Image Signal Processing
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Lecture No. 48
1D DCT – Definition, Motivation

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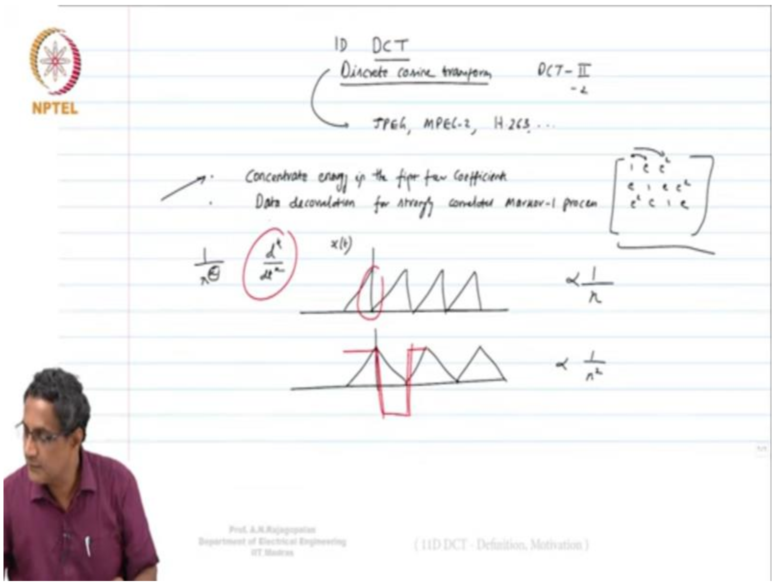

Now let us start. So, so the idea is not to kind of go through all properties of every transform and all mainly with respect to let us say decorrelation and energy compaction that is where our focus will be, not in terms of studying what happens to the transformer if you shift the signal, what happens if you, DFT I thought I will do it because DFT all of you will tend to use. This is a cosine transform. This is somewhat now this some of you may have heard about it, some of you may not may have not seen it before and again, we will start with 1D and same.

So, since we have already laid the laid the foundation for going from 1D to 2D to 3D so all that will follow just that will do the 1D in more detail. Then 2D and all is just straightforward. Now, a discrete cosine transform this we study because it is been used in in a JPEG and JPEG and MPEG-2 and then H.263 and all of this it has been used. Then you might wonder why was that picked over a DFT when DFT seems to be the ones that it does the rounds most of the time but when it came to doing all this compression, this coding so it was a DCT that got picked and so we would just like to know what is it that makes makes a DCT tick.

Now so, so this revolves around 2 properties. One is one is what is called the ability to to concentrate energy. So, the ability to concentrate energy in the first few coefficients energy in the first few coefficients and the second thing is its ability to decorrelate and by the way, DCT has many forms. There is not like 1 DCT. The one that we are doing is kind of a most standard one that is called DCT 2. That is what it is called, but there are several forms of it. One cannot go through all of that and all. So, we will just look at the most standard one. This is what is used in compression and all, DCT-2. So, whatever I am writing is for this particular DCT 2.

So, data decorrelation for strongly correlated for strongly correlated Markov-1 process, strongly correlated Markov-1 process. So, Markov-1 process is something like this, in 1D. It look like 1 rho, rho square and so on where this rho is a number between 0 and 1 and then you can have rho 1, rho and rho square and rho square, rho 1 rho and so on. So, it is like saying that so so you have a correlation as you look at this with itself of course it is 1 but then as you go to your neighbouring neighbouring neighbouring value, then this kind of decreases and then if you go further off, it decreases even more which is natural.

Your dependence as you look at farther off entities, all will automatically go down. So, for such process it is not immediately apparent as to what so if you asked what will diagonalize this, if you had covariance matrix of this type. If you ask what will diagonalize it is it is not obvious as to what will diagonalize it. So, turns out that DCT 2, the one that we are going to do, if you look at the basis vectors of this one, it turns out that that the basis vectors come very close to being the being the eigenvectors of this matrix which is which means that analytically also they are very close.

