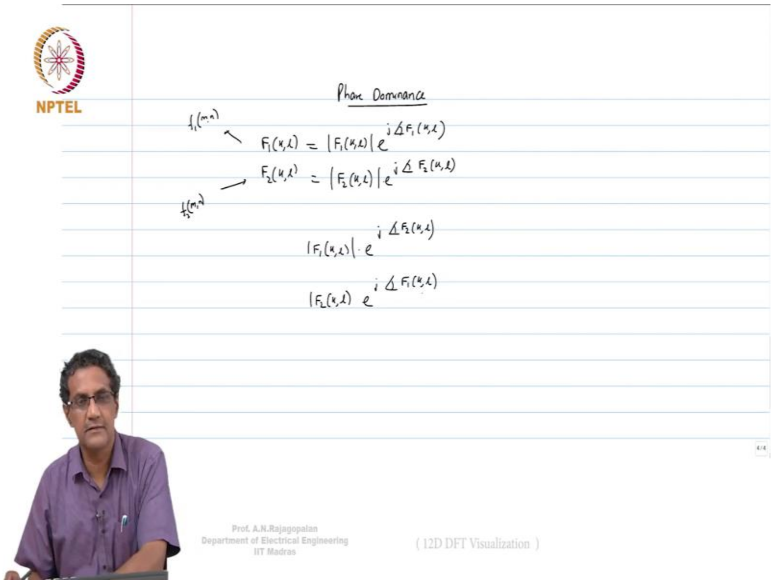

So it is like saying that if you take, it is like saying if you take the, so this is called, this called phase dominance. So what does it actually means is if you take, let us say, suppose I have F1 k, l which is like magnitude F1 k, l, and e j, into e j angle F1 k, l, and then if I had F2 k, l. This is, of course, coming from some image f1 m, n, this is coming from some other image f2 m, n. This is magnitude F2 k, l e j angle F2 k, l.

Then in order to show this magnitude dominance property, what we do is you can take magnitude of F1 that is, let us say, I take the magnitude of the first image but then take the phase of the second F2 of k, l. Or I take magnitude of F2 and the phase of first.

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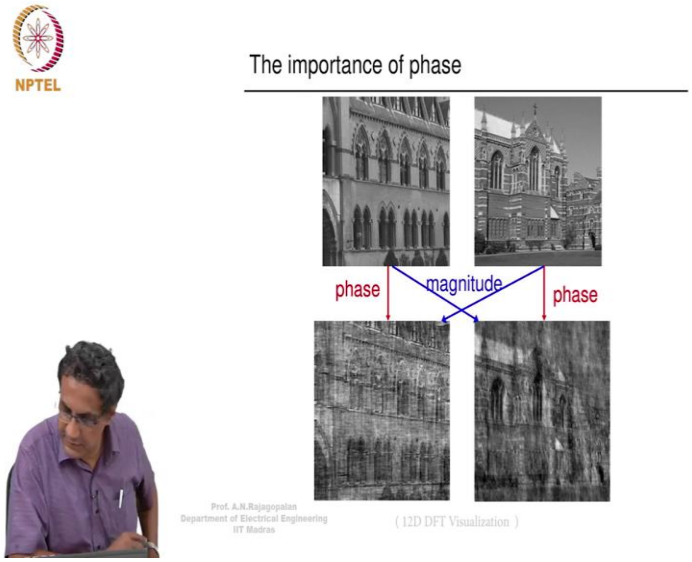

So what you can, what you can see is, so what this figure that I am showing here is showing is, so what it is showing is, if you take the phase of the first so, of course, in this case, that building and this building are not identical there is difference between the two. If we take the phase of first and the magnitude of second, you can see that this building's picture is more dominant when you reconstruct.

When you are reconstructing, you are taking absolute value and all that, there is no guarantee that you will get all real number, so that and all has been sort of, there is some handwaving because when you reconstruct, we have taken absolute value. Similarly, there if you look at instead if I take magnitude of that and phase of that, then you see that that is the one that kind of comes up more dominantly. So this is the well-known fact.

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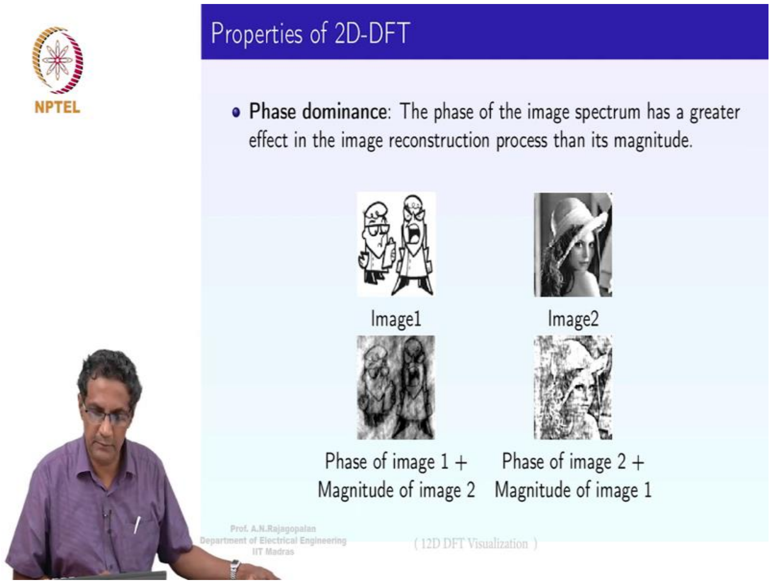

In fact, for people who like cartoons, Lena and this, so you can see that if you take array; Dexter, right, this is? So if you take that and then take Lena, then you see phase of image 1 is that of Dexter. So you see that Dexter is clearly there. The other one, Lena is clearly there.

So, this is kind of a dominance, phase dominance property. But, of course, it depends upon where you want to use what, but in general, it is known fact. But then it is also a known fact for 1D.