It is not just the sort of determinants it is also like, on the whole class of such process, on the whole class of such data, which follow this Markov-1 property with a with astrong correlation. DCT 2 will in fact be the most optimal. Although it is not equal to the KLT, it is not like it exactly diagonalizes. It becomes very close to diagonalizing. It is something that you will see. In fact what it diagonalizes really, I thought we will talk about later. It is like a Toeplitz Hankel kind of matrix. I think I talked about toeplitz that it is this way. So hankle is of course, the other way.

So, toeplitz hankel is sort of a combination in terms of that toeplitz hankle structure has something to do with this kind of the covariance matrix. We will see that down the line. So, the idea was well I can write down the transform but before I write down what what a 1D DCT looks like or what the equation for that is like before that let me just let me just talk about a few things about about the point number 1, which is about concentrating energy in the first few coefficients. I mean so this kind of goes back to your signals and systems and Fourier series and all.

If I gave you a waveform of this type, let us say it is a sawtooth waveform, I think this called sawtooth. Now, if I give you give you a periodic waveform like that and suppose I give you give you a periodic waveform like this. The second one is called a triangular wave. The first one we call this a sawtooth wave. So, between the sawtooth wave and the triangular wave, which one do you think I mean, if you compute the Fourier series this periodic shows the other one.

So, if I compute the Fourier series coefficients, which one do you think will decay faster?

Student: The below one.

Professor: The below one.

Student: (())(6:17)

Professor: It will it will die off relatively more slowly, but can you tell me in what order will it go down and then what will be the rate of decay? For example, you are correct, the second one will decay faster but but then can you say something about at what rate will be the decay of the coefficients of the courses in the first case versus the second? You must have seen the Fourier series coefficients for this. Have you seen it?

So, for the first one, this Fourier series coefficients will be will be inversely proportional to 1 by n. There will be some other terms and all. This will be a proportional to 1 by n square actually and where do you know what is that, why that means how do you figure out if I give you give you a waveform whether it is proportional to whatever in n bar k. Let us say, how do you find out that smallest integer k that you can claim is the one for which that this will decay. If you take take take this x of, suppose you call this as x of t, if you take d xt by dt, can you take d square xt by dt square and so on.

The first time an impulse appears when you take when you take when you take sort of a derivative the smallest d, what you say dk, whatever you say dk by dt k, the smallest k at which an impulse appears is the k that will sit here. For example, the first case if you will take dst by dt, immediately you see an impulse because of the jump there in the sort two. Here you already have a jump no, so so at this point at this point there is there will be an impulse. If you take d xt by dt. Of course, it will be pointing downwards because it is a fall but but then if you take d xt by dt at that point, it is an impulse.

The second, if you look at it, if you if you take d xt by dt, it will be a square wavew I mean if you take d xt by dt, in this case it is rising slope, this is a falling slope, rising slope, falling slope and so on. So the first derivative, you will not see any impulses. Next time when you take a take a derivative because here, here it is a positive jump that is an infinite negative jump. So, you start seeing impulses. So, the smallest k. So, look at it.

So, d xt by dts xt by dt, itself you start seeing an impulse or originally itself it has an impulse then it means that it will not decay at all like if you have begin with an impulse, but if you have a signal and then the first derivative gives you an impulse then you know that it is going to decay as proportional to 1 by n then 1 by n square, 1 by n cube and so on. Now if you why why I am quoting this example is because if you look at the, if you now come back to come back to now a discrete sequence.

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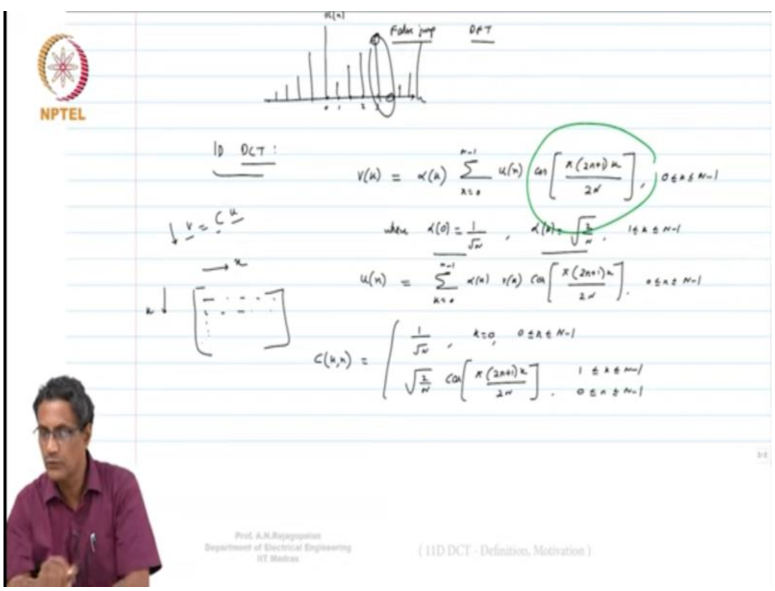

Our ideas is of course to look at DFT DCT and sort of compare why why a DCT got picked. So suppose, I tell you that I have a sequence of number that kind of look like this and I am just plotting something. Suppose n equal to 0, n equal to 1, I have some value n equal to 2, I have some value and let us say 1 2 3. Now now if I, so it is like this. So, if I suppose some sequence x fn versus n. Now, if I compute compute a DFT, computing a DFT means that means that the sequence will start to cyclically repeat. DFT implicitly will kind of make that happen which then which then means that this will again again start to repeat and then of course again from from here to whatever and then starts to repeat.

Now, that is okay but then the problem with this is that what happens is even this repeats the value let us say x of n minus 1 in general and the value of x at 0 these do not have to be the same at all. In general, there is no no no reason to believe that when a sequence ends and begins and ends with the same value. In general, it does not have to. So, so I I sort of I sort of wantonly took took a a case where I have a jump rate a big jump in fact from here to here. So, so this kind of a jump rate which is there so this kind of a jump is what is not really a sort of a sort of a desirable thing in the sense that this is this is like that sawtooth wave, you know, where you are kind of making the jump happen now.

Now so a DFT implicitly will do this. When you give it a sequence it will implicitly repeat it and therefore if you look at computing the Fourier series for this, a DFS for this and compare it with another case where probably this does not happen like big wave. You had a triangular wave, you would expect that this would this would take more time to die. Now a DCT is kind of smart in the sense that one can actually show that, when you compute compute a DCT and DCT is sort when, well we will show that today. DCT, you can show that it is actually it can be arrived at by computing the DFT of x of n symmetrically extended.

So when you extend x of n symmetrically, you actually end up ensuring that x0 and xn minus, I mean, so when you call this a new sequence yn, So, yn and yn minus 1 tend to be y0 and yn minus 1 will tend to be the same. So, that is how you symmetrically extended it. There are various ways to symmetrically extend but the one way in which DCT 2 kind of now do it is a particular way in which it does it and you can show that extension is what we will kind of remove this.

Some people call this is a false jump. This is called this is called a discontinuity or a kind of a false jump in the sense that this is something that is undesirable, but then it will inherently happen, if you compute compute a DFT and then the other property. So, here there is no kind of notion of, there is no statistical notion or anything. This is valid for any x of n you take any x of n and then if you compute a DFT if you complete a DCT will find that you are packing more energy in the first few coefficients. Does not have anything to do with where xn is coming from.

The second second property that I mentioned is purely statistical property. That it is in fact true that DCT basis will come close even though it is independent independent data independent transform but that transform that basis it turns out turns out that it is close to being the basis for a covariance matrix of that type I showed in the showed in the earlier one. So, so so these 2 are very very important thing and they work in the favour of a DCT and then on top of this there are various things. DCT of a real signal is real therefore, you do not have to end up with these complex entries.

Then the other thing is that if take a take a DFT, if you take an n point DFT, you actually exhaust I mean n by 2 on the with respect to whatever the imaginary side and another n by 2, which is itself is to the real side. Whereas here everything is real so you are using all of the all of the n and coefficients. So, there are various other properties that a DCT has. We do not want to, like I said, if you start doing that then we need a whole course for just doing transforms.

So, the idea was simply to simply to motivate the fact that because we are looking at data decorrelation, the diagonalization, I told you going back again that 1 2 3 points that we are kind of looking at looking at how to how to how to see this as a data independent transform. Then also see what are the advantages which you have to use it and the advantages will stem from what about its diagonalization property and whether you can kind of whether you can whether you can see preserve energy, most of the energy in the first few coefficients.

And thirdly, does it is it again something that can be computed quickly, is it separable so that going to going to higher different dimensions easy through simply a product. Then again going to revolve only around those 3 points if you are not going to step out of that. Let me actually write down what what what a 1D DCT looks like. Then we will move on.

So, 1D DCT looks like this. So v k, which are your DCT coefficients are given by alpha k summation n equals 0 to n minus 1 u n cos pi 2 n plus 1. This is again DCT 2. Whatever I am saying is all for DCT type 2. 2 n 0 less than equal to k less than equal to N minus 1 where alpha 0 is 1 by root n and alpha k is equal to root 2 by n, 1 less than equal to k less than equal to n minus 1 and the inverse is given by for the reconstruct un from vk is given as summation k equals 0 to n minus 1 alpha k vk cos pi 2n plus 1 k by 2 n then 0n less than or equal to n minus 1.

So this is like a DCT pair, forward and the inverse. So, of course as you can see, this is this is simply clearly not taking simply a real part of a DFT. If you think of e power j 2 pi by n kn, if you write it as cos 2 pi by n kn plus j sine 2 pi by k, it is not simply taking the real part of the DFT. This this surely is not that but on the face of it therefore it is not clear as to what kind of a relation it might have with not really a DCT, but then before we go into all that, let us go with respect to DFT before we go into that, let us just look at how do we construct.

So again, going back the same way we will now look at V and then suppose you give me a sequence u, I would like to act a DCT on that. So, I will call it as C. So, I need to know how should I construct this C matrix. So we go the same way there there just as in DFT, we had something here and then we said v will be equal to that phi times u. Now similarly what do you find here, I mean you have something like c kn. There it was phi of k comma n. So here, it is c kn. So, this is what 1 by root n for k equals 0.

So if you see VK, when you when you when you expand it so if k equal to 0 and 0 less than or equal to n less than or equal to n minus 1 and this is root 2 by n because of because of this. Because of the fact that alpha 0 is 1 by root n alpha is 1 by root n for k equal to 0 and alpha k is root 2 by n. So, if you expand it, it will be like alpha 0 u n whatever cos and plus alpha 1 and so on. No, not plus alpha 1 plus u 1 and so on. 1 by root n and then this is root 2 by n cos pi 2 n 1 k by 2 n.

This is for k between 1 and N minus 1 and N again is between 0 and N minus 1. So, it means that if you want to do a forward transformation, this is how this is how you would do. So, if you again go back and think about the c matrix just as we thought about the phi matrix. So, where was K? K was running down because that is how the K in v reruns down, call a vector and this is the way n runs.

n, this is n and therefore clearly what you can see is so, k if equal to 0, therefore all these entries will be simply 1 by root n and then when you come to k equal to 1 and then depending upon depending upon the value of N and K, these entries will start to change again for k equal to 2 and so on. So, we have to start filling those entries.

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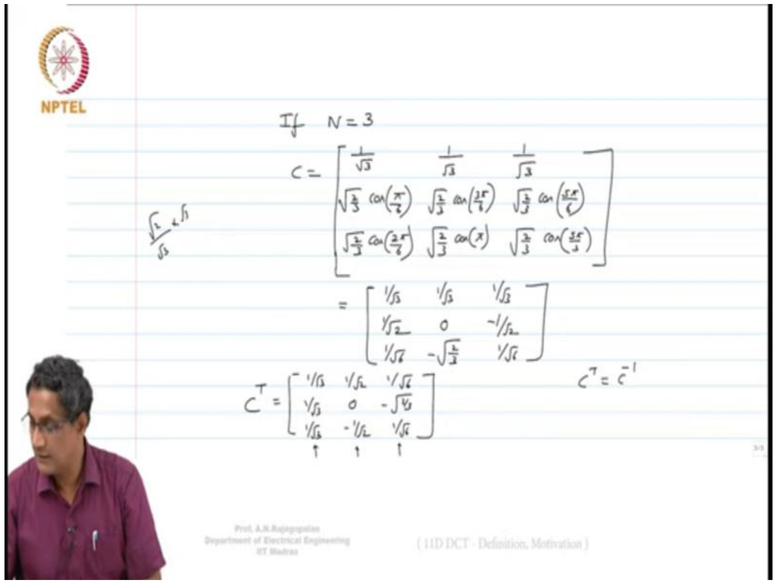

So, the first row of course will be simply this one by so C is equal to so what will you have, 1 by root n for the first row so that is like k equal to 0. So, 1 by root 3, 1 by root 3, 1 by root 3. Then the next entry is what? So, you got k equal to 1. So, what do you have? So, alpha for k equal to 1 and onwards is all root 2 by n? So, if you look at your c k, c k as we found like this for k equal to 1 and onwards. So, what do you have?

So next one will be root 2 by 3 cos n is 0 so pi by 6 because your 2 n, so pi by 6. Then root 2 by 3 cos now n equal to 1, so 2 n plus so 3 pi by 6. Then root 2 by 3 cos 5 pi by 6. So, n equal to 5 pi by 6. Then you come down to k equal to 2 now. So, you will get root 2 by 3 k equal to so cos 2 pi by n equal to 0. So, cos 2 pi by 6 root 2 by 3 cos. So, k equal to 2, n equal to 1. So, what is it? 6 pi by 6, n equal to 1. So, 2 plus 1 3 into 2 6. So, 6 pi by 6.

Actually, that is just cos pi then root 2 by 3. So, k equal to 2, N equal to 2 of 10 pi by 6 which is 5 pi by 3. So, that will be that will be the construction of this c. So, if you look at, so what does it this c equal to can somebody quickly tell me? So, 1 by root 3, 1 by root 3, 1 by root 3 cos pi by 6 is what, root 3 by 2. So root 3 by 2 it is 1 by root 6. So, 1 by root 6. This is 0 pi by 2. Which one? This 1 by root 2 root 2 by root 3 no. Pi by 6 is what? 30. So, root 3 by 1 by root 2. What about 5 pi by 6?

Minus if you do not want to strain your brains, let me see whether I have the minus 1 by root 2. Anyway, these are things that you can do. 1 by root 2 then 1 by root 6 for this and minus root 2 by 3 for this and 1 by root 6 for this. This is your how your C will look like and therefore if you are interested in how the bases will be like, then you have to find out C transpose. So, C transpose will be and of course you can see that C is not equal to C transpose. So, 1 by root 3, 1 by root 2, 1 by root 6, 1 by root 3, 0 minus root 2 by 3, 1 by root 3, minus 1 by root 2, 1 by root 6.

So, each of these columns is now like your AK star. Going back this is the first basis vector which is simply all constants and then this one is the next and that one is the next and all this you can plot actually because they are all real numbers. So, one can see how it varies and so on and again everyone can go. The thing to notice is that all entries are real. Even if you go to higher dimensions, everything is real.

So, the signal is real then after you multiply you will get only real value so that is one advantage of this guy. Another thing is just like it is easy to show that that C transpose is in fact C inverse. Again, going back to the same thing and if you had A Hermitian if unitary then A, which is in this case C star transpose. There is no effect of star. It is all real. So, C transpose. So, I just leave it to you to verify that the inverse of C is simply the transpose of C and the column of the basis vectors can be obtained as columns of C transpose.

Now these basis vectors so they are the implications of what they mean and all with respect to optimality in terms of data decorrelation that we will see later but first just try to understand. So, let us first try to understand as to what you say so as to what is its relation to actually a DFT because it looks like some things look very very familiar but at the same time it is not it is not obvious as to what is that relation